THE PRIMITIVE FACTOR RINGS OF THE ENVELOPING ALGEBRA OF $sl(2, \mathbb{C})$

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1. *Introduction*

Let R denote a non-artinian primitive factor ring of the enveloping algebra of $sl(2, \mathbb{C})$. Arnal-Pinczon [**1**] and Roos [**10**] have shown that if R is simple then it has Krull dimension 1. Roos also shows that "most" of the simple R have global dimension 1. In this paper we prove that if R is not simple then it has Krull dimension 1 (thus all non-artinian primitive factor rings of $sl(2, \mathbb{C})$ have Krull dimension 1) and does not have global dimension 1.

Notation and the basic properties of these factor rings are described in §2. In particular, if R denotes such a non-simple primitive factor ring, then R has a unique proper two-sided ideal M of finite codimension, and R embeds in the Weyl algebra A_1. In §3 we prove that R has Krull dimension 1. The proof illustrates and depends on the close relationship between R and A_1. In §4 the relationship between certain R-modules and certain A_1-modules is examined more closely. The results in §4 are used in §5 to describe the generators of M as a left ideal. We also show in §5 that the grading on A_1 (defined by the semi-simple element) induces a grading on R, and that both R and M are graded by the induced grading. Finally, in §6 it is proved that R is not hereditary. In particular, it is shown that R/M has projective dimension 2. (The primitive ideal of the enveloping algebra corresponding to R is the annihilator of a Verma module of length two and both composition factors of this Verma module have projective dimension at most two as R-modules.) The precise global dimension of R remains an open question.

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2. *Preliminaries*

All modules are tacitly assumed to be left modules; global dimension and Krull dimension are both calculated on the left but of course a similar argument will give corresponding results on the right. The Krull dimension of a module, M, is denoted by $|M|$, and global dimension is abbreviated to gl.dim. The annihilator of a module M is denoted by ann (M).

Let U denote the enveloping algebra of $sl(2, \mathbb{C})$. The basic properties of U appear in Dixmier [**3**] and Nouazé–Gabriel [**7**]. Let

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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be a basis for $sl(2, \mathbb{C})$. Let

$$Q = 4EF + H^2 - 2H = 4FE + H^2 + 2H \in U.$$

The centre of U is $\mathbb{C}[Q]$. For all $c \in \mathbb{C}$ let $I_c = U(Q-c)$. The map $c \to I_c$ is a bijection from $\mathbb{C}$ onto the set of primitive ideals of U of infinite codimension; I_c is maximal if and only if c is not of the form n^2+2n, $n \in \mathbb{N} = \{0,1,2,\ldots\}$. As mentioned above, if c is not of the form n^2+2n ($n \in \mathbb{N}$) then $|U/I_c| = 1$ ([1], [10]). Roos also proved that if c is transcendental over $\mathbb{Q}$ then gl.dim. $(U/I_c) = 1$. Let $A = A_1(\mathbb{C})$ denote the Weyl algebra over $\mathbb{C}$ with two generators p, q subject to the relation $pq - qp = 1$. For $t \in \mathbb{Z}$ set

$$D(t) = \{x \in A \mid [qp, x] = tx\}$$

(where $[a, b] = ab - ba$). It is obvious that $A = \bigoplus_{-\infty}^{\infty} D(t)$, and

$$D(t) = \begin{cases} q^t\mathbb{C}[qp] & t \geqslant 0 \\ p^{-t}\mathbb{C}[qp] & t < 0. \end{cases}$$

It is easy to see that $D(s)D(t) \subset D(s+t)$; frequent use is made of this fact. We also make frequent use of the identity $q^t p^t = (qp-t+1)\ldots(qp-1)(qp)$.

Roos [10] has shown that U/I_c embeds in A_1. There are a number of different such embeddings and we shall use the embedding defined by the map

$$E \to q(qp-\mu), \quad F \to -p, \quad H \to 2qp - \mu$$

where $\mu \in \mathbb{C}$ satisfies $\mu^2 + 2\mu = c$.

Henceforth fix $n \in \mathbb{N}$, put $c = n^2+2n$, and let R be the subring of A_1 isomorphic to U/I_c, which is defined by the map above with $\mu = n$. Put

$$e = q(qp-n); \quad f = -p; \quad h = 2qp - n.$$

The ring R has a unique proper two sided ideal M; this ideal M is of codimension $(n+1)^2$ in R, and is the annihilator of the unique finite dimensional simple R-module, S, and S has dimension $(n+1)$. We have that $S \cong R/I$ where $I = Re + R(h-n) + Rf^{n+1}$, (see for example [4; 7.2.7]).

Let h denote the subspace of $sl(2, \mathbb{C})$ spanned by H, let h^* denote the dual space of h and let $\lambda : h \to \mathbb{C}$ be the element of h^* defined by $\lambda(H) = n$. In the notation of Dixmier [4; Chapter 7], I_c is the annihilator of the Verma module $M(\lambda+\delta)$ (where δ is the half-sum of positive roots). The finite dimensional simple U-module, $L(\lambda+\delta)$, is isomorphic as an R-module to S, and as mentioned above has annihilator M (as an R-module); $L(\lambda+\delta)$ has highest weight λ and the weights of $L(\lambda+\delta)$ are precisely those $\mu \in h^*$ such that $\mu(H) = n - 2j$ ($j \in \mathbb{N}, 0 \leqslant j \leqslant n$). Thus, $t = \prod_{j=0}^{n} (h - n + 2j)$ is an element of M (this fact is used in Lemma 3.1).

3. *Krull dimension*

Recall Corollaire 1 of Roos [**10**]. R may be localised at $\mathbb{C}[e]\setminus\{0\}$ and at $\mathbb{C}[f]\setminus\{0\}$ to obtain the rings R_e and R_f, say. Both R_e and R_f have Krull dimension 1 (being isomorphic to partial localisations of A); both R_e and R_f are flat as either right or left R-modules. Consider R as a subring of $T = R_e \oplus R_f$ via the diagonal embedding $r \to (r, r)$. It is implicit in Roos' Corollaire 1 that if I and J are left ideals of R with $I \subseteq J$ and $TI = TJ$ then J/I is finite dimensional as a $\mathbb{C}$-vector space.

LEMMA 3.1. *Let I and J be left ideals of R with $I \subseteq J$, J/I a simple left R-module and $AI = AJ$. Then J/I is infinite dimensional as a $\mathbb{C}$-vector space.*

Proof. Suppose not—that is, let J and I be as above with J/I finite dimensional.

Now $AI = I + qI + q^2I + \dots$. For $x \in AI$, define the length of x to be $l(x) = \min\{m \in N \mid x \in I + qI + q^2I + \dots + q^mI\}$. Pick $x \in J$, $x \notin I$ of least length (such x exist because $I \neq J$ and $l(x)$ is defined because $J \subseteq AJ = AI$). Let $x = b_0 + qb_1 + \dots + q^m b_m$ where each $b_i \in I$ and $l(x) = m \geqslant 1$. Put $a = x - b_0$, so $a \in J$, $a \notin I$ and $l(a) = m$.

Put $t = \prod_{j=0}^{n} (h - n + 2j)$. By the remark in §2 $t \in M$. Because J/I is a finite dimensional simple module, $MJ \subseteq I$; in particular $ta \in I$.

Look at tq^i. For $i \geqslant 1$, we have $h = 2qp - n$, so $(h-n+2j)q^i = 2(qp-n+j)q^i = 2q^i(qp-n+j+i) = q^i(h-n+2(j+i))$. So $tq^i = q^i s_i$ where $s_i = \prod_{j=0}^{n} (h-n+2(j+i))$. Thus, $ta = qs_1b_1 + q^2s_2b_2 + \dots + q^m s_m b_m$.

Considering s_i as a polynomial in $(h-n)$ it has non-zero constant term because $j+i \geqslant i \geqslant 1$. Write $s_i = (h-n)r_i + \alpha_i$ where $r_i \in R$ and $0 \neq \alpha_i \in \mathbb{C}$. Now $q(h-n) = 2q(qp-n) = 2e \in R$. Write $b_i' = q(h-n)r_ib_i \in I$. Now

$$ta - \alpha_m a = b_1' + qb_2' + \dots + q^{m-1}b_m' + (\alpha_1 - \alpha_m)qb + \dots + (\alpha_{m-1} - \alpha_m)q^{m-1}b_{m-1}.$$

However, as $\alpha_m \neq 0$, it follows that $ta - \alpha_m a \in J$, $ta - \alpha_m a \notin I$ and $ta - \alpha_m a$ is of shorter length than x. This contradicts the choice of x. Hence J/I cannot be finite dimensional.

THEOREM 3.2. *The Krull dimension of R is* 1.

Proof. Consider the natural embedding of R in $T \oplus A = R_e \oplus R_f \oplus A$ via the map $r \to (r, r, r)$. Suppose that I and J are left ideals of R with $I \subsetneq J$. Then $(T \oplus A)I \neq (T \oplus A)J$ because if $TI = TJ$ then J/I is finite dimensional by the comments prior to Lemma 3.1, but then by the lemma, $AI \neq AJ$.

Thus the lattice of left ideals of R may be embedded in the lattice of left ideals of $T \oplus A$. Consequently, $|R| \leqslant |T \oplus A|$. However,

$$|T \oplus A| = \max\{|R_e|, |R_f|, |A|\} = 1.$$

4. *Relation between certain R-modules and A-modules*

The Weyl algebra A is easier to deal with than R, and the purpose of this section is to exhibit some of the connections between R and A so that the simpler structure of A may be exploited. In particular, Lemma 4.3 will enable us to describe the generators of M (Theorem 5.2) by using the properties of A. Lemma 4.1 and Lemma 4.2 exhibit the remarkably close relationship between R and A. Unfortunately, though, A is not flat as an R-module.

LEMMA 4.1. *If J is a left ideal of A, then $J = A(J \cap R)$.*

Proof. Clearly $J \supseteq A(J \cap R)$.

For $m \in \mathbb{N}$, put $J_m = J \cap (R+qR+\ldots+q^mR)$. It is obvious that $A = R+qR+q^2R+\ldots$, and so $J = \bigcup_{m=0}^{\infty} J_m$. We will show that $J_m \subseteq A(J \cap R)$ for all m. The proof is by induction on m. Clearly $J_0 \subseteq A(J \cap R)$.

Suppose that $J_m \subseteq A(J \cap R)$ and pick $x \in J_{m+1}$. We will show that both $(qp-n-m-1)x$ and $(qp)x$ are in $A(J \cap R)$. Let $x = y+q^{m+1}r$ with $r \in R$ and $y \in R+qR+\ldots+q^mR$. Now,

$$(qp-n-m-1)x = (qp-n-m-1)y+q^{m+1}(qp-n)r = (qp-n-m-1)y+q^m r'$$

where $r' = q(qp-n)r \in R$. It is easy to check that $(qp-n-m-1)y \in R+qR+\ldots+q^mR$, and so $(qp-n-m-1)x \in J_m \subseteq A(J \cap R)$. Now

$$px = py+(qp+1)q^m r = py+q^m(qp+m+1)r$$

and because $py \in R+qR+\ldots+q^mR$ it follows that $px \in J_m \subseteq A(J \cap R)$. Consequently, $qpx \in A(J \cap R)$, and thus $(n+m+1)x \in A(J \cap R)$ which implies $x \in A(J \cap R)$ as $n+m+1 \neq 0$. Hence $J_{m+1} \subseteq A(J \cap R)$.

LEMMA 4.2. *Let K be a maximal left ideal of R and suppose $AK \neq A$. Then AK is a maximal left ideal of A, and $K = AK \cap R$.*

Proof. Now $K \subseteq AK \cap R$ implies that either $AK \cap R = R$ or $AK \cap R = K$. If $AK \cap R = R$ then $1 \in AK$ which implies $AK = A$, contradicting the hypothesis. So $AK \cap R = K$.

Suppose now that J is a left ideal of A and $AK \subseteq J \neq A$. Clearly $K \subseteq J \cap R$, so either $J \cap R = R$ or $J \cap R = K$. But $J \cap R \neq R$ (as $J \neq A$) so $J \cap R = K$. However, by Lemma 4.1, $J = A(J \cap R)$, so $J = AK$ and consequently AK is a maximal left ideal of A.

LEMMA 4.3. *If K is a maximal left ideal of R and $AK \neq A$, then A/AK (considered as an R-module) contains an isomorphic copy of R/K, namely $(R+AK)/AK$, and*

$$\operatorname{ann}_R(R/K) = R \cap \operatorname{ann}_A\big((R+AK)/AK\big).$$

Proof. As R-modules, $(R+AK)/AK \cong R/(R \cap AK) = R/K$ (by Lemma 4.2). The second part of the Lemma follows easily from the fact that

$$\operatorname{ann}_R(R/K) = \operatorname{ann}_R\big((R+AK)/AK\big).$$

5. *Description of R and M*

Recall the definition of the $D(t)$ in §2. It is well known that the $D(t)$ define a gradation on A. Because R is generated as an algebra by homogeneous elements of A (with respect to the gradation defined by the $D(t)$) it follows that

$$R = \bigoplus_{-\infty}^{\infty} (R \cap D(t)).$$

This fact together with the proposition below describes R in detail.

PROPOSITION 5.1. (i) *If $t > 0$ then*

$$D(t) \cap R = q^t(qp-n)(qp-n+1)\dots(qp-n+t-1)\mathbb{C}[qp]$$

$$= (qp-n-t)(qp-n-t+1)\dots(qp-n-1)q^t\mathbb{C}[qp].$$

(ii) *If $t \leqslant 0$ then*

$$D(t) \cap R = D(t) = p^{-t}\mathbb{C}[qp].$$

Proof. (i) Recall that R is spanned (as a $\mathbb{C}$-vector space) by the homogeneous elements $\{e^if^jh^l \mid i,j,l \geqslant 0\}$, and so $R \cap D(t)$ is spanned by $\{e^if^jh^l \mid i,j,l \geqslant 0\} \cap D(t)$. Now $e^if^jh^l \in D(i-j)$ so $e^if^jh^l \in D(t)$ if and only if $i-j = t$. It follows that $e^t\mathbb{C}[qp] \subseteq D(t)$; also if $i-j = t$, then $e^if^jh^l = e^te^jf^jh^l \in e^tD(0) \subset R$. Hence $R \cap D(t) = e^t\mathbb{C}[qp]$, and the result follows because

$$e^t = [q(qp-n)]^t = q^t(qp-n)\dots(qp-n+t-1) = (qp-n-t)\dots(qp-n-1)q^t.$$

(ii) As $p \in R$, so $p^t \in R$ and consequently $p^t\mathbb{C}[qp] \subseteq R$ as $qp \in R$.

THEOREM 5.2. *The unique proper ideal M of R is generated as a left ideal by*

$$X = \{q^ip^{n+1} \mid 0 \leqslant i \leqslant 2(n+1)\}.$$

Proof. We show first that $X \subseteq R$. Now $q^ip^{n+1} \in D(i-n-1)$, so if $0 \leqslant i \leqslant n+1$ then $q^ip^{n+1} \in R$ by Proposition 5.1 (ii). Suppose $i = j+n+1$ with $1 \leqslant j \leqslant n+1$. Then

$$q^ip^{n+1} = q^jq^{n+1}p^{n+1} = q^j(qp-n)\dots(qp-1)(qp)$$

and because $j \leqslant n+1$ this is an element of $q^j(qp-n)\dots(qp-n+j-1)\mathbb{C}[qp]$ which by Proposition 5.1 (i) is contained in R. Hence $X \subseteq R$.

Recall that $M = \operatorname{ann}_R(R/I)$ (where I is as in §2). Now $AI = Ae+A(h-n)+Af^{n+1} = A(qp-n)+Ap^{n+1}$, which by McConnell–Robson [5; Proposition 5.11] is a maximal left ideal of A. In particular $AI \neq A$, and so by Lemma 4.3

$$M = R \cap \operatorname{ann}_A[(R+AI)/AI].$$

But $R/I = [\mathbb{C}+\mathbb{C}p+\dots+\mathbb{C}p^n+I]$, so $(R+AI)/AI = [\mathbb{C}+\mathbb{C}p+\dots+\mathbb{C}p^n+AI]$. It is easy to see that $p^{n+1}[(R+AI)/AI] = 0$ (considering $(R+AI)/AI$ as a subset of the

A-module A/AI). Thus $Ap^{n+1} \subseteq \text{ann}_A[(R+AI)/AI]$ which implies that $X \subseteq M$ and so $RX \subseteq M$.

Finally we show that the left ideal generated by X is also a right ideal. To show this it suffices to prove that $Xf \subseteq RX$, $Xh \subseteq RX$, and $Xe \subseteq RX$. These three cases are proved separately:

(i) $p^{n+1} . p \in RX$; if $i \geqslant 1$ then

$$q^i p^{n+1} . p = q^{i-1}(qp)p^{n+1} = (qp-i+1)q^{i-1}p^{n+1} \in RX;$$

(ii) $q^i p^{n+1} . (qp) = (qp+n+1-i)q^i p^{n+1} \in RX$;

(iii) $q^i p^{n+1} . q(qp-n) = q^i p^n (qp+1)(qp-n) = (qp+n-i+1)q^{i+1}p^{n+1}$, and if $i \leqslant 2n+1$ then this is in RX; should $i = 2(n+1)$ then this is equal to

$$(qp-n-1) . q . q^{2(n+1)}p^{n+1} = q(qp-n)q^{2(n+1)}p^{n+1} \in RX.$$

It is well known that if J is a two sided ideal of a factor ring of an enveloping algebra of a finite dimensional lie algebra, then any set which generates J as a left ideal also generates J as a right ideal. Thus $M = RX = XR$. Because the generators of M are homogeneous we have the following.

COROLLARY 5.3. $M \mp \bigoplus_{-\infty}^{\infty} (M \cap D(t))$.

We now have a good enough description of M to show that R is not hereditary (Theorem 6.2) but to show that R/M has projective dimension 2 a slightly better description of M is required. This is given in Theorem 5.5.

LEMMA 5.4. *Let K be the left ideal of R. Let $m \in N$ and suppose $a \in K$ and $q^m a \in K$. Then $a, qa, q^2 a, \ldots, q^m a$ are all in K.*

Proof. The proof is by induction on m. Clearly the result holds if $m = 1$. Suppose $m \geqslant 2$.

If $a \in K$ then $[q(qp-n)]^{m-1} a \in K$, and

$$[q(qp-n)]^{m-1}a = q^{m-1}(qp-n)\ldots(qp-n+m-2)a$$

$$= (qp-n-m+1)\ldots(qp-n-1)q^{m-1}a$$

If $q^m a \in K$ then $pq^m a = (qp+1)q^{m-1}a \in K$. It is now easy to see that $q^{m-1}a \in K$ because when viewed as polynomials in (qp) the expressions $(qp-n-m+1)\ldots(qp-n-1)$ and $(qp+1)$ do not have a common root.

THEOREM 5.5. *As a left ideal M is generated by $f^{n+1} = p^{n+1}$ and by $e^{n+1} = [q(qp-n)]^{n+1} = q^{2(n+1)}p^{n+1}$.*

Proof. By Theorem 5.2 both these elements are in M, and a single application of Lemma 5.4 shows that $X \subseteq Rp^{n+1} + Rq^{2(n+1)}p^{n+1}$.

6. *Global dimension*

Let $S = \{a \in Q(R) \mid aM \subseteq M\}$ where $Q(R)$ is the quotient division ring of R. In fact $Q(R) = D_1$ the quotient division ring of A_1. From Theorem 5.2 we see that $p^{n+1} \in M$ and $AM = Ap^{n+1}$, so $aM \subseteq M$ implies $ap^{n+1} \in Ap^{n+1}$ whence $a \in A$. Thus $R \subseteq S \subseteq A$.

THEOREM 6.1. *The rings S and R are equal.*

Proof. Because $A = \bigoplus_{-\infty}^{\infty} D(t)$ and $R = \bigoplus_{-\infty}^{\infty} (R \cap D(t))$ with $R \cap D(t)$ as given in Proposition 5.1, it follows that

$$A = R \oplus \left[\bigoplus_{t=1}^{\infty} q^t V_t \right]$$

where V_t is the $\mathbb{C}$-subspace of $\mathbb{C}[qp]$ generated by $\{1, (qp), (qp)^2, \ldots, \ldots, (qp)^{t-1}\}$. Suppose now that $S \neq R$. Then there exists $0 \neq a \in S \cap \left(\bigoplus_{t=1}^{\infty} q^t V_t \right)$ with $a = \sum_{t=1}^{m} a_t$ where each $a_t \in q^t V_t \subseteq D(t)$. Clearly $aM \subseteq M$ if and only if $aX \subseteq M$; because X consists of homogeneous elements and $M = \bigoplus_{-\infty}^{\infty} (M \cap D(t))$ we have for some t that $a_t \neq 0$ and $a_t X \subseteq M$. So we may assume, without loss of generality, that there exists $0 \neq a \in S \cap q^t V_t$ for some t.

Let $a = q^t f(qp)$ where $f(qp)$ is a polynomial in qp of degree $\leqslant t-1$. Now,

$$aq^{2(n+1)}p^{n+1} = q^{n+t+1}(qp-n)\ldots(qp)f(qp+n+1)$$

and as $aX \subseteq M$ this is an element of $R \cap D(n+t+1)$. Now by Proposition 5.1 (i) $aq^{2(n+1)}p^{n+1} \in R \cap D(n+t+1)$ if and only if $f(qp) \in (qp+1)\ldots(qp+t)\mathbb{C}[qp]$. However, $f(qp) \neq 0$ and the degree of f is $\leqslant t-1$, so $f(qp)$ cannot be an element of this ideal of $\mathbb{C}[qp]$. Thus $aq^{2(n+1)}p^{n+1} \notin M$. This contradiction shows that no such a can exist and hence $S = R$.

COROLLARY 6.2. *The global dimension of R is either 2 or infinity.*

Proof. Suppose gl.dim. R is finite. It is well known that gl.dim. $U = 3$, so by Kaplansky [**5**; Theorem 4, p. 173], gl.dim. $R \leqslant 2$. It is clear that gl.dim. $R \neq 0$. It remains to show that gl.dim. $R \neq 1$.

Suppose gl.dim. $R = 1$; that is R is hereditary. Because M is idempotent (that is, $M^2 = M$) and R is prime noetherian, Theorem 4 (iii) of Robson [**9**] applies. This says that $MS = S$, but this contradicts Theorem 6.1 as $M \neq R$. Hence gl.dim. $R = 2$.

If K is a left ideal of R put $K^* = \{a \in Q(R) \mid Ka \subseteq R\}$. Because R is a prime noetherian ring $K^* \cong \text{Hom}(K, R)$ and so K is projective if and only if $K^*K = \text{End}_R(K)$; and for this to happen it is sufficient that $1 \in K^*K$. This provides a reasonable method to prove Theorem 6.4. First we require an easy lemma.

LEMMA 6.3. *If s and sq^m are both elements of R, then $sq, sq^2, \ldots, sq^{m-1}$ are also elements of R.*

Proof. It is enough to prove the lemma when $sq^m \in R \cap D(t)$. Suppose $t > 1$; then $sq^m = x(qp-n-t)\ldots(qp-n-1)q^t$ for some $x \in \mathbb{C}[qp]$ by Proposition 5.1 (i). Thus $sq^{m-1} = x(qp-n-t)\ldots(qp-n-1)q^{t-1}$ which is an element of R by Proposition 5.1 (i). Suppose $t \leqslant 1$; then $sq^{m-1} \in D(t-1)$ and $t-1 \leqslant 0$, but by Proposition 5.1 (ii), $D(t-1) \subseteq R$. An induction argument now shows that the lemma holds.

THEOREM 6.4. *The R-module R/M has projective dimension* 2.

Proof. Because R is uniform as an R-module, R/M is not projective. Notice that the ring S is the endomorphism ring of M, so M is projective if and only if $M^*M = S$. However, $S = R$ by the previous theorem. Thus if M were projective then $M^*M = R$; but this cannot happen as M is idempotent, viz. $M^*M = R$ implies $M^*M^2 = M$. So M is not projective.

Now look at a projective resolution for $M = Rp^{n+1} + Rq^{2(n+1)}p^{n+1}$. Let

$$0 \longrightarrow K \longrightarrow R \oplus R \xrightarrow{\pi} M \longrightarrow 0$$

be a short exact sequence where $\pi(r, s) = rp^{n+1} + sq^{2(n+1)}p^{n+1}$. It is obvious that the kernel K is given by

$$K = \{(-sq^{2(n+1)}, s) \mid s \in R \quad \text{and} \quad sq^{2(n+1)} \in R\};$$

K is isomorphic to the left ideal $J = \{s \in R \mid sq^{2(n+1)} \in R\}$. To show that R/M has projective dimension two it suffices to show that $1 \in J^*J$ (that is J is projective).

Clearly $1, q^{2(n+1)} \in J^*$, so $1, q, q^2, \ldots, q^{2(n+1)}$ are elements of J^* by Lemma 6.3. It is easily checked that J contains $p^{2(n+1)}$ and $p^t(qp-n-2(n+1))\ldots(qp-n-t-1)$ for $0 \leqslant t < 2(n+1)$. Hence J^*J contains the following elements:

$$q^{2(n+1)}p^{2(n+1)},\ q^t p^t(qp-n-2(n+1))\ldots(qp-n-t-1)$$

for $0 \leqslant t < 2(n+1)$. Putting $x = qp$ these elements are

$$x(x-1)\ldots(x-2n-1);$$

$$\{(x-3n-2)\ldots(x-n-t-1)(x-t+1)\ldots(x-1)x \mid 1 \leqslant t \leqslant 2n+1\};$$

$$(x-3n-2)\ldots(x-n-1).$$

Look at the commutative polynomial ring $\mathbb{C}[x]$. The ideal of $\mathbb{C}[x]$ generated by the elements above is $\mathbb{C}[x]$ itself, so $1 \in J^*J$ (because $\mathbb{C}[x] \subseteq J^*$).

COROLLARY 6.5. *The unique finite dimensional simple R-module has projective dimension* 2.

Proof. The R-module R/M is a direct sum of copies of this simple module so Exercise 4 of Kaplansky [5; p. 169] gives the result.

7. *Remarks*

1. It is possible for corresponding results to be established on the right rather than the left. However, for this to be done a different embedding of R in A_1 is required. This embedding is given by the map

$$E \to -(n+pq)q, \quad F \to p, \quad H \to n+2pq.$$

2. As mentioned in §4, A is not flat as an R-module. To see this let I be the left ideal of R generated by p and qp. Then $q \otimes p - 1 \otimes qp$ is a non-zero element of $A \otimes I$, but the image of this element in $A \otimes R$ under the natural homomorphism is zero. Thus A is not flat as a right R-module. A similar example using the right ideal of R generated by p and pq shows that A is not flat as a left R-module either.

3. As mentioned in the introduction and in §2 (using the notation of §2), $M(\lambda+\delta)$ is an artinian R-module of length two and is faithful over R. Moreover, $M(\lambda+\delta)$ has a unique simple submodule which is itself faithful over R. This simple submodule is isomorphic to R/K where

$$K = Re+R(h+n+2) = Rq(qp-n)+R(qp+1).$$

To show that K has projective dimension $\leqslant 1$ is straightforward. Let

$$0 \longrightarrow J \longrightarrow R \oplus R \xrightarrow{\ \pi\ } K \longrightarrow 0$$

be the short exact sequence given by $\pi(r, s) = rq(qp-n)+s(qp+1)$, so the kernel, J, is given by

$$J = \{(r, s) \mid \pi(r, s) = 0\} \cong \{r \in R \mid rq(qp-n)(qp+1)^{-1} \in R\} = J_2$$

where J_2 is a left ideal of R. It is easy to see that $p \in J_2$, and that $q(qp-n)(qp+1)^{-1} \in J_2^*$, so $q(qp-n)(qp+1)^{-1}p \in J_2^*J_2$. This element is equal to $qp-n-1$, and this together with the fact that $qp \in J_2 \subseteq J_2^*J_2$ shows that $1 \in J_2^*J_2$ (as $n \geqslant 0$). So J_2, and hence J, is projective.

4. Dixmier [2] has shown that every left ideal of A_1 can be generated by two elements. More generally, Stafford [11] has shown that every left ideal of a simple noetherian ring with Krull dimension 1 can be generated by two elements. Considering how close R is to satisfying these conditions, we are prompted to ask whether every left ideal of R can be generated by two elements. The answer is no.

Let $J = \{s \in R \mid sq^{2(n+1)} \in R\}$ be the left ideal which occurs in Theorem 6.4 as the kernel of the resolution of M. We will show that J requires at least three generators. Clearly if J can be generated by less than three elements so too can the R-module J/MJ. It will be shown that J/MJ requires at least three generators.

We use the techniques and notation developed in Ratliff and Robson [8]. To begin recall the definitions of [8]. Let B be a finitely generated R-module. Let $J(B)$ denote the Jacobson radical of B (that is, the intersection of the maximal submodules of B), and $\lambda(B)$ denotes the length of a composition series for $B/J(B)$, or ∞ if no composition series exists. Suppose $\lambda(B) < \infty$ and $B \neq 0$. For each simple R-module T occurring in a composition series for $B/J(B)$, let $e(T)$ denote the number of copies of T in the composition series and let $f(T) = \lambda(R/\text{ann } T)$. Define $\nu(B)$ to be the least integer such that

$$\nu(B) \geqslant \sup\{1, e(T)/f(T)\}$$

where T varies over all simple composition factors of $B/J(B)$. The following theorem forms the basis for our proof.

THEOREM 7.1 [8]. *If $\lambda(B) < \infty$, then B can be generated by $\nu(B)$ elements and no fewer.*

Here J/MJ is a finitely generated left R/M module, so is certainly of finite length as R/M is simple artinian. It follows that J/MJ is semi-simple and has zero Jacobson radical. The only module possibly occurring in a composition series for J/MJ is the finite dimensional simple module S (of dimension $n+1$). It is clear that $f(S) = n+1$ and $e(S)$ is precisely $\dim(J/MJ)/(n+1)$ so we shall show that $\dim(J/MJ) > 2(n+1)^2$, implying $\nu(J/MJ) \geqslant 3$, and hence by the theorem, J has at least three generators.

The idea behind the proof is simple but a tedious amount of calculation is required. The first step is to show that $MJ \subseteq J \cap Ap^{3(n+1)}$, and the problem is reduced to showing that we can find at least $2(n+1)^2+1$ elements of J which are linearly independent (over $\mathbb{C}$) modulo $Ap^{3(n+1)}$.

LEMMA 7.2. $MJ \subseteq Ap^{3(n+1)}$.

Proof. After Theorem 5.5 it is enough to show that $p^{n+1}J \subseteq Ap^{3(n+1)}$. Because $q^{2(n+1)}$ is a homogeneous element of A it follows that $J = \bigoplus_{-\infty}^{\infty} (J \cap D(t))$, and accordingly we prove that $p^{n+1}(J \cap D(t)) \subseteq Ap^{3(n+1)}$ for all t. Let $a \in J \cap D(t)$.

Suppose $t > 0$. Put $a = (qp-n-t)\dots(qp-n-1)f(qp)q^t$ (after Proposition 5.1) where $f(qp) \in \mathbb{C}[qp]$. Because $aq^{2(n+1)} \in R$, it follows (by Proposition 5.1) that there exists $g(qp)$ in $\mathbb{C}[qp]$ such that $f(qp) = (qp-n-2(n+1)-t)\dots(qp-n-t-1)g(qp)$. Now

$$p^{n+1}a = (qp-n-(n+1)-t)\dots(qp)p^{n+1}g(qp)q^t = g(qp+n+1)q^{2(n+1)+t}p^{3(n+1)+t}q^t$$
$$= g(qp+n+1)q^{2(n+1)+t}p^{3(n+1)}p^tq^t$$

which is an element of $Ap^{3(n+1)}$ as $p^tq^t \in \mathbb{C}[qp]$ and $p^{3(n+1)}\mathbb{C}[qp] = \mathbb{C}[qp]p^{3(n+1)}$.

Suppose $t \leqslant 0$. If $t \leqslant -2(n+1)$ then it is clear that

$$p^{n+1}a \in D(t-(n+1)) \subseteq Ap^{3(n+1)}$$

by Proposition 5.1. So suppose $-2(n+1) < t \leqslant 0$, and put $s = -t$. Let $a = f(qp)p^s$ where $f(qp) \in \mathbb{C}[qp]$. Because $aq^{2(n+1)} \in R$, and is equal to $f(qp)p^sq^sq^{2(n+1)-s}$, it follows that

$$f(qp)p^sq^s = (qp-n-2(n+1)+s)\dots(qp-n-1)g(qp)$$

for some $g(qp) \in \mathbb{C}[qp]$ (this is by Proposition 5.1). Now $p^sq^s = (qp+1)\dots(qp+s)$ and so $g(qp) = (qp+1)\dots(qp+s)h(qp)$ with $h(qp) \in \mathbb{C}[qp]$. In particular, if follows that $f(qp) = (qp-n-2(n+1)+s)\dots(qp-n-1)h(qp)$ and now

$$p^{n+1}a = p^{n+1}(qp-n-2(n+1)+s)\dots(qp-n-1)h(qp)p^s = q^{2(n+1)-s}p^{3(n+1)-s}h(qp)p^s$$

which is an element of $Ap^{3(n+1)}$ because $p^{3(n+1)-s}h(qp) \in \mathbb{C}[qp]p^{3(n+1)-s}$.

This completes the proof of the lemma.

LEMMA 7.3. *The vector space* $J/J \cap Ap^{3(n+1)}$ *has dimension at least* $n+1+2(n+1)^2$.

Proof. In Theorem 6.4 it was shown that J contains the elements $p^{2(n+1)}$ and $p^t(qp-n-2(n+1))\dots(qp-n-t-1)$ for $0 \leqslant t < 2(n+1)$. Because J is a left ideal J contains the following elements:

$$x(s,t) = \begin{cases} p^{2(n+1)+s} & 0 \leqslant s < n+1, \quad t = 2(n+1) \\ p^{t+s}(qp-n-2(n+1))\dots(qp-n-t-1) & 0 \leqslant s < n+1, \quad 0 \leqslant t < 2(n+1) \end{cases}$$

There are precisely $n+1+2(n+1)^2$ of the elements and the proof will show that the images of these elements in $J/J \cap Ap^{3(n+1)}$ are linearly independent. Notice that each $x(s,t)$ may be written uniquely in the form $y(s,t)p^{s+t}$ where $y(s,t)$ is a polynomial in qp of degree $2(n+1)-t$.

Suppose that $\sum \alpha_{s,t}x(s,t) \in Ap^{3(n+1)}$ where $\alpha_{s,t} \in \mathbb{C}$ and the sum over all pairs (s,t) with $0 \leqslant s < n+1$, $0 \leqslant t < 2(n+1)$. Because $p^{3(n+1)}$ is a homogeneous element of A, it follows that $Ap^{3(n+1)} = \bigoplus_{-\infty}^{\infty} (Ap^{3(n+1)} \cap D(r))$. Fix an integer r with $0 \leqslant r \leqslant 3(n+1)$. Splitting the above sum into its homogeneous parts, it must be the case that

$$\sum_{s+t=r} \alpha_{s,t}x(s,t) \in Ap^{3(n+1)} \cap D(-r).$$

Now, if $x \in A$, then $xp^{3(n+1)} \in D(-r)$ if and only if $x \in D(3(n+1)-r)$; that is if and only if $x = f(qp)q^{3(n+1)-r}$ for some $f(qp) \in \mathbb{C}[qp]$. Hence,

$$\sum_{s+t=r} \alpha_{s,t}x(s,t) = f(qp)q^{3(n+1)-r}p^{3(n+1)}$$

and, dividing on the right by p^r,

$$\sum_{s+t=r} \alpha_{s,t}y(s,t) = f(qp)q^{3(n+1)-r}p^{3(n+1)-r}.$$

Now, $q^{3(n+1)-r}p^{3(n+1)-r}$ is an element of $\mathbb{C}[qp]$ of degree $3(n+1)-r$, so the above equation may be interpreted as being a relationship of linear dependence between certain polynomials in $\mathbb{C}[qp]$. However, each term on the left is of degree $2(n+1)-t = 2(n+1)-(r-s) < 3(n+1)-r$ because $s < n+1$. So we have a linear dependence relation between elements of $\mathbb{C}[qp]$, each element being of different degree—this of course can only happen if all $\alpha_{s,t} = 0$.

COROLLARY 7.4. *The left ideal* $J = \{s \in R \mid sq^{2(n+1)} \in R\}$ *requires at least three generators.*

Proof. It is simply a matter of putting together all the above. By the two foregoing lemmas, $\dim(J/MJ) > \dim(J/J \cap Ap^{(n+1)}) > 2(n+1)^2$, and then the discussion prior to Lemma 7.2 completes the argument.

Note added in proof. Stafford has recently shown that every infinite dimensional simple R-module is of projective dimension 1. Consequently gl.dim. $R = 2$.

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