Math. Z. 215, 169–177 (1994)

Mathematische Zeitschrift

Point modules over Sklyanin algebras

S.P. Smith*
University of Washington, Seattle, WA 98195, USA

Received 21 October 1992; in final form 22 January 1993

1 Introduction and definitions

We work over a fixed algebraically closed base field k. We also fix an integer $n \geqq 3$, an elliptic curve E defined over k, and a point $\tau \in E$ which is not in the nm-torsion subgroup E_{nm} for any $m = 1, 2, \ldots, n-2$.

Following Odesskii and Feigin [4, 5] we now define a class of graded algebras determined by this data.

Definition 1.1 [5] Fix a degree n line bundle $\mathscr{L}$ on E. Set $V = H^0(E, \mathscr{L})$. Identify $V \otimes V$ with $H^0(E \times E, \mathscr{L} \boxtimes \mathscr{L})$. Define the shifted diagonal $\Delta_\tau := \{(x, x + (n-2)\tau) \mid x \in E\}$. Denote by M the set of fixed points for the involution $(x, y) \to (y + 2\tau, x - 2\tau)$ on $E \times E$. We say that a divisor D on $E \times E$ is *allowable* if D is stable under this involution, and M occurs in D with even multiplicity. The *n-dimensional Sklyanin algebra* associated to (E, τ) is defined to be the quotient of the tensor algebra,

$$A(E, \tau) := T(V)/\langle R_A \rangle$$

where

$$R_A := \{f \in V \otimes V \mid f = 0, \text{ or } (f)_0 = \Delta_\tau + D \text{ where } D \text{ is allowable}\}.$$

Since E and τ will be fixed throughout the paper, we will just write A for $A(E, \tau)$. The algebra A is denoted $Q_{n,1}(E, \tau)$ by Odesskii and Feigin. The dependence of the algebra on $\mathscr{L}$ is illusory, since any two line bundles of degree n are pullbacks of one another along suitable translations.

A *linear module* over A is a cyclic graded A-module having the same Hilbert series as a polynomial ring. The linear modules over the 3- and 4-dimensional Sklyanin algebras are rather well understood now [1, 2, 3, 7]. A *point module* is a linear module with the same Hilbert series as the polynomial ring in 1 variable. Our main result is that the point modules over $A(E, \tau)$ are parametrized by E, if

---

* Supported by NSF grant DMS-9100316

$n \geqq 5$. If $p \in E$ then we write $M(p)$ for the corresponding point module: an explicit description of $M(p)$ is given by Proposition 1.2 below.

There is already quite a bit of information about linear modules in [4] and [5], although few proofs are provided. In particular, it is stated in [5, Sect. 3.3] that for all n the point modules are precisely the $M(p)$ for $p \in E$. Unfortunately it is not proved there that these are *all* the point modules, and since the claim is false when $n = 4$, it seems sensible to offer a proof for $n \geqq 5$.

Define $\sigma \in \mathrm{Aut}(E)$ by $p^\sigma = p + (n-2)\tau$. Thus Δ_τ is the graph of σ.

Proposition 1.2 *Let $p \in E$. Define an action of the tensor algebra $T(V)$ on the graded vector space $M := kv_0 \oplus kv_1 \oplus \ldots$ by*

$$x \cdot v_j = x(p^{\sigma^{-j}})v_{j+1}$$

for each $x \in V$. (More precisely, for each j fix a representative of $p^{\sigma^{-j}}$ in V^* and evaluate each $x \in V$ at this representative.) Then M is an A-module, and is a point module if $\deg(v_j) = j$.

Proof. It is obvious that M is a cyclic $T(V)$-module, and hence a point module for $T(V)$. To see that M is an A-module, it suffices to check that $f \cdot v_j = 0$ for all $j \geqq 0$, and all relations $f \in R_A \subset V \otimes V$. This is clear since $(p^{\sigma^{-(j+1)}}, p^{\sigma^{-j}}) \in \Delta_\tau$, and all the relations vanish on Δ_τ by definition. □

One of the main steps in proving these are all the point modules, is to prove that the subvariety $\mathscr{V}(R_A) \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ cut out by R_A equals the shifted diagonal Δ_τ. Once this is proved, a simple argument involving truncated point modules (Lemma 4.2 and Theorem 4.3) completes the classification of the point modules. To establish this intermediate result, we must show that there are certain relations in R_A which are closely related to the geometry of $E \subset \mathbb{P}(H^0(E, \mathscr{L})^*)$. This is the substance of Sect. 2.

2 Rank 2 tensors in R_A

By viewing elements of $V \otimes V$ as linear maps $V^* \to V$ we may speak of their rank. The main result in this section is the following

Theorem 2.1 *The (projectivized) space of rank 2 tensors in R_A is in bijection with the secant $(n-3)$-planes to E in $\mathbb{P}(V^*) \cong \mathbb{P}^{n-1}$.*

In particular, if $\mathscr{V}(u,v)$ is a secant $(n-3)$-plane then the corresponding rank 2 relation is of the form $a \otimes v - b \otimes u$ with

$$(u)_0 = \sum_{i=1}^{n-2} (p_i) + (q_1) + (q_2)$$

$$(v)_0 = \sum_{i=1}^{n-2} (p_i) + (r_1) + (r_2)$$

$$(a)_0 = \sum_{i=1}^{n-2} (p_i + 2\tau) + (q_1 - (n-2)\tau) + (q_2 - (n-2)\tau)$$

$$(b)_0 = \sum_{i=1}^{n-2} (p_i + 2\tau) + (r_1 - (n-2)\tau) + (r_2 - (n-2)\tau).$$

The p_i, q_i, r_i may be arbitrary elements of E.

It is obvious that R_A cannot contain any rank 1 tensors, because a non-zero element $u \otimes v \in V \otimes V$ cannot vanish on Δ_τ. The importance of rank 2 tensors in R_A is already apparent from [1] and [3].

From now on we will identify E with its image in $\mathbb{P}(V^*) \cong \mathbb{P}^{n-1}$ obtained via the ample line bundle $\mathscr{L}$. Thus E is a degree n curve.

Whenever we intersect E with a linear subspace of $\mathbb{P}(V^*)$, we will be interested in the scheme-theoretic intersection. Hence we always treat this intersection as an element of $\mathrm{Div}(E)$, the group of divisors on E. For convenience we will choose our identity element $0 \in E$ such that $\mathscr{L} \cong \mathcal{O}_E(n.(0))$. Hence by Abel's Theorem n points of E span a hyperplane of $\mathbb{P}(V^*)$ if and only if their sum is 0.

Definition 2.2 If $H \cong \mathbb{P}^k$ is a linear subspace of $\mathbb{P}(V^*)$, we call H a secant k-plane to E, if H meets E at $k+1$ points counted with multiplicity i.e. $H \cap E$ is a divisor on E of degree $k+1$. Secant 1-planes and secant 2-planes, will be called simply *secant lines* and *secant planes*. The union of all the secant lines (which includes the tangent lines to E since we are counting intersections with multiplicity) is denoted $\mathrm{Sec}(E)$.

Lemma 2.3 *If $D = (p_1) + \cdots + (p_k) \in \mathrm{Div}(E)$ then there is a unique linear subspace $H \cong \mathbb{P}^{k-1}$ of $\mathbb{P}(V^*)$ such that $H \cap E = D$.*

Proof. This is obvious. It is enough to prove it when $k = n-1$. In that case $n-1$ points certainly do not span $\mathbb{P}^{n-1}$, so suppose they lie in two distinct hyperplanes. The intersection of these hyperplanes gives $H \cong \mathbb{P}^{n-3}$ such that $H \cap E \supset \sum_{i=1}^{n-1} (p_i)$. Choose a $p \in E$ in general position, and let H' be the hyperplane spanned H and p. Thus $H' \cap E \supset \sum_{i=1}^{n-1} (p_i) + (p)$, so the sum of these n points must be zero. This is absurd, as p may vary. $\square$

Proof of Theorem 2.1 Suppose that $f = a \otimes v - b \otimes u \in R_A$ is of rank 2. We will prove that $\mathscr{V}(u,v)$ meets E at $n-2$ points counted with multiplicity. Since $\deg(E) = n$ we may write $(u)_0 = \sum_{i=1}^n (p_i)$ for the divisor of zeroes of u on E, and similarly $(v)_0 = \sum_{i=1}^n (q_i)$. By our choice of $0 \in E$, it follows that $\sum_{i=1}^n p_i = 0$.

First we show that u and v have a common zero on E. Suppose to the contrary that $v(p_i) \neq 0$ for all i. Since $f(\Delta_\tau) = 0$, it follows that $(a)_0 = \sum_{i=1}^n (p_i - (n-2)\tau)$. Since the sum of these points is also 0, $n(n-2)\tau = 0$ which contradicts the hypothesis on τ. Thus u and v have a common zero on E.

Suppose that $\mathscr{V}(u,v) \cap E = \sum_{i=1}^k (p_i)$ and that $q_i = p_i$ for all $1 \leqq i \leqq k$. Since $f(\Delta_\tau) = 0$, we have

$$\left(\frac{a}{b}\right)(x) = \left(\frac{u}{v}\right)(x + (n-2)\tau)$$

for a dense set of $x \in E$. Hence a and b also have k common zeroes on E (counted with multiplicity). We will label these $r_1, \ldots, r_k$. Since $f(\Delta_\tau) = 0$, the other zeroes of a on E are $p_i - (n-2)\tau$ for $k+1 \leqq i \leqq n$.

Notice that $(f)_0$, the divisor of f on $E \times E$, contains $\sum_{i=1}^k \{r_i\} \times E$. Since the part of $(f)_0$ which is supported outside Δ_τ is stable under the involution $(p,q) \to (q + 2\tau, p - 2\tau)$ it follows that $(f)_0$ also contains $\sum_{i=1}^k E \times \{r_i - 2\tau\}$. Therefore for all $x \in E$, the divisor of zeroes of the linear form $a(x)v - b(x)u$ on E contains $\sum_{i=1}^k (r_i - 2\tau)$. Since the rational function $\frac{a}{b}$ is not constant, there exist $x, y \in E$ such that $\{a(x)v - b(x)u, a(y)v - b(y)u\}$ is linearly independent.

It follows that

$$\mathscr{V}(u,v) \cap E = \sum_{i=1}^{k} (r_i - 2\tau) = \sum_{i=1}^{k} (p_i) .$$

Therefore

$$(a)_0 = \sum_{i=1}^{k} (p_i + 2\tau) + \sum_{i=k+1}^{n} (p_i - (n-2)\tau) .$$

But $\sum_{i=1}^{n} p_i = 0$, so $n(n-k-2)\tau = 0$. This contradicts the hypothesis on the order of τ, unless $k = n-2$. We conclude that $\mathscr{V}(u,v)$ meets E at $n-2$ points counted with multiplicity.

We now prove the converse. Suppose that $u, v \in V$ are linearly independent and that $\mathscr{V}(u,v) \cap E = \sum_{i=1}^{n-2} \{p_i\}$; i.e. $\mathscr{V}(u,v)$ is a secant $(n-3)$-plane. We will prove that there exist $0 \neq a, b \in V$ such that $a \otimes v - b \otimes u \in R_A$. Write $(u)_0 = \sum_{i=1}^{n} (p_i)$ and $(v)_0 = \sum_{i=1}^{n} \{q_i\}$ with $q_i = p_i$ for $1 \leqq i \leqq n-2$. Choose $a, b \in V$ such that

$$(a)_0 = \sum_{i=1}^{n-2} (p_i + 2\tau) + (p_{n-1} - (n-2)\tau) + (p_n + (n-2)\tau)$$

and

$$(b)_0 = \sum_{i=1}^{n-2} (p_i + 2\tau) + (q_{n-1} - (n-2)\tau) + (q_n + (n-2)\tau) .$$

Thus a and b are determined up to scalar multiples.

Since translation by τ is a morphism there exists forms $f_1, \ldots, f_n$ (of some degree d say) such that $x + \tau = (f_1(x), \ldots, f_n(x))$ for all x in some dense open subset of E. In particular, there exists forms f and g of degree d on $\mathbb{P}(V^*)$ such that

$$\left(\frac{u}{v}\right)(x) = \left(\frac{f}{g}\right)(x - (n-2)\tau)$$

for a dense set of $x \in E$. Both f and g have $N := nd$ zeroes on E. But $\frac{u}{v}$ has only 2 zeroes on E, so f and g have $N-2$ common zeroes, say $\mathscr{V}(f,g) \cap E = \sum_{i=1}^{N-2} (r_i)$. Since

$$\operatorname{div}\left(\frac{u}{v}\right) = (p_{n-1}) + (p_n) - (q_{n-1}) - (q_n)$$

it follows that

$$\operatorname{div}\left(\frac{f}{g}\right) = (p_{n-1} - (n-2)\tau) + (p_n - (n-2)\tau) - (q_{n-1} - (n-2)\tau) - (q_n - (n-2)\tau) .$$

Therefore

$$(ag)_0 = (bf)_0 = \sum_{i=1}^{n-2} (p_i + 2\tau) + (p_{n-1} - (n-2)\tau) + (p_n - (n-2)\tau) + \sum_{i=1}^{N-2} (r_i) + (q_{n-1} - (n-2)\tau) + (q_n - (n-2)\tau) .$$

Thus ag and bf are forms of degree $d+1$, having $n(d+1)$ common zeroes on E. Since $\deg(E) = n$ there exists a scalar λ such that $ag - \lambda bf$ vanishes identically on E.

Replace b by λb, whence $(ag - bf)(E) = 0$. Therefore

$$\left(\frac{a}{b}\right)(x) = \left(\frac{f}{g}\right)(x) = \left(\frac{u}{v}\right)(x + (n-2)\tau)$$

for a dense set of $x \in E$. Therefore $(a \otimes v - b \otimes u)(\Delta_\tau) = 0$.

Set $z = -\sum_{i=1}^{n-2} p_i$ and define the divisors

$$D' = \sum_{i=1}^{n-2} (\{p_i + 2\tau\} \times E + E \times \{p_i\})$$

$$D'' = \{(p, z - p - (n-2)\tau) \mid p \in E\}$$

in $E \times E$. We will now show that

$$(a \otimes v - b \otimes u)_0 = \Delta_\tau + D' + D'' .$$

We have already seen that $a \otimes v - b \otimes u$ vanishes on Δ_τ and on D'. For $p \in E$ in general position, $a(p)v - b(p)u$ is a (non-zero) linear form on E which vanishes on a hyperplane H satisfying

$$H \cap E \supset \sum_{i=2}^{n-2} (p_i) + (p + (n-2)\tau) .$$

The other point of $H \cap E$ is determined by the fact that the n points of $H \cap E$ sum to 0 in E. Hence $z - p - (n-2)\tau$ is the other point of $H \cap E$. Thus $(a \otimes v - b \otimes u)(D'') = 0$. Hence $(a \otimes v - b \otimes u)_0$ contains $\Delta_\tau + D' + D''$. By looking at the n zeroes of $a(p)v - b(p)u$ and the n zeroes of $v(p)a - u(p)b$ it is now clear that we actually have equality as claimed.

Both D' and D'' are stable under the involution $(p, q) \to (q + 2\tau, p - 2\tau)$, and the fixed point set of the involution, namely $M = \{(p, p - 2\tau) \mid p \in E\}$, occurs in $D' + D''$ with multiplicity 0. Hence $D' + D''$ is an allowable divisor, so $a \otimes v - b \otimes u$ is a relation as required.

The final step is to check the uniqueness (up to scalar multiple) of the relation associated to the secant plane $\mathscr{V}(u, v)$. Suppose that both $f = a \otimes v - b \otimes u$ and $f' = a' \otimes v - b' \otimes u$ are relations. By the first part of the proof, it follows that $(a)_0 = (a')_0$ and $(b)_0 = (b')_0$. Since E spans $\mathbb{P}^{n-1}$, a is a scalar multiple of a', and b is a scalar multiple of b'. Hence if f and f' are linearly independent then $a \otimes v$ is a linear combination of f and f'. However, as remarked earlier there are no rank 1 relations (since $a \otimes v$ cannot vanish on Δ_τ). □

3 Geometry of E for $n \geqq 5$

In this section we take $n \geqq 5$.

If $y \in \mathbb{P}(V^*) \cong \mathbb{P}^{n-1}$ write $\pi_y \colon \mathbb{P}^{n-1} \backslash \{y\} \to \mathbb{P}^{n-2}$ for the projection with center y. The image of E under this projection will be denoted E_y. If $y \notin \operatorname{Sec}(E)$ then E_y is a smooth elliptic curve of degree n. If $y \in E$ then E_y is an elliptic curve of degree $n - 1$, which is embedded by a complete linear system of degree $n - 1$. In particular, when $y \in E$, then we may apply an induction argument to E_y.

Notation. If $p_1, \ldots, p_k \in E$ we will write $\overline{p_1 \ldots p_k}$ for the linear space $H \cong \mathbb{P}^{k-1}$ such that $H \cap E = \sum_{i=1}^{k} \{p_i\}$. Thus $\overline{pp}$ is the tangent line to E at p, and if the p_i are distinct $\overline{p_1 \ldots p_k}$ is the linear span of the p_i.

If $y \in \mathbb{P}^{n-1} \setminus E$, and X is the set of all secant $(n-3)$-planes to E which contain y, then X can be realised as a closed subvariety of $S^{n-3}E$, the $(n-3)$-th symmetric power of E:

$$X \cong \{(p_1, \ldots, p_{n-3}) \in S^{n-3}E \mid \overline{p_1 \ldots p_{n-3} y} \text{ meets } E \text{ with multiplicity } n-2\}$$

where $\overline{p_1 \ldots p_{n-3} y}$ denotes the $(n-3)$-plane spanned by y and the secant $(n-4)$-plane $\overline{p_1 \ldots p_{n-3}}$.

Lemma 3.1 $(n = 5)$ *Fix $y \notin E$. Then*

$$X := \{\textit{secant planes to } E \textit{ containing } y\}$$

is a 1-dimensional variety.

Proof. Define $Z \subset E \times X$ by

$$Z := \{(p, \mathfrak{h}) \mid p \in \mathfrak{h}\} .$$

The projection onto the second factor is a surjection $\mathrm{pr}_2 \colon Z \to X$ with finite fibers, so it suffices to prove that $\dim(Z) = 1$. However, projection to the first factor, fibers Z over E with fibers isomorphic to

$$X_p := \{\text{secant planes containing } p \text{ and } y\} ,$$

for $p \in E$. To prove the lemma it suffices to prove that $\dim(X_p) = 1$ for at most a finite number of p, and $\dim(X_p) = 0$ otherwise.

Consider E_p. This is a quartic elliptic curve, lying on a pencil of quadric surfaces in $\mathbb{P}^3$, four of which are cones, and the rest of which are smooth. The elements of X_p are in bijection with the secant lines to $E_p \subset \mathbb{P}^3$ which contain $\pi_p(y)$.

Suppose that $\pi_p(y) \in E_p$. Thus $y \in \overline{pq}$ for some $q \in E$. Hence the secant planes containing p and y are those of the form $\overline{pqr}$ for $r \in E$. Hence $\dim(X_p) = 1$ in this case. Furthermore, there is at most one $p \in E$ for which $\pi_p(y) \in E_p$, because y can lie on at most one secant line: if it lay on two such, then the span of those secant lines would be a $\mathbb{P}^2$ meeting E with multiplicity 4, which contradicts Lemma 2.3.

Now suppose that $\pi_p(y) \notin E_p$. Then $\pi_p(y)$ lies on a unique quadric, Q say, containing E_p. If Q is smooth there are exactly 2 secant lines to E_p passing through $\pi_p(y)$, whence $\dim(X_p) = 0$. If Q is a cone, and $\pi_p(y)$ is a smooth point of Q, then there is a unique secant line through $\pi_p(y)$, whence $\dim(X_p) = 0$. If Q is singular, and $\pi_p(y)$ is the vertex of Q then there is a 1-dimensional family of secant lines through $\pi_p(y)$, whence $\dim(X_p) = 1$.

It remains to show that this last possibility can occur for only a finite number of p. It is clear that $\pi_p(y)$ is the vertex of a quadric cone containing E_p if and only y and p both lie on the singular locus of a rank 3 quadric containing E. This is equivalent to the condition, that $\pi_y(p)$ be the vertex of a quadric cone containing E_y. Since E_y is a quintic curve, it can lie on at most one quadric in $\mathbb{P}^3$. Since y can lie on at most one secant line, there are at most 2 points of E mapping to the vertex of that cone. □

Remark 3.2 The next result does not hold for $n = 4$, and is the reason that the 4-dimensional Sklyanin algebra differs from the higher dimensional Sklyanin algebras in terms of point modules.

A general $y \in \mathbb{P}^3 \setminus E$ lies on either 1 or 2 secant lines depending on whether y lies on a singular or non-singular quadric containing E. However, if y is the vertex of a quadric cone containing E, then there is a 1-dimensional family of secant lines through y.

Proposition 3.3 *Suppose that $n \geqq 5$. Let $y \in \mathbb{P}^{n-1} \backslash E$. Define*

$$X_n(y) := \{\textit{secant } (n-3)\textit{-planes containing } y\}.$$

Then $\dim(X_n(y)) \leqq n-4$.

Proof. The result is true for $n=5$ by Lemma 3.1. We proceed by induction. Suppose the result is true for $n-1$.

Define $Z \subset E \times X_n(y)$ by

$$Z := \{(p, \mathfrak{h}) \mid p \in \mathfrak{h}\}.$$

Fix $p \in E$. The projection π_p with center p, maps E to the normal degree $(n-1)$ curve $E_p \subset \mathbb{P}^{n-2}$, and the induction hypothesis applies to this situation. Write Z_p for the fiber of the projection $\mathrm{pr}_1 \colon Z \to E$ over $p \in E$.

If $(p, \mathfrak{h}) \in Z_p$ then $\pi_p(\mathfrak{h}) \in X_{n-1}(\pi_p(y))$. Conversely, every secant $(n-4)$-plane to E_p which contains $\pi_p(y)$ is the image under π_p of a secant $(n-3)$-plane to E which contains y. Thus Z_p is in bijection with $X_{n-1}(\pi_p(y))$, so by the induction hypothesis, $\dim(Z_p) \leqq n-5$. It follows that $\dim(Z) \leqq n-4$. Since the projection $\mathrm{pr}_2 \colon Z \to X_n(y)$ has finite fibers, $\dim(X_n(y)) = \dim(Z)$. □

Lemma 3.4 *Let $y \in \mathbb{P}^{n-1}$. Then there is a hyperplane H containing y, and $n-1$ distinct points $p_1, \ldots, p_{n-1} \in H \cap E$ such that y does not lie in any of the $(n-3)$-secant planes spanned by these points.*

Proof. First suppose that $y \in E$. If $p_1, \ldots, p_{n-1} \in E$ are any $(n-1)$ distinct points whose sum is $-y$ then $H := \overline{p_1 \ldots p_{n-1}}$ contains y and the lemma is true.

Now suppose that $y \notin E$. Let X denote the variety of all secant $(n-3)$-planes to E which contain y. By Proposition 3.3 $\dim(X) \leqq n-4$. Let Y denote the variety of all hyperplanes through y. Thus $Y \cong \mathbb{P}^{n-2}$. Define $Z \subset X \times Y$ by

$$Z := \{(\mathfrak{h}, H) \mid \mathfrak{h} \subset H\}.$$

Let $\mathrm{pr}_1 \colon Z \to X$ and $\mathrm{pr}_2 \colon Z \to Y$ be the projections. If $\mathfrak{h} \in X$ then $\mathrm{pr}_1^{-1}(\mathfrak{h}) \cong \mathbb{P}^1$, whence $\dim(Z) \leqq n-3$. Therefore $\mathrm{pr}_2(Z) \neq Y$. Choose $H \notin Y$ such that $H \notin \mathrm{pr}_2(Z)$. Hence H does not contain any secant $(n-3)$-plane which contains y. In fact, there is a dense open set of Y consisting of such H. In particular, there will be such an H which meets E at n distinct points. Take such an H, and any $n-1$ of the points $p_1, \ldots, p_{n-1} \in H \cap E$. This will satisfy the lemma. □

4 Point modules for $n \geqq 5$

In this section we prove our main result. It will follow from the next theorem.

Theorem 4.1 *If $n \geqq 5$ then $\mathscr{V}(R_A) = \Delta_\tau$.*

Proof. This is a rather simple consequence of Lemma 3.4. Suppose that $(x, y) \in \mathscr{V}(R_A)$. Pick a hyperplane H containing y and distinct points $p_1, \ldots, p_{n-1} \in H$ as in Lemma 3.4. Let p_n be the other point of $H \cap E$. Fix a linear form u vanishing on H.

Fix $m \in \{1, \ldots, n-1\}$ and write $\sum_{i=1}^{n-1}(p_i) = (p_m) + \sum_{j=1}^{n-2}(q_j)$. Choose a linear form v_m such that $\mathscr{V}(u, v_m) = \overline{q_1 \ldots q_{n-2}}$. By Theorem 2.1 there exist $0 \neq a_m, b_m \in A_1$ such that $a_m \otimes v_m - b_m \otimes u \in R_A$. Since $y \notin \overline{q_1 \ldots q_{n-2}}$, it follows that $v_m(y) \neq 0$ and hence that $a_m(x) = 0$. Thus $x \in \mathscr{V}(a_1, \ldots, a_{n-1})$. By Theorem 2.1

$$(a_m)_0 = \sum_{j=1}^{n-2} (q_j + 2\tau) + (p_m - (n-2)\tau) + (p_n - (n-2)\tau)\,. \tag{4.1}$$

For all $1 \leqq k < n-2$, notice that $\overline{p_{k+1} + 2\tau, p_n - (n-2)\tau} \subset \mathscr{V}(a_1, \ldots, a_k)$, but is not contained in $\mathscr{V}(a_{k+1})$. Hence $\mathscr{V}(a_1, \ldots, a_{k+1})$ is properly contained in $\mathscr{V}(a_1, \ldots, a_k)$. Therefore $\mathscr{V}(a_1, \ldots, a_{n-1}) = \{p_n - (n-2)\tau\}$, whence $x = p_n - (n-2)\tau$.

In particular, $x \in E$. A similar argument (applying Lemma 3.4 to x rather than y) shows that $y \in E$ also. Now, returning to the first three paragraphs of this proof, if $y \in E$, then $y = p_n$ by the careful choice of H and the p_i. But we just showed that $x = p_n - (n-2)\tau$, whence $(x, y) \in \Delta_\tau$. □

For the 3-dimensional Sklyanin algebra one also has $\mathscr{V}(R_A) = \Delta_\tau$, but this fails for the 4-dimensional Sklyanin algebra. For the 4-dimensional Sklyanin algebra, $\mathscr{V}(R_A) = \Delta_\tau \cup \{(e_i, e_i) \mid 0 \leqq i \leqq 3\}$ where the e_i are the singular points of the 4 quadric cones containing E [3, Theorem 1.1].

Our main result is now an immediate consequence of the following lemma which applies to any finitely generated quadratic algebra.

Lemma 4.2 [1, Sect. 3] *If $\mathscr{V}(R_A)$ is the graph of an automorphism of $P := \mathrm{pr}_1(\mathscr{V}(R_A))$ then the point modules for A are in bijection with the points of P via the construction given in Proposition* 1.2.

Proof. We give the details for completeness. A *truncated point module* is a cyclic graded A-module with Hilbert series $1 + t + \cdots + t^m$ for some $m \geqq 1$. It is easy to see that the truncated point modules of length 3 are in bijection with the points of $\mathscr{V}(R_A) \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$. The bijection is implemented as follows. If $(p, q) \in \mathscr{V}(R_A)$ then the corresponding truncated point module is $kv_0 \oplus kv_1 \oplus kv_2$ with $x \in A_1$ acting via

$$x \cdot v_0 = x(q)v_1, \quad x \cdot v_1 = x(p)v_2, \quad x \cdot v_2 = 0\,. \tag{4.2}$$

Now suppose that $\mathscr{V}(R_A)$ is the graph of an automorphism σ of P. For every $p \in P$ one may construct a point module $M(p)$ as in Proposition 1.2. To see that these are all of them, suppose that $M = \bigoplus_{i=0}^{\infty} kv_i$ is a point module for A. The truncated point module $N := M/Av_3$ corresponds to a point of $\mathscr{V}(R_A)$ as explained above. This point is of the form $(p^{\sigma^{-1}}, p)$ for some $p \in P$. The action of A on N is given by (4.2). This also gives the action of A on M_0 and M_1. Now consider the truncated point module $Av_1/Av_4[1]$ (we have to shift the degree). Again it corresponds to a point of $\mathscr{V}(R_A)$, and since $x \cdot v_1 = x(p^{\sigma^{-1}})v_2$ for all $x \in A_1$, that point must be $(p^{\sigma^{-2}}, p^{\sigma^{-1}})$. Continue in this way looking at the truncated point modules $Av_i/Av_{i+3}[i]$. One finds that $x \cdot v_{i+2} = x(p^{\sigma^{-i-2}})\tau)v_{i+3}$. Thus, by induction, $M \cong M(p)$ as required. □

Theorem 4.3 *Let $n \geqq 5$. Then every point module over the n-dimensional Sklyanin algebra, is of the form $M(p)$ for some $p \in E$.*

References

1. Artin, M., Tate, J., van den Bergh, M.: Some algebras related to automorphisms of elliptic curves. In: Caitier, P. et al. (eds.) The Grothendieck Festschrift, vol. 1, pp. 33–85. Boston: Birkhäuser 1990
2. Artin, M., Tate, J., van den Bergh, M.: Modules over regular algebras of dimension 3. Invent. Math. **106**, 335–388 (1991)
3. Levasseur, T., Smith, S.P.: Modules over the 4-dimensional Sklyanin algebra. Bull. Soc. Math. Fr. **121**, 35–90 (1993)
4. Odesskii, A.V., Feigin, B.L., Sklyanin algebras associated with an elliptic curve (in Russian). (Preprint 1989)
5. Odesskii, A.V., Feigin, B.L.: Elliptic Sklyanin algebras (in Russian). Funkts. Anal. Prilozh. **23**(3), 45–54 (1989)
6. Sklyanin, E.K.: Some algebraic structures connected to the Yang–Baxter equation. Funct. Anal. Appl. **16**, 27–34 (1982)
7. Smith, S.P., Stafford, J.T.: Regularity of the 4-dimensional Sklyanin algebra. Compos. Math. **83**, 259–289 (1992)