Fibers in Ore extensions

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Abstract

Let R be a finitely generated commutative algebra over an algebraically closed field k and let $A = R[t; \sigma, \delta]$ be the Ore extension with respect to an automorphism σ and a σ-derivation δ. We view A as the coordinate ring of an affine non-commutative space X. The inclusion $R \to A$ gives an affine map $\xi : X \to \operatorname{Spec} R$, and X is a non-commutative analogue of $\mathbb{A}^1 \times \operatorname{Spec} R$. We define the fiber X_p of ξ over a closed point $p \in \operatorname{Spec} R$ as a certain full subcategory $\mathsf{Mod} X_p$ of $\mathsf{Mod} A$. The category $\mathsf{Mod} X_p$ has the following structure. If p has infinite σ-orbit, then $\mathsf{Mod} X_p$ is equivalent to the category of graded modules over the polynomial ring $k[x]$ with $\deg x = 1$. If p is not fixed by σ, but has finite σ-orbit, say of size n, then $\mathsf{Mod} X_p$ is equivalent to the representations of the quiver $\tilde{A}_{n-1}$ with the arrows all going in the same direction. If p is fixed by σ then $\mathsf{Mod} X_p$ is equivalent to either $\mathsf{Mod} k$ or $\mathsf{Mod} k[x]$. It is also shown that X is the disjoint union of the fibers X_p in a certain sense.

1 Introduction

The algebraic structure of Ore extensions has been studied by many people; for example, see [3], [4], and [5]. In this paper we use the language developed by Rosenberg [9] and Van den Bergh [13] to discuss a simple geometric question concerning an Ore extension of a commutative ring.

Throughout let R be a finitely generated commutative algebra over an algebraically closed field k. Let $R[t]$ be the polynomial extension. It is a tautology that the inclusion $R \to R[t]$ induces a morphism of schemes $\xi : \operatorname{Spec} R[t] \to \operatorname{Spec} R$ with the following properties: $\operatorname{Spec} R[t]$ is the disjoint union of the fibers $\xi^{-1}(p)$ over the closed points of $\operatorname{Spec} R$, and each fiber is isomorphic to the affine line over k.

In this paper we prove a non-commutative analogue of this.

The ring R remains as before, but we replace $R[t]$ by an Ore extension $A = R[t; \sigma, \delta]$ where σ is a k-algebra automorphism of R and δ is a k-linear σ-derivation of R. We will show that there are a limited number of possibilities for the fibers, and that the non-commutative space with coordinate ring A is a disjoint union of these fibers in a suitable sense. Our results can be viewed as an exercise in non-commutative algebraic geometry. The first problem is to define the terms.

We define spaces $X = \mathsf{Mod}A$ and $Y = \mathsf{Mod}R$. The inclusion $R \to A$ induces an affine map of spaces $\xi : X \to Y$, by which we mean an adjoint triple of functors $(\xi^*, \xi_*, \xi^!)$, where each is right adjoint to the preceding one. If p is a closed point of $\operatorname{Spec} R$, and $\mathcal{O}_p$ is the corresponding simple R-module, we call $\xi^*\mathcal{O}_p$, which is isomorphic to $A/\mathfrak{m}_p A$, the fiber module over p. We define a certain full subcategory $\mathsf{Mod}X_p$ of $\mathsf{Mod}A$ associated to $\xi^*\mathcal{O}_p$ and call X_p the fiber over p (Definition 2.3).

Theorem 1.1 *The fibers X_p have the following structure.*

1. $X_p \cong \mathbb{A}^1$ *if* $p = p^\sigma$ *and* $f(\delta)(R) \subset \mathfrak{m}_p$ *for some non-zero polynomial* $f(t)$*;*
2. $X_p \cong \operatorname{Spec} k$ *if* $p = p^\sigma$ *and the previous case does not occur;*
3. $X_p \cong \mathsf{GrMod}k[u]$, $\deg u = 1$, *if the* σ*-orbit of* p *is infinite;*
4. $X_p \cong \mathsf{Mod}Q$ *where* Q *is the quiver (6-1) of type* $\tilde{A}_n$ *with cyclic orientation if the* σ*-orbit of* p *has size* n, $2 \leq n < \infty$.

If k is uncountable, then X is the disjoint union of the fibers.

With the exception of $\mathsf{Mod}k$ in part 2, the categories in Theorem 1.1 have global dimension one and Krull dimension one. They are therefore reasonable analogues of smooth curves. It is easy to see that all possibilities can occur. These categories also turn up as affine pieces of the exceptional curve in Van den Bergh's non-commutative blowing up [13]. They should therefore be considered as appropriate non-commutative analogues of the affine line.

It is a tautology in the commutative case that every closed point of $\operatorname{Spec} R[t]$ lies on one of the fibers. It is rather easy to prove a non-commutative analogue of this (Proposition 3.4): every finite dimensional simple A-module belongs to some fiber. More precisely, every such module is a quotient of some fiber module. We prefer to state this geometrically: every closed point of X lies on some fiber X_p. The closed points of X are in bijection with the finite dimensional simple A-modules, and the degree of a closed point is the dimension of the corresponding simple module. Analysis of the fibers gives complete information about the finite dimensional simple A-modules.

Theorem 1.2 *The degrees of the closed points on each fiber, equivalently the dimensions of the simple A-modules on each fiber, are as follows (the numbering of the four cases is the same as that in Theorem 1.1):*

1. *the closed points are parametrized by $x \in \mathbb{A}^1$, and $\deg x$ is the minimal degree of a polynomial $f(t) \in k[t]$ with the property that $f(\delta)(R) \subset \mathfrak{m}_p$;*

2. *there are no closed points on these fibers;*

3. *the closed points are indexed by $\mathbb{Z}$, and all have degree one;*

4. *there are n points of degree one and the other points, which are parametrized by $\mathbb{A}^1 \backslash \{0\}$, have degree n.*

In section 8 we give a precise meaning to the phrase "X is the disjoint union of the fibers". We define the words "union" and "disjoint" in terms of Ext-groups. The fact that X is the disjoint union of the fibers implies that there are no non-split extensions between simple modules lying on different fibers. Although the results in that section are satisfying, we remain uncertain whether we really do have good notions of "union" and "disjoint". Other test cases need to be examined.

2 Definitions

We recall some definitions from [13] and [11]. A non-commutative space, or quasi-scheme, is a Grothendieck category $X = \mathsf{Mod}X$. Objects in $\mathsf{Mod}X$ are called X-modules. Two spaces X and X' are isomorphic if $\mathsf{Mod}X$ is equivalent to $\mathsf{Mod}X'$.

The full subcategory of $\mathsf{Mod}X$ consisting of the noetherian X-modules is denoted by $\mathsf{mod}X$. If $\mathsf{Mod}X$ is locally noetherian, then $\mathsf{mod}X$ determines $\mathsf{Mod}X$.

A map $\xi : X \to Y$ of non-commutative spaces is a natural equivalence class of an adjoint pair of functors $\xi_* : \mathsf{Mod}X \to \mathsf{Mod}Y$, and its left adjoint $\xi^* : \mathsf{Mod}Y \to \mathsf{Mod}X$. If ξ_* has a right adjoint, it is denoted by $\xi^!$. A weakly closed subspace Z of X is a full subcategory $\mathsf{Mod}Z$ of $\mathsf{Mod}X$ that is closed under subquotients and direct sums, and such that the inclusion functor $i_* : \mathsf{Mod}Z \to \mathsf{Mod}X$ has a right adjoint. If i_* also has a left adjoint, Z is called a closed subspace of X.

If A is a ring, we denote by $\mathsf{Mod}A$ its category of right modules. We say that A is a coordinate ring of this space. When A is a ring, we will abuse language and use the notations $\operatorname{Spec} A$ and $\mathsf{Mod}A$ interchangeably.

Let X be a k-space, meaning that $\mathsf{Mod}X$ is k-linear. A closed point x of X is a closed subspace $\mathsf{Mod}x$ that is isomorphic to $\operatorname{Spec} k$. The simple module in $\mathsf{Mod}x$ is denoted by $\mathcal{O}_x$. Thus, $\mathsf{Mod}x$ is the full subcategory of $\mathsf{Mod}X$ consisting of all direct sums of $\mathcal{O}_x$. Since the inclusion $i_* : \mathsf{Mod}x \to \mathsf{Mod}X$ has a right adjoint, every direct product of copies of $\mathcal{O}_x$ is isomorphic to a direct sum of copies of $\mathcal{O}_x$. When $X = \operatorname{Spec} A$, we call $\dim_k \operatorname{Hom}_A(A, \mathcal{O}_x)$ the degree of x.

It follows from [9, Proposition 6.4.1, p.127] that every closed subscheme of $\operatorname{Spec} A$ is of form $\mathsf{Mod}A/I$ for some two sided ideal $I \subset A$. In particular, if x is a closed point of $\operatorname{Spec} A$, then $\mathsf{Mod}x$ is equal to $\mathsf{Mod}A/I$ where $A/I \cong M_n(k)$ for some n. In particular, $\mathcal{O}_x$ is finite dimensional. If A is an algebra over an algebraically closed field k, it follows that there is a bijection between the closed points of $\mathsf{Mod}A$ and the finite dimensional simple A-modules.

Definition 2.1 *Let X be a space endowed with a dimension function ∂. For each X-module L we define $\mathsf{Mod}L$ to be the smallest full subcategory of $\mathsf{Mod}X$ satisfying the following conditions:*

1. *if M is a noetherian submodule of the injective envelope of L such that $\partial(M+L/L) < \partial(L)$, then M is in $\mathsf{Mod}L$;*
2. *$\mathsf{Mod}L$ is closed under direct limits;*
3. *$\mathsf{Mod}L$ is closed under subquotients.*

Hence $\mathsf{Mod}L$ is a weakly closed subspace of X. The main result in [11] is to show that if L is a curve module in *good position* [11, Definition 5.1], then $\mathsf{Mod}L$ is isomorphic to $\mathsf{GrMod}_{\mathbb{Z}^n} k[x_1, \cdots, x_n]/(\mathrm{Kdim} \le n-2)$ for some n. The case discussed in Section 5 of this paper is the case when $n = 1$. The cases in the other sections of this paper are like L not in good position.

Let σ be a k-algebra automorphism of R, and let δ be a k-linear σ-derivation of R. We write r^σ for the image of $r \in R$ under σ. The Ore extension $A = R[t;\sigma,\delta]$ is generated by R and t subject to the relations

$$tr = r^\sigma t + \delta(r)$$

for all $r \in R$. The ring A is noetherian on both sides because R is. If R is a domain, so is A.

Let p be a closed point in $\operatorname{Spec} R$. We write $\mathfrak{m}_p$ for the maximal ideal of R vanishing at p, and $\mathcal{O}_p$ for the corresponding simple module $R/\mathfrak{m}_p$. We write p^σ for the image of p under σ, and adopt the convention that $r^\sigma(p) = r(p^\sigma)$. Thus $\sigma^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{p^\sigma}$.

For each $n \ge 0$, define A_n to be the R-submodule of A generated by $1, t, t^2, \ldots, t^n$. This is a free right (and left) R-module with basis given by the $\{t^i \mid 0 \le i \le n\}$. The filtration $A_0 \subset A_1 \subset \ldots$ makes A a filtered k-algebra. Each slice A_n/A_{n-1} is an invertible R-R-bimodule. It is isomorphic to ${}_{\sigma^{-n}}R_1 \cong {}_1R_{\sigma^n}$; this is defined as the free right R-module with basis ε_n, and left action of R given by

$$r.\varepsilon_n = \varepsilon_n r^{\sigma^{-n}}.$$

It is easy to check that $\mathcal{O}_p \otimes_R (A_n/A_{n-1}) \cong \mathcal{O}_{p^{\sigma^n}}$.

If L is an R-module, we define L^σ to be the R-module which is equal to L as an abelian group, but with a new R-action defined by $\ell * r = \ell r^\sigma$ for $\ell \in L^\sigma$ and $r \in R$. For example, $(\mathcal{O}_p)^\sigma \cong \mathcal{O}_{p^\sigma}$.

We write $X = \mathsf{Mod}A$ and $Y = \operatorname{Spec} R$, and $\xi : X \to Y$ for the projection. Explicitly, $\xi^* = - \otimes_R A$, ξ_* is the restriction of scalars, and $\xi^! = \operatorname{Hom}_R(A, -)$.

Let L be an R-module. The ascending filtration on A by the R-R-bimodules A_n induces a filtration on $\xi^* L = L \otimes_R A$ by the right R-submodules $L \otimes_R A_n$.

We call this the standard filtration on ξ^*L. The slices associated to the filtration are

$$L \otimes_R (A_n/A_{n-1}) \cong L^{\sigma^n}.$$

Definition 2.2 *If $p \in \operatorname{Spec} R$ is closed we call $\xi^*\mathcal{O}_p = \mathcal{O}_p \otimes_R A = A/\mathfrak{m}_p A$ the* fiber module *over p. We denote it by F_p.*

There is a right $k[t]$-module decomposition $A = \mathfrak{m}_p A \oplus k[t]$, so as a right $k[t]$-module, $F_p \cong k[t]$. We write ε for a generator of F_p as a right $k[t]$-module.

The degree $\leq n$ R-submodule of the canonical filtration on F_p is the linear span of $\{\varepsilon t^i \mid 0 \leq i \leq n\}$. The successive slices are simple R-modules, and starting at the bottom these are

$$\mathcal{O}_p,\ \mathcal{O}_{p^\sigma}, \mathcal{O}_{p^{\sigma^2}}, \ldots.$$

Thus, if p has infinite σ-orbit, then as an R-module F_p is isomorphic to $\mathcal{O}_p \oplus \mathcal{O}_{p^\sigma} \oplus \ldots$.

Because F_p is a free $k[t]$-module of rank one, its Krull dimension is either zero or one, and every proper quotient of it is finite dimensional.

The natural dimension function on R-modules is Krull dimension, denoted by Kdim. Because R is a finitely generated commutative k-algebra this equals the Gelfand-Kirillov dimension, which is denoted by GKdim. We will use GK-dimension as our dimension function on $X = \mathsf{Mod}A$, and we define the fiber X_p to be $\mathsf{Mod}F_p$ as at the end of previous section. The precise definition is the following.

Definition 2.3 *The* fiber *X_p over a closed point $p \in \operatorname{Spec} R$ is defined by requiring $\mathsf{Mod}X_p$ to be the smallest full subcategory of $\mathsf{Mod}X$ satisfying the following conditions:*

1. *if M is a submodule of the injective envelope of F_p such that $M + F_p/F_p$ is finite dimensional, then M is in $\mathsf{Mod}X_p$;*
2. *$\mathsf{Mod}X_p$ is closed under direct limits;*
3. *$\mathsf{Mod}X_p$ is closed under subquotients.*

Sometimes it is simpler to describe the noetherian category $\mathsf{mod}X_p$. It is the smallest full subcategory that is closed under finite direct sums, subquotients, and satisfies condition (1).

Condition (3) ensures that the inclusion $i_* : \mathsf{Mod}X_p \to \mathsf{Mod}X$ is exact. Condition (2) ensures that i_* has a right adjoint $i^!$, so X_p is weakly closed in X.

If x is a closed point of X, we say that x lies on X_p if $\mathcal{O}_x$ is in $\mathsf{Mod}X_p$. In that case we simply write $x \in X_p$.

3 Structure of the fiber modules

From now on we fix the following notation: $A = R[t, \sigma, \delta]$, $X = \mathsf{Mod}A$, F_p is the fiber module $\xi^*\mathcal{O}_p = A/\mathfrak{m}_p A$, and X_p is the fiber over $p \in \operatorname{Spec} R$.

In several of this section's proofs we let ε denote a $k[t]$-basis for F_p.

Lemma 3.1 *Suppose that $r \in R$, and $f \in k[t]$. If $\deg f = n$, then $\varepsilon f.r = \varepsilon(r(p^{\sigma^n})f + g)$ where $\deg g \leq n-1$.*

Proof. Since the action of r on F_p is k-linear, it suffices to prove the result for $f = t^n$. We argue by induction on n. If $n = 0$, the result is true because $\varepsilon.R \cong R/\mathfrak{m}_p$. If $n \geq 1$, then

$$\begin{aligned}\varepsilon t^n.r &= \varepsilon t^{n-1}(r^\sigma t + \delta(r)) \\ &= \varepsilon(r^\sigma(p^{\sigma^{n-1}})t^{n-1} + h)t + \varepsilon t^{n-1}\delta(r) \\ &= \varepsilon r(p^{\sigma^n})t^n + \varepsilon(ht + t^{n-1}\delta(r)).\end{aligned}$$

The induction hypothesis gives $\deg h \leq n-2$ and $\deg t^{n-1}\delta(r) \leq n-1$, so the result is proved. □

Lemma 3.2 *Let $f \in k[t]$ be of degree n. The following are equivalent:*

1. *$\varepsilon f.k[t]$ is an A-submodule of F_p;*
2. *$\varepsilon f.r \in k\varepsilon f$ for all $r \in R$;*
3. *$\varepsilon f.r = r(p^{\sigma^n})\varepsilon f$ for all $r \in R$.*

In this case, $\varepsilon fA \cong F_{p^{\sigma^n}}$.

Proof. (3) ⇒ (2) This is obvious.

(2) ⇒ (3) By Lemma 3.1, $\varepsilon f.r = \varepsilon(r(p^{\sigma^n})f + g)$ for some $g \in k[t]$ of degree $\leq n-1$, so the only possibility is that $g = 0$ and $\varepsilon f.r = r(p^{\sigma^n})\varepsilon f$.

(2) ⇒ (1) We must show that $\varepsilon f t^i.r \in \varepsilon f.k[t]$ for all $i \geq 0$, and all $r \in R$. This is true for $i = 0$ by hypothesis. Now $\varepsilon f t^i.r = \varepsilon f t^{i-1}.tr = \varepsilon f t^{i-1}.(r^\sigma t + \delta(r))$. By the induction hypothesis applied to ft^{i-1}, this is in $f.k[t]$.

(1) ⇒ (3) By Lemma 3.1, $\varepsilon f.r = \varepsilon(r(p^{\sigma^n})f + g)$ for some $g \in k[t]$ of degree $\leq n-1$. But $\varepsilon f.r \in \varepsilon f.k[t]$, so $g = 0$. Hence (3) holds.

This completes the proof that the three conditions are equivalent. Now suppose that the conditions hold. It follows at once from (3) that εf is annihilated by $r - r(p^{\sigma^n})$ for all $r \in R$. Thus $\varepsilon f \mathfrak{m}_{p^{\sigma^n}} = 0$. Hence εfA is a quotient of $F_{p^{\sigma^n}}$. However, as a $k[t]$-module, $\varepsilon fA \cong k[t]$, so the map $F_{p^{\sigma^n}} \to \varepsilon fA$ must be bijective. □

One consequence of the preceding result is that a non-zero submodule of F_p is necessarily isomorphic to $F_{p^{\sigma^n}}$ for some $n \geq 0$.

Lemma 3.3 *Let p and q be closed points of* $\operatorname{Spec} R$. *Then $F_p \cong F_q$ if and only if $p = q$.*

Proof. Let ε be a $k[t]$-module basis for F_p, and let ε' be a $k[t]$-module basis for F_q. Suppose that $\varphi : F_q \to F_p$ is an isomorphism. Let $f \in k[t]$ be such that $\varphi(\varepsilon') = \varepsilon f$. Then $\varepsilon f \mathfrak{m}_q = 0$ and

$$F_p = \varepsilon f A = \varepsilon f(\mathfrak{m}_q A \oplus k[t]) = \varepsilon f k[t].$$

Therefore $\deg f = 0$, and it follows from Lemma 3.2, that $q = p$. □

Recall that the closed points of X are in bijection with the finite dimensional simple A-modules. The next result shows that each closed point of X lies on some fiber X_p.

Proposition 3.4 *Let x be a closed point of X. Then there is a closed point $p \in \operatorname{Spec} R$ such that $x \in X_p$ and, if $\deg x = n$, there is an exact sequence*

$$0 \to F_{p^{\sigma^n}} \to F_p \to \mathcal{O}_x \to 0. \tag{3-1}$$

Proof. If N is a finite dimensional simple A-module, then it contains a simple R-submodule. That submodule is isomorphic to some $\mathcal{O}_p$, and the inclusion $\mathcal{O}_p \to N$ induces an A-module map $\xi^* \mathcal{O}_p \to N$. This map is surjective because N is simple, and it is a right $k[t]$-module map, so its kernel equals $\varepsilon f.k[t]$ for some $f \in k[t]$. Clearly $\deg f = \dim N$. The result now follows from Lemma 3.2. □

Lemma 3.5 *If $q \notin \{p, p^\sigma, p^{\sigma^2}, \dots\}$, then $\operatorname{Ext}^i_A(F_q, F_p) = 0$ for all i.*

Proof. Let N be an arbitrary right A-module. Since A is a free left R-module, the change of rings spectral sequence [10, Theorem 11.54] shows that

$$\operatorname{Ext}^i_A(\mathcal{O}_q \otimes_R A, N) \cong \operatorname{Ext}^i_R(\mathcal{O}_q, \xi_* N). \tag{3-2}$$

This is zero if q is not in the support of $\xi_* N$. □

Proposition 3.6 *Suppose that $p \neq p^\sigma$. Then*

1. *F_p contains a copy of F_{p^σ} of codimension one.*
2. *$\delta^\alpha(R) \subset \mathfrak{m}_p$ for a unique $\alpha \in k$ where $\delta^\alpha = \delta + \alpha(1 - \sigma)$.*

Because $1 - \sigma$ is a σ-derivation, the map δ^α in part (2) is a σ-derivation. Furthermore $R[t, \sigma, \delta] = R[(t + \alpha), \sigma, \delta^\alpha]$. This is sometimes a useful change of variables.

Proof. It follows from the remarks in section two that the composition factors of the R-submodule $k\varepsilon + k\varepsilon t$ of F_p are isomorphic to $\mathcal{O}_p$ and $\mathcal{O}_{p^\sigma}$. Since $p \neq p^\sigma$, it is isomorphic to $\mathcal{O}_p \oplus \mathcal{O}_{p^\sigma}$. Hence $\varepsilon(t + \alpha)\mathfrak{m}_{p^\sigma} = 0$ for a unique $\alpha \in k$. Therefore $\varepsilon(t + \alpha)k[t]$ is a codimension one A-submodule isomorphic to F_{p^σ}.

(2) Suppose that $\varepsilon(t + \alpha)$ generates F_{p^σ}. Changing the variable t to $t + \alpha$, we have the same σ, but the new derivation is δ^α. So it reduced to the case when $\alpha = 0$. Since εt generates F_{p^σ}, the relation $ta = a^\sigma t + \delta^\alpha(a)$ shows that $\delta^\alpha(R) \subset \mathfrak{m}_p$. □

Lemma 3.7 *Let f and g be elements in $k[t]$ of the same degree. If $\varepsilon f.k[t]$ and $\varepsilon g.k[t]$ are A-submodules of F_p, so is $\varepsilon(\alpha f + \beta g).k[t]$ for all $\alpha, \beta \in k$.*

Proof. This follows from criterion (3) in Lemma 3.2. □

Proposition 3.8 *Let p and q be closed points of* Spec R. *If* $\dim_k \operatorname{Hom}_A(F_q, F_p) \geq 2$*, then p has a finite σ-orbit.*

Proof. Let ψ and θ be linearly independent maps $F_q \to F_p$. Let ε be a $k[t]$-basis for F_p, and let $f, g \in k[t]$ be such that $\operatorname{im}\psi = \varepsilon f k[t]$ and $\operatorname{im}\theta = \varepsilon g k[t]$. Since ψ and θ are linearly independent, so are f and g. Set $m = \deg f$ and $n = \deg g$. By Lemma 3.2, $q = p^{\sigma^n} = p^{\sigma^m}$. If $m \neq n$, then p has a finite σ-orbit. On the other hand, suppose that $\deg f = \deg g = n$. Then there is a non-zero $\lambda \in k$ such that $\deg(\lambda f + g) < n$. However, $\varepsilon(\lambda f + g)k[t] = \operatorname{im}(\lambda\psi + \theta)$, whence $q = p^{\sigma^d}$ for some $0 \leq d < n$. It follows that p has a finite σ-orbit. □

Lemma 3.9 *A fiber module F_p can have either zero, one, or infinitely many, 1-dimensional quotients. In particular,*

1. *F_p has infinitely many if and only if $p^\sigma = p$ and $\delta(R) \subset \mathfrak{m}_p$; this is equivalent to the condition that $\mathfrak{m}_p A$ is a two-sided ideal of A;*
2. *F_p has exactly one 1-dimensional quotient if and only if $p \neq p^\sigma$;*
3. *F_p has no 1-dimensional quotient if and only if $p = p^\sigma$ and $\delta(R) \not\subset m_p$.*

Proof. By Lemma 3.2(3), the 1-dimensional quotients of F_p are those quotients $F_p/\varepsilon(t+\lambda).k[t]$ for which $\varepsilon(t+\lambda).r = r(p^\sigma)\varepsilon(t+\lambda)$ for all $r \in R$. However, $\varepsilon(t+\lambda).r = \varepsilon(r^\sigma(p)t + \lambda r(p) + \delta(r)(p))$. So λ must satisfy $\lambda(r(p) - r(p^\sigma)) + \delta(r)(p) = 0$ for all $r \in R$. Viewing this as a system of equations in the unknown λ, the system has either zero, one, or infinitely many solutions.

(1) The linear system has infinitely many solutions if and only if $r(p) - r(p^\sigma) = \delta(r)(p) = 0$ for all r. That is, if and only if, $r - r^\sigma \in \mathfrak{m}_p$ and $\delta(r) \in \mathfrak{m}_p$ for all $r \in R$. Writing any r as the sum of an element in $\mathfrak{m}_p$ and a constant, one sees that the first condition is equivalent to $\sigma(\mathfrak{m}_p) \subset \mathfrak{m}_p$; but this is just the condition that $p^\sigma = p$.

If $\mathfrak{m}_p A$ is a two-sided ideal, then, as k-algebras, $A/\mathfrak{m}_p A \cong k[t]$, so it has infinitely many one-dimensional quotients. Conversely, if F_p has infinitely many one-dimensional quotients then $\sigma(\mathfrak{m}_p) \subset \mathfrak{m}_p$, and $\delta(\mathfrak{m}_p) \subset \mathfrak{m}_p$, so $t\mathfrak{m}_p \subset \mathfrak{m}_p A$, whence $\mathfrak{m}_p A$ is two-sided.

(2) If the linear system has only one solution, then there is an r such that $r(p) - r(p^\sigma) \neq 0$. This means that $p \neq p^\sigma$. Conversely, if $p \neq p^\sigma$, then F_p has a 1-dimensional quotient by Proposition 3.6(1). By case (1), F_p can not have more that one quotient. Case (3) follows from (1) and (2) □

4 Degree one points, and non-fixed points of $\operatorname{Spec} R$

By Lemma 3.9, if p is fixed by σ, then there are either no closed points x such that $\xi_*\mathcal{O}_x$ is isomorphic to $\mathcal{O}_p$, or an affine line of them. When there is an affine line of them, that line is $\operatorname{Spec} A/\mathfrak{m}_p A$, and the points on the line have degree one.

Proposition 4.1 *Suppose that $p \neq p^\sigma$. Then there is a unique degree one closed point $x \in X$ such that $\xi_*\mathcal{O}_x \cong \mathcal{O}_p$. The annihilator of $\mathcal{O}_x$ is $A\mathfrak{m}_p A$.*

Proof. If M is a 1-dimensional A-module that is isomorphic to $\mathcal{O}_p$ as an R-module, then M is necessarily a quotient of F_p. However, by Lemma 3.9, F_p has a unique one dimensional quotient A-module. Define x to be the closed point of X for which $\mathcal{O}_x$ is that quotient. It follows from the proof of Proposition 3.6 that $\mathcal{O}_x\mathfrak{m}_p = 0$, whence $\xi_*\mathcal{O}_x \cong \mathcal{O}_p$. Certainly $\mathcal{O}_x$ is annihilated by $A\mathfrak{m}_p A$. Since p is not fixed by σ, there is $r \in \mathfrak{m}_p$ such that $r^\sigma \notin \mathfrak{m}_p$. Now, evaluating the image of $0 = r^\sigma t + \delta(r) - rt$ in $A/A\mathfrak{m}_p A$, it follows that $t + \alpha \in A\mathfrak{m}_p A$ for some scalar $\alpha \in k$; this is the scalar α in the proof of Proposition 3.6. Hence $\dim_k(A/A\mathfrak{m}_p A) = 1$. Thus the annihilator of $\mathcal{O}_x$ is precisely $A\mathfrak{m}_p A$. □

Let U be the open subscheme of $\operatorname{Spec} R$ on which σ is not the identity. By Proposition 4.1, the degree one closed points of X that lie above U are in bijection with the closed points of U. It is reasonable to ask if X contains a copy of U.

The proof of the following result is due to C. Ingalls. We thank him for allowing us to include it here.

Proposition 4.2 *Suppose that R is a domain and that σ does not fix any closed points of* $\operatorname{Spec} R$. *Then $\xi : X \to \operatorname{Spec} R$ has a section, the image of which consists of the degree one closed points of X.*

Proof. It suffices to show that there is a two-sided ideal J in A such that the composition $R \to A \to A/J$ is a ring isomorphism. We will do this by exhibiting an element $a \in R$ such that $t + a$ is a normal element in A.

By Proposition 3.6, there is a k-valued map $p \mapsto \alpha_p$ on the closed points of $\operatorname{Spec} R$ such that $\delta(r) + \alpha_p(r - r^\sigma) \in \mathfrak{m}_p$ for all $r \in R$. Since p is not fixed by σ, there is an r in $\mathfrak{m}_p$ such that $r^\sigma \notin \mathfrak{m}_p$. Hence $\delta(r)(r - r^\sigma)^{-1}$ is regular at p, and so in an open neighborhood of p. Since it takes the value α_p, which is uniquely determined by p, we can glue these to find a single $a \in R$ such that $\delta(r) + a(r - r^\sigma) \in \mathfrak{m}_p$ for all $r \in R$ and all p. Since R is a domain, it follows that $\delta(r) + a(r - r^\sigma) = 0$. It follows that $(t + a)r = r^\sigma(t + a)$ for all $r \in R$. It is clear that $A/(t + a) \cong R$.

Since $A/(t+a)$ is commutative, the section consists of the degree one closed points of X. □

The proof of Proposition 4.2 shows that $A \cong R[t+a; \sigma]$. This can be restated as follows.

Corollary 4.3 *Suppose that R is a domain and that σ does not fix any closed points of* $\operatorname{Spec} R$. *Then every σ-derivation of R is σ-inner.*

5 When p has infinite σ-orbit

In this section we suppose that the σ-orbit of p is infinite. We will show that $X_p \cong \mathsf{GrMod}k[x]$, where $\deg x = 1$. We use some of the ideas in [11].

Proposition 5.1 *Suppose that the σ-orbit of p is infinite.*

1. *There is a unique descending chain of submodules in F_p, namely*

$$F_p \supset F_{p^\sigma} \supset F_{p^{\sigma^2}} \supset \dots .$$

2. *There are elements $\alpha_i \in k$ such that the copy of $F_{p^{\sigma^n}}$ in F_p is the $k[t]$-submodule generated by $(t+\alpha_1)\dots(t+\alpha_n)$.*

3. *The scalars α_n satisfy $(\delta + \alpha_n(1-\sigma))(R) \subset \mathfrak{m}_{p^{\sigma^n}}$.*

4. $\operatorname{Hom}_A(F_p, F_p) \cong k$.

Proof. (1) By Proposition 3.6(1) and induction on n, F_p has a submodule isomorphic to $F_{p^{\sigma^n}}$. By Proposition 3.8 there is only one such submodule.

(2) If $f_n \in k[t]$ is such that $\varepsilon f_n A \cong F_{p^{\sigma^n}}$, then $\deg f_n = n$ by Lemma 3.2.

(3) A computation shows that $\varepsilon(t+\alpha)k[t]$ is an A-submodule of F_p if and only if $\delta(r)(p) = \alpha(r^\sigma - r)(p)$ for all $r \in R$; that is, if and only if $(\delta + \alpha(1-\sigma))(R) \subset \mathfrak{m}_p$. An induction argument now establishes the result.

(4) Let $\varphi : F_p \to F_p$ be a non-zero map. Let $f \in k[t]$ be such that $\varphi(\varepsilon) = \varepsilon f$. If $r \in R$, then

$$\varepsilon f \cdot r = \varphi(\varepsilon) r = \varphi(\varepsilon r) = \varphi(\varepsilon r(p)) = r(p)\varepsilon f,$$

whence $\deg f = 0$ by Lemma 3.2. □

Corollary 5.2 *Suppose that the σ-orbit of p is infinite. Then there is a unique finite dimensional simple quotient of F_p, and that quotient has dimension one.*

Lemma 5.3 *Let x be a closed point of degree n on X. Suppose there is a non-split extension $0 \to F_p \to E \to \mathcal{O}_x \to 0$. Then $\mathcal{O}_x \cong F_q/F_{q^{\sigma^n}}$ for some $q^{\sigma^n} \in \{p, p^\sigma, p^{\sigma^2}, \dots\}$. If the σ-orbit of p is infinite, then $n = 1$.*

Proof. By Proposition 3.4, there is a short exact sequence

$$0 \to F_{q^{\sigma^n}} \to F_q \to \mathcal{O}_x \to 0$$

for some closed point q in $\operatorname{Spec} R$. This induces a long exact sequence

$$\begin{aligned} 0 \to &\operatorname{Hom}_A(\mathcal{O}_x, F_p) \to \operatorname{Hom}_A(F_q, F_p) \to \operatorname{Hom}_A(F_{q^{\sigma^n}}, F_p) \to \\ &\operatorname{Ext}^1_A(\mathcal{O}_x, F_p) \to \operatorname{Ext}^1_A(F_q, F_p) \to \operatorname{Ext}^1_A(F_{q^{\sigma^n}}, F_p) \to \cdots . \end{aligned} \tag{5-1}$$

Since $\mathrm{Ext}^1_A(\mathcal{O}_x, F_p) \neq 0$, either $\mathrm{Hom}_A(F_{q^{\sigma^n}}, F_p)$ or $\mathrm{Ext}^1_A(F_q, F_p)$ is nonzero. It follows from Lemma 3.5 that either q^{σ^n} or q is in $\{p, p^\sigma, p^{\sigma^2}, \dots\}$. If $q \in \{p, p^\sigma, p^{\sigma^2}, \dots\}$, then of course $q^{\sigma^n} \in \{p, p^\sigma, p^{\sigma^2}, \dots\}$.

The last assertion of the lemma follows from Proposition 5.1. □

Proposition 5.4 *Suppose that the σ-orbit of p is infinite. Let x be a closed point on X. If $0 \to F_p \to E \to \mathcal{O}_x \to 0$ is a non-split extension, then $E \cong F_{p^{\sigma^{-1}}}$.*

Proof. Since the sequence does not split, E has no nonzero finite dimensional A-submodule. By Lemma 5.3, $\mathcal{O}_x \cong F_q/F_{q^s}$ and $q \in \{p^{\sigma^{-1}}, p, p^\sigma, p^{\sigma^2}, \dots\}$.

As an R-module, $F_p \cong \mathcal{O}_p \oplus \mathcal{O}_{p^\sigma} \oplus \mathcal{O}_{p^{\sigma^2}} \oplus \dots$. Since F_q/F_{q^σ} is isomorphic to $\mathcal{O}_q$ as an R-module, E is isomorphic to a direct sum of indecomposable R-modules, each of which has length at most two, at most one of which has length two. If E has an indecomposable R-submodule of length two, that submodule has support $\{q\}$.

Now we show that E is a free $k[t]$-module of rank one. Since F_p is a $k[t]$-submodule of E and is free of rank one, it suffices to show that E has no $k[t]$-torsion. Suppose to the contrary that there is a non-zero element $e \in E$ such that $e(t+\lambda) = 0$ for some $\lambda \in k$. But E is a union of finite dimensional R-submodules, so $eI = 0$ for some ideal I in R such that $\dim_k R/I < \infty$. Therefore $eA = e(R \oplus (t+\lambda)A) = eR$ is finite dimensional. But this cannot happen because E is non-split. Hence E is a free $k[t]$-module of rank one.

Next we show that E is a semisimple R-module. If not, then it contains an indecomposable R-submodule of length two, say L. Furthermore, $\mathrm{Supp}\, L = \{q\}$. Now $\xi^* L$ has a filtration by R-submodules $(\xi^* L)_n := L \otimes_R A_n$. Since $(\xi^* L)_n/(\xi^* L)_{n-1}$ is isomorphic to L^{σ^n}, which has support $\{q^{\sigma^n}\}$, it follows that, as an R-module,

$$\xi^* L \cong L \oplus L^\sigma \oplus L^{\sigma^2} \oplus \dots.$$

If K denotes the socle of L, then $\xi^* K \cong K \oplus K^\sigma \oplus K^{\sigma^2} \oplus \dots$. Since the socle of a direct sum of modules is the direct sum of the individual socles, $\xi^* K$ is the R-socle of $\xi^* L$. Further, since K is an essential R-submodule of L, $\xi^* K$ is an essential A-submodule of $\xi^* L$. Since $K \cong L/K \cong \mathcal{O}_q$, there is an exact sequence

$$0 \to F_q \cong \xi^* K \to \xi^* L \to F_q \cong \xi^* L/\xi^* K \to 0.$$

The submodule LA of E is a quotient of $\xi^* L$, and has no non-zero finite dimensional A-submodules, so the only possibility is that $LA \cong \xi^* L/\xi^* K \cong F_q$. But this is a semisimple R-module.

Therefore $e\mathfrak{m}_q = 0$ for some $e \in E \backslash F_p$. Thus $eA \cong F_q$. If $eA = E$, then F_p embeds in F_q with codimension one, so $q = p^{\sigma^{-1}}$ by Proposition 5.1, and the proof is complete.

Suppose to the contrary that $eA \neq E$. Set $n = \dim_k(E/eA)$. Since $eA + F_p = E$, $eA \cap F_p$ has codimension one in eA and codimension n in F_p. Hence

$eA \cap F_p \cong F_{q^\sigma} \cong F_{p^{\sigma^n}}$. Therefore $q = p^{\sigma^{n-1}}$. Let ε be a $k[t]$-basis for eA and ε' a $k[t]$-basis for the copy of $F_{p^{\sigma^{n-1}}}$ in F_p. It follows from Proposition 5.1 that there is some $\alpha \in k$ such that $\varepsilon(t+\alpha)$ is a scalar multiple of $\varepsilon'(t+\alpha)$. Thus $(\varepsilon - \lambda\varepsilon')(t+\alpha) = 0$ for some $\lambda \in k$. But E has no $k[t]$-torsion, so ε and ε' are scalar multiples of each other. Hence $eA \subset F_p$. This is a contradiction. $\square$

Theorem 5.5 *[11] Suppose a k-linear category C has the following properties:*

1. *C is generated by $\{\mathcal{O}(i) \mid i \in \mathbb{Z}^n\}$ in the sense that the set of all subobjects of all finite direct sums of the $\mathcal{O}(i)$s generates C;*
2. *if $j \leq i$ (i.e., $j_s \leq i_s$ for $s = 1, \cdots, n$), then $\operatorname{Hom}_{\mathsf{C}}(\mathcal{O}(j), \mathcal{O}(i)) = k$;*
3. *the only submodules of $\mathcal{O}(i)$ are copies of $\mathcal{O}(j)$ for $j \leq i$;*
4. *if $e_u = (0, \ldots, 1, \ldots, 0)$ for $u = 1, \ldots, n$ are the obvious basis elements for $\mathbb{Z}^n$, then there are short exact sequences*
$$0 \to \mathcal{O}(i - e_u) \to \mathcal{O}(i) \to p_{u,i_u} \to 0,$$
and $\{p_{u,w} \mid u \in \{1, \cdots, n\}, w \in \mathbb{Z}\}$ is a complete set of isomorphism classes of the simple objects in C;
5. *$\operatorname{End}_{\mathsf{C}}(p_{u,w}) \cong k$.*

Then C is equivalent to $\mathsf{GrMod}_{\mathbb{Z}^n} k[x_1, \cdots, x_n]/(\mathrm{Kdim} \leq n-2)$.

Proof. This was proved in [11], but was not stated in quite this way.

All the proofs in [11, Section 6] apply to the present situation. For example, the proof of [11, Proposition 6.1] can be applied to show that every submodule of a finite direct sum of various copies of $\mathcal{O}(i)$ is also a finite direct sum of such modules. Therefore the assertion follows from [11, Theorem 6.9]. $\square$

Here we only need the case $n = 1$ of this theorem. In that case C is equivalent to $\mathsf{GrMod}k[x]$, the category of graded modules over the polynomial ring generated by an element of degree one.

Theorem 5.6 *Suppose that the σ-orbit of p is infinite.*

1. *$\mathsf{Mod}X_p$ is equivalent to $\mathsf{GrMod}k[x]$.*
2. *The closed points in X_p can be labelled x_n, $n \in \mathbb{Z}$, in such a way that $\xi_* \mathcal{O}_{x_n} \cong \mathcal{O}_{p^{\sigma^n}}$, and for each n there is an exact sequence*
$$0 \to F_{p^{\sigma^{n+1}}} \to F_{p^{\sigma^n}} \to \mathcal{O}_{x_n} \to 0.$$

Proof. Write C for the full subcategory of $\mathsf{Mod}X$ consisting of all subquotients of finite direct sums of various copies of $F_{p^{\sigma^i}}$. Lemma 3.2 and Proposition 5.4 shows that C satisfies the first condition in Definition 2.3, namely, every noetherian 1-critical X_p-module is isomorphic to $F_{p^{\sigma^i}}$ for some i. Since C is closed under finite direct sums and subquotients, it follows that $\mathsf{C} = \mathsf{mod}X_p$.

(2) Since every noetherian X_p-module is a subquotient of a finite direct sum of $F_{p^{\sigma^i}}$, every simple X_p-module is a quotient of some $F_{p^{\sigma^i}}$ [12, Lemma 2.1]. By Proposition 5.1, $F_{p^{\sigma^n}}$ has a unique simple quotient, and that quotient is one dimensional. By Proposition 3.6 that quotient is isomorphic to $\mathcal{O}_{p^{\sigma^n}}$ as an R-module. In other words $\xi_*\mathcal{O}_{x_n}$ is isomorphic to $\mathcal{O}_{p^{\sigma^n}}$.

(1) We now check that the hypotheses in Theorem 5.5 hold. Condition (1) follows from the fact that every noetherian 1-critical is isomorphic to $F_{p^{\sigma^i}}$ for some i. Conditions (2) and (3) follow from Propositions 3.8 and 5.1(1). Condition (4) is part (2) of the present result, and condition (5) is clear. Therefore the assertion follows from Theorem 5.5.

Under the equivalence of categories $F_{p^{\sigma^n}}$ corresponds to $k[x](-n)$, the free module with its generator in degree n. □

6 When p has finite σ-orbit

In this section we suppose that the σ-orbit of p has size n for some integer $n \geq 2$.

Lemma 6.1 *Suppose that the σ-orbit of p has size $n \geq 2$. Then*

$$\delta(\mathfrak{m}_p\mathfrak{m}_{p^\sigma}\dots\mathfrak{m}_{p^{\sigma^{n-1}}}) \subset \mathfrak{m}_p\mathfrak{m}_{p^\sigma}\dots\mathfrak{m}_{p^{\sigma^{n-1}}}.$$

Proof. Write $I = \mathfrak{m}_p\mathfrak{m}_{p^\sigma}\dots\mathfrak{m}_{p^{\sigma^{n-1}}}$. Every element in I is a sum of elements of the form ab where $a \in \mathfrak{m}_p$ and $b \in \mathfrak{m}_{p^\sigma}\dots\mathfrak{m}_{p^{\sigma^{n-1}}}$. It is clear that $\delta(ab)$, which equals $\delta(a)b + a^\sigma\delta(b)$, vanishes at p^σ. Thus $\delta(I) \subset \mathfrak{m}_{p^\sigma}$. Replacing p by p^{σ^i}, and repeating the argument, we see that $\delta(I) \subset \mathfrak{m}_{p^{\sigma^j}}$ for all j. The ideals $\mathfrak{m}_{p^{\sigma^j}}$, $j \in \mathbb{Z}_n$, are pairwise distinct, so their intersection equals their product. □

Lemma 6.1 does not extend to the case $n = 1$; to see this, take $R = k[x]$, $\sigma = \mathrm{id}_R$, and $\delta = d/dx$.

Proposition 6.2 *Suppose that the σ-orbit of p has size $n \geq 2$. Then $J := \mathfrak{m}_p\mathfrak{m}_{p^\sigma}\dots\mathfrak{m}_{p^{\sigma^{n-1}}}A$ is a two-sided ideal of A, and $A/J \cong S[t;\sigma,\delta]$ where S is a product of n copies of the field k, σ is a generator of $\mathrm{Aut}_k S$, and δ is a σ-derivation of S.*

Proof. Write $I = \mathfrak{m}_p\mathfrak{m}_{p^\sigma}\dots\mathfrak{m}_{p^{\sigma^{n-1}}}$. Thus $J = IA$. It is clear that J is closed under left multiplication by R, so we need only check that $tJ \subset J$ in order to show that it is a two-sided ideal. However, $tI \subset It + \delta(I)$, and and this is contained in IA by Lemma 6.1. Hence $tJ \subset J$.

Since I is stable under σ and under δ, σ induces an automorphism of R/I, which we denote by σ also, and δ induces a σ-derivation of R/I, which we denote by δ. Set $S = R/I$. Thus S is a product of n copies of k. Since σ has order n, it generates $\mathrm{Aut}_k S$. Since $A/I \cong R/J \otimes_R A$ as a right A-module, it follows that $A/I \cong S[t;\sigma,\delta]$ as claimed. □

If the σ-orbit of p has size $n \geq 2$, F_p is annihilated by $\mathfrak{m}_p \mathfrak{m}_{p^\sigma} \dots \mathfrak{m}_{p^{\sigma^{n-1}}}$ so, as an R-module, it decomposes as a direct sum

$$F_p = (F_p)_0 \oplus (F_p)_1 \oplus \dots \oplus (F_p)_{n-1},$$

where $(F_p)_i$ is the R-submodule supported at p^{σ^i}. We will show that $A/\operatorname{Ann} F_p$ is isomorphic to the representations of a certain quiver. The vertices of the quiver can be thought of as the points in the orbit of p, and the component of F_p at the vertex p^{σ^i} is the submodule supported at p^{σ^i}.

Lemma 6.3 *Let S be a product of $n \geq 2$ copies of the field k. Let σ be a generator of $\operatorname{Aut}_k S$. Let $\{e_i \mid i \in \mathbb{Z}_n\}$ be a complete set of primitive orthogonal idempotents for S, labeled so that $e_i^\sigma = e_{i+1}$. Let $\{\nu_i \mid i \in \mathbb{Z}_n\}$ be arbitrary scalars. Then the linear map $\delta : S \to S$ defined by*

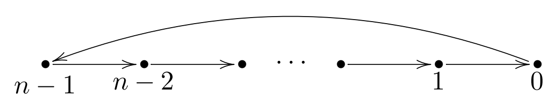
$$\delta(e_i) = \nu_{i+1} e_{i+1} - \nu_i e_i$$

is a σ-derivation of S, and every σ-derivation arises in this way.

Proof. See [1, Lemma 1.4], [2, Lemma 6.1] and [3, Lemma 3.7]. □

Although the following is known in various forms (e.g., [3, Proposition 7.1] and [2, Proposition 6.1]), we include a proof for the convenience of the reader.

Proposition 6.4 *Let S be a product of $n \geq 2$ copies of the field k. Let σ be a generator of $\operatorname{Aut}_k S$, and let δ be a σ-derivation of S. Then $S[t; \sigma, \delta]$ is isomorphic to the path algebra of the quiver*



(6-1)

Proof. Let Q denote the quiver, and write $B = S[t; \sigma, \delta]$. Clearly $\operatorname{Aut}_k S \cong \mathbb{Z}_n$. Let $\{e_i \mid i \in \mathbb{Z}_n\}$ be a complete set of primitive orthogonal idempotents for S, labeled so that $e_i^\sigma = e_{i+1}$. Thus

$$te_i = e_{i+1} t + \delta(e_i)$$

for all i. Let ν_i, $i \in \mathbb{Z}_n$, be such that $\delta(e_i) = \nu_{i+1} e_{i+1} - \nu_i e_i$ for all i. Define

$$u = t + \sum_i \nu_i e_i.$$

Then

$$ue_j = te_j + \nu_j e_j = e_{j+1} t + \delta(e_j) + \nu_j e_j = e_{j+1} t + \nu_{j+1} e_{j+1} = e_{j+1} u$$

for all j.

Clearly, B is generated by u and $e_0, \dots, e_{n-1}$. Because $ue_j = e_{j+1} u$ for all j, $uS = Su$. Thus the powers of u form a basis for B as a left-, and as a right-,

S-module. Therefore $B \cong S[u;\sigma]$. We can thus make B a graded k-algebra with $B_0 = S$, and $B_i = Su^i = u^i S$ for all $i \geq 0$.

Now the isomorphism becomes apparent. If ε_i denotes the trivial path at the vertex labelled i, and x_i denotes the arrow from the vertex $i+1$ to the vertex i, then the isomorphism is given by sending e_i to ε_i and u to $x_0 + \ldots + x_{n-1}$. Under the isomorphism, x_i corresponds to $ue_i = t + \nu_i e_i$. □

We are now able to describe the fiber X_p.

Proposition 6.5 *Suppose that p is not fixed by σ, but has a finite σ-orbit. Let $B = A/\operatorname{Ann} F_p$. If $\operatorname{GKdim} B = 1$, then $\mathsf{Mod} X_p = \mathsf{Mod} B$.*

Proof. Because F_p is critical with respect to Krull dimension, its annihilator is prime [7, Theorem 6.8.26]. So B is a prime noetherian ring. Since $\operatorname{GKdim} B = 1$, B satisfies a polynomial identity. Thus B embeds in a finite direct sum of copies of F_p. Hence B is in $\mathsf{Mod} X_p$, and therefore $\mathsf{Mod} B$ is contained in $\mathsf{Mod} X_p$. To prove equality it remains to show that a module M as in criterion (1) of Definition 2.3 is a B-module. It suffices to show that if $0 \to F_p \to E \to N \to 0$ is an essential extension of F_p and $\dim_k N < \infty$, then E is a B-module. Because F_p is essential in E, E is also critical with respect to GKdim. By [7, Theorem 6.8.26], $\operatorname{Ann} E = \operatorname{Ann} F_p$, whence E is a B-module. □

Theorem 6.6 *Suppose that the σ-orbit of p has size n and that $2 \leq n < \infty$. Then $\mathsf{Mod} X_p = \mathsf{Mod} A/\operatorname{Ann} F_p$, and this is equivalent to the category of representations of the quiver (6-1).*

The representation theory of this quiver is completely understood, so one has complete information about X_p. For example, there are n closed points on X_p of degree one, and all other closed points on X_p have degree n. The indecomposable projectives in $\mathsf{Mod} X_p$ are, up to isomorphism, the $F_{p^{\sigma^i}}$. For each i, there is an injective map $F_{p^{\sigma^{i+1}}} \to F_{p^{\sigma^i}}$, and the cokernel is one-dimensional.

Suppose that the primitive orthogonal idempotents $e_i \in S = R/I$ are labelled so that $\mathcal{O}_p e_0 \neq 0$. Then e_1, which is equal to e_0^σ, is in $\sigma(\mathfrak{m}_p) = \mathfrak{m}_{p^{\sigma^{-1}}}$, whence

$$\mathcal{O}_{p^{\sigma^i}} e_{-i} \neq 0.$$

Therefore $\mathcal{O}_{p^{\sigma^i}} \cong e_{-i} S$, and $F_{p^{\sigma^i}} \cong e_{-i} B$.

7 When p is fixed by σ

In this section we suppose that $p^\sigma = p$. The following is an immediate consequence of Proposition 3.6.

Proposition 7.1 *If F_p is simple, then p is fixed by σ.*

It is possible for F_p to be a simple module. This happens, for example, when A is the Weyl algebra and k has characteristic zero.

Proposition 7.2 *If F_p is simple, then $X_p \cong \operatorname{Spec} k$.*

Proof. We will show that the full subcategory, say C, of $\mathsf{Mod}X$ consisting of all direct sums of copies of F_p satisfies the three conditions in Definition 2.3. It is clear that C satisfies conditions (2) and (3). To show that C satisfies condition (1) it suffices to show that any exact sequence $0 \to F_p \to E \to \mathcal{O}_x \to 0$ in which $\mathcal{O}_x$ is a finite dimensional simple A-module splits.

Suppose to the contrary that there is a non-split sequence of this form. By Lemma 5.3, $\mathcal{O}_x = F_q/F_{q^{\sigma^n}}$ for some $q^{\sigma^n} \in \{p, p^\sigma, p^{\sigma^2}, \dots, \}$. Since F_p is simple, $p = p^\sigma$ and $q = p$. Thus F_q is simple, contradicting the hypothesis that $\mathcal{O}_x$ is finite dimensional.

Therefore $\mathsf{Mod}X_p$ consists of all direct sums of copies of F_p. To complete the proof we must show that the endomorphism ring of F_p is isomorphic to k. Any non-zero endomorphism of F_p is an automorphism because F_p is simple, and is a $k[t]$-module map, so is given by multiplication by a non-zero scalar. □

Although $X_p \cong \operatorname{Spec} k$, X_p is *not* a closed point in X. The inclusion functor $\mathsf{Mod}X_p \to \mathsf{Mod}X$ has a right adjoint but does *not* have a left adjoint. Thus X_p is a weakly closed, but not a closed, subspace of X.

In the rest of this section we consider the case when F_p is not simple. This happens, for example, when σ is identity and δ is zero.

Proposition 7.3 *Let*

$$D = \{d \in k[t] \mid d\mathfrak{m}_p \subset \mathfrak{m}_p A\}.$$

Then there is an isomorphism $\Phi : D \to \operatorname{End}_A F_p$ defined by

$$\Phi(d)(\varepsilon a) = \varepsilon d a$$

for all $a \in A$. As a consequence, D is a subring of $k[t]$.

Proof. We write $\mathbf{I}(\mathfrak{m}_p A)$ for the idealizer of $\mathfrak{m}_p A$ in A. Since $A = \mathfrak{m}_p A \oplus k[t]$, $\mathbf{I}(\mathfrak{m}_p A) = \mathfrak{m}_p A \oplus D$. It is easy to check that $\operatorname{End}_A(A/\mathfrak{m}_p A) \cong \mathbf{I}(\mathfrak{m}_p A)/\mathfrak{m}_p A$ via the map Φ [7, 1.1.11]. □

From now on we view F_p as a D-A-bimodule. If $m \in M$ and $d \in D$, then we write $d.m$ to denote $\Phi(d)(m)$. Thus, if $m = \varepsilon a$, then $d.m = \varepsilon d a$.

The right action of A on F_p restricts to give a right action of D on A also. Each element of F_p is of the form εf for some $f \in k[t]$. But D is a subalgebra of $k[t]$, and $k[t]$ is commutative, so $d.\varepsilon f = \varepsilon d f = \varepsilon f d$, whence $dm = md$ for all $m \in F_p$ and all $d \in D$. Thus we can unambiguously speak of *the* D-module structure on F_p without specifying whether we mean the left or the right structure.

Lemma 7.4 *If $p = p^\sigma$, then every non-zero submodule of F_p is isomorphic to F_p, and is equal to dF_p for some $d \in D$.*

Proof. This is an immediate consequence of Lemma 3.2 because every A-submodule of F_p is a $k[t]$-submodule. $\square$

Lemma 7.5 *If $p = p^\sigma$, then $D = k[u]$ for some $u \in k[t]$. If F_p is not simple, then $u \notin k$.*

Proof. Write $F = F_p$. We show that D is closed under greatest common divisors. If $c, d \in D$, then $cF + dF$ is an A-submodule of F, so equals eF for some $e \in D$. However, viewing F as a $k[t]$-module, eF is generated by the greatest common divisor of c and d in $k[t]$. Hence e is a non-zero scalar multiple of that greatest common divisor. Let $u \in D$ be a polynomial of minimal t-degree and with zero constant term. Let $d \in D$ be another polynomial with zero constant term. Then t divides the greatest common divisor of u and d, so by minimality of $\deg u$, $u|d$. Induction shows that d is a polynomial in u. $\square$

As a right $k[t]$-module, F_p is isomorphic to $k[t]$. If $\deg u = n > 0$, then $k[t]$, and hence F_p is a free D-module of rank n. A basis is given by $\varepsilon, \varepsilon t, \dots, \varepsilon t^{n-1}$. In particular, $\operatorname{End}_D F_p \cong \mathsf{Mod} M_n(k[u])$. Because F_p is a free D-module, if $a, b \in D$, then $aF_p \subset bF_p$ if and only if b divides a.

Lemma 7.6 *Suppose $p = p^\sigma$. Then $\operatorname{Hom}_A(F_p, S) = k$ for every simple quotient S of F_p.*

Proof. If F_p is simple, this follows from Proposition 7.2. If F_p is not simple, then $S = F_p/(u+\alpha)F_p$ for some $\alpha \in k$. Since $u + \alpha$ acts centrally on F_p, every non-zero homomorphism from F_p to S has kernel $(u+\alpha)F_p$. Thus $\operatorname{Hom}_A(F_p, S) = \operatorname{Hom}_A(S, S) = k$, where the last equality follows from the fact the only finite dimensional division algebra over k is k itself. $\square$

We will use the following result [12, Corollary 1.2].

Proposition 7.7 *[12] Let k be algebraically closed field and C a k-linear category. Suppose that*

1. *C is equivalent to $\mathsf{Mod} L$ for some object L in C;*
2. *L is noetherian and 1-critical with respect to the Krull dimension;*
3. *L is the unique noetherian 1-critical object up to isomorphism;*
4. *$\operatorname{End}_{\mathsf{C}} L$ is isomorphic to a polynomial ring $k[u]$;*
5. *$\operatorname{Hom}_{\mathcal{C}}(L, S) \cong k$ for every simple quotient S of L.*

Then L is a progenerator in C, and C is equivalent to $\mathsf{Mod} k[u]$.

Proposition 7.8 *Suppose that $p = p^\sigma$. If F_p is not simple, then $X_p \cong \mathbb{A}^1$ and every closed point on X_p has degree equal to that of the element u appearing in Lemma lem.D=k[u].*

Proof. If $n = \deg u$, then $\operatorname{End}_D F_p$ is isomorphic to $M_n(k[u])$. The action of A on F_p gives an injective map from $B = A/\operatorname{Ann} F_p$ to $\operatorname{End}_D F_p$. By an argument similar to that in Proposition 6.5, $\mathsf{Mod} X_p = \mathsf{Mod} B$. Since B is a noetherian prime ring satisfying a polynomial identity, there are nonzero morphisms between any two uniform right ideals. So every noetherian 1-critical B-module is isomorphic to a submodule of F_p. The lemmas in this section show that the hypotheses of Proposition 7.7 are satisfied, so $X_p \cong \mathbb{A}^1$. It follows from Lemma 7.4 that the simple quotients of F_p have dimension $\deg u$. □

The four cases in Theorem 1.1 now follow from Propositions 5.4, 6.5, 7.2, and 7.8.

Finally we identify the element u that generates $\operatorname{End}_A F_p$ when $p = p^\sigma$. This allows one to determine the degree of the points on the fibers over the fixed point, and thus completes the description of the finite dimensional simple A-modules.

Proposition 7.9 *Suppose that $p = p^\sigma$. The element $u(t)$ in Lemma 7.5 can be chosen to be a non-constant polynomial of minimal degree such that $u(\delta)(\mathfrak{m}_p) \subset \mathfrak{m}_p$.*

Proof. Define $D' = \{f(t) \in k[t] \mid f(\delta)(\mathfrak{m}_p) \subset \mathfrak{m}_p\}$. We will show that $D \subset D'$, and if $f(t) \in D'$ is chosen to be a non-constant polynomial of minimal degree, then $f(t) \in D$. Then, because Lemma 7.5 implies that u is an element in D of minimal positive degree, it will follow that D is generated by this particular $f(t)$.

We define k-linear operators $\nu_{ij} : R \to R$ for all integers $i, j \geq 0$ by

$$\nu_{ij} := \text{the sum of all words in } \sigma \text{ and } \delta \text{ having } i\ \delta\text{s and } j\ \sigma\text{s}. \tag{7-1}$$

For example, $\nu_{22} = \delta^2\sigma^2 + \delta\sigma\delta\sigma + \delta\sigma^2\delta + \sigma\delta^2\sigma + \sigma\delta\sigma\delta + \sigma^2\delta^2$. We define $\nu_{00} = \mathrm{id}_R$.

Let $r, s \in R$. An induction argument on m shows that

$$\delta^m(rs) = \sum_{j=0}^{m} \nu_{m-j,j}(r)\delta^j(s)$$

and

$$t^m r = \sum_{j=0}^{m} \nu_{m-j,j}(r)t^j.$$

Let $f(t) = \sum_{m=0}^{n} \lambda_m t^m$ be any element of $k[t]$. Then

$$f(\delta)(rs) = \sum_{m=0}^{n} \lambda_m \sum_{j=0}^{m} \nu_{m-j,j}(r)\delta^j(s) = \sum_{j=0}^{n}\left(\sum_{m=j}^{n} \lambda_m \nu_{m-j,j}(r)\right)\delta^j(s) \tag{7-2}$$

and, computing in the ring A,

$$f(t)r = \sum_{m=0}^{n} \lambda_m \sum_{j=0}^{m} \nu_{m-j,j}(r)t^j = \sum_{j=0}^{n}\left(\sum_{m=j}^{n} \lambda_m \nu_{m-j,j}(r)\right)t^j. \tag{7-3}$$

From the second of these equations, one sees that $f(t)\mathfrak{m}_p \subset \mathfrak{m}_p A$ if and only if $\sum_{m=j}^{n} \lambda_m \nu_{m-j,j}(\mathfrak{m}_p) \subset \mathfrak{m}_p$ for all $j = 0, \dots, n$. Therefore, if $f(t) \in D$, then $f(\delta)(rs) \subset \mathfrak{m}_p$ for all $r \in \mathfrak{m}_p$ and $s \in R$ so, setting $s = 1$, $f(\delta)(\mathfrak{m}_p) \subset \mathfrak{m}_p$. Thus $D \subset D'$.

Now suppose that $f(t) \in D'$ is chosen to have minimal positive degree. Therefore the right hand side of (7-2) belongs to $\mathfrak{m}_p$ if either r or s does. In particular, if we fix $r \in \mathfrak{m}_p$ and set

$$\mu_j =: \left(\sum_{m=j}^{n} \lambda_m \nu_{m-j,j}(r)\right)(p),$$

then $\sum_{j=0}^{n} \mu_j \delta^j(s) \in \mathfrak{m}_p$ for all $s \in R$. However, $\mu_n = \lambda_n \nu_{0,n}(r)(p) = \lambda_n \sigma^n(r)(p) = 0$. Therefore $\sum_{j=0}^{n-1} \mu_j \delta^j(s) \in \mathfrak{m}_p$ for all $s \in R$. In other words, $\sum_{j=0}^{n-1} \mu_j t^j$ is in D'. But $f(t)$ was chosen to have minimal positive degree, so $\mu_1 = \mu_2 = \dots = \mu_{n-1} = 0$. Equivalently, if $r \in \mathfrak{m}_p$, then $\sum_{m=j}^{n} \lambda_m \nu_{m-j,j}(r) \in \mathfrak{m}_p$ for $j = 1, \dots, n-1$. However, this expression is also in $\mathfrak{m}_p$ when $j = 0$ and $j = n$; for example, $\sum_{m=0}^{n} \lambda_m \nu_{m,0}(r) = \sum_{m=0}^{n} \lambda_m \delta^m(r) = f(\delta)(r)$ which is in $\mathfrak{m}_p$ by hypothesis. Therefore all the coefficients of t^j in (7-3) belong to $\mathfrak{m}_p$, whence $f(t)r \in \mathfrak{m}_p A$. Therefore $f(t) \in D$. $\square$

In general, D is not equal to D'. For example, take $A = k[x][t; \mathrm{id}, \delta]$ where $\delta = d/dx$ and $char k = 2$. Then $\delta^i = 0$ for all $i > 1$ so $\delta^3(R) \subset \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$. Hence $t^3 \in D'$, but it is not in D for any $\mathfrak{m}$.

8 Disjoint union of fibers

We would like to say that X is the disjoint union of the fibers X_p. For such a statement to have some substance it must mean something a little different than it does in the commutative case. In the commutative case it says that every point of X lies on some fiber. We proved an analogue of this in Proposition 3.4. However, that result is vacuous if A has no finite dimensional simple modules (for example, if A is the Weyl algebra over a field of characteristic zero). We therefore seek a result which says something about *all* A-modules, a result which is a little like saying that every subvariety of X meets some fiber, and that distinct fibers do not meet. Following ideas in [6] and [8] we use Ext groups to give meaning to the words "meet" and "disjoint".

Proposition 8.1 *Let X_p be the fiber over a closed point $p \in \operatorname{Spec} R$.*

1. *$\mathsf{Mod}X_p$ is the full subcategory of $\mathsf{Mod}X$ generated by all subquotients of all possible direct sums of the $F_{p^{\sigma^n}}$s for $n \in \mathbb{Z}$.*
2. *If $M \in \mathsf{Mod}X_p$, then the support of $\xi_* M$ is contained in the σ-orbit of p.*
3. *$X_p = X_q$ if and only if the σ-orbits of p and q are equal.*

Proof. (1) By the analysis of all cases, every essential extension of $F_{p^{\sigma^n}}$ by a finite dimensional module is isomorphic to $F_{p^{\sigma^m}}$ for some m (see Propositions 5.4, 6.5, 7.2, and 7.8). Hence the assertion follows from the definition of $\mathsf{Mod}X_p$.

(2) We observed in section 2 that if p is a closed point of $\operatorname{Spec} R$, then the support of $\xi_* F_{p^{\sigma^n}}$ is contained in the σ-orbit of p. Therefore (1) implies (2).

(3) This follows from (1) and Lemma 3.3. □

Although (2) seems to say that f sends the fiber X_p to the σ-orbit of p that statement should be interpreted carefully. For example, if $R = k[x]$ and $A = k[x, \partial]$ where $\partial = d/dx$, then $F_0 = A/xA$ is isomorphic as an R-module to the injective envelope of $R/(x)$ so, although $\xi_* F_0$ is supported at 0, it is not a $k[x]/(x)$-module. Parts (2) and (3) are a weak way of saying that distinct fibers are disjoint. The next result is a stronger way of saying that distinct fibers are disjoint.

Lemma 8.2 *Let X_p and X_q be distinct fibers. Let L be an X_p-module and M an X_q-module. Then $\operatorname{Ext}^i_X(L, M) = 0$ for all i.*

Proof. Since X_p is locally noetherian, it suffices to prove this when L is a noetherian X_p-module. Using the long exact sequence, it is reduced to the case when L is critical. By the first part of Theorem 1.1, L is isomorphic to $F_{p^{s^n}}$ or a simple quotient of $F_{p^{s^n}}$. In both cases there is a short exact sequence $0 \to L_1 \to L_0 \to L \to 0$ in which L_1 and L_0 are isomorphic to (possibly different) $F_{p^{\sigma^j}}$. Therefore it suffices to prove the result when L is isomorphic to some $F_{p^{\sigma^j}}$. For simplicity, we may assume that $L = F_p$. But $\operatorname{Ext}^i_A(F_p, M) \cong \operatorname{Ext}^i_R(\mathcal{O}_p, M)$ 3-2, and this is zero because the support of $\xi_* M$ is contained in the σ-orbit of q which does not contain p. □

We need two lemmas before proving that the fibers "cover" X.

Lemma 8.3 *Let B be a right noetherian k-algebra and let M be a right B-module. If $\dim_k M$ is countable, then $\dim_k \operatorname{Ext}^i_B(N, M)$ is countable for all finitely generated right B-module N.*

Proof. Take a resolution of N by finitely generated free modules. The assertion follows from the fact $\operatorname{Hom}_B(B, M) = M$ has countable dimension over k. □

Lemma 8.4 *Let k be uncountable and let B be a right noetherian k-algebra satsifying a polynomial identity. Let M be a non-zero right B-module such that $\dim_k M$ is countable. Then $\operatorname{Ext}^i_B(N, M)$ is non-zero for some i and some simple right B-module N.*

Proof. Suppose to the contrary that $\operatorname{Ext}^i_B(N, M) = 0$ for all simple N and all i. We will show that $\operatorname{Ext}^i_B(L, M) = 0$ for all finitely generated L and all i. By noetherian induction we may assume that L is critical with respect to Krull dimension. If $\operatorname{Kdim} L = 0$, then L is simple so $\operatorname{Ext}^i_B(L, M) = 0$ by hypothesis. Suppose that $\operatorname{Kdim} L > 0$, and that the Ext groups vanish for those L of smaller Krull dimension. Using the long exact sequence for Ext and filtering L by appropriate submodules, we may assume that $L \cong B/P$ for some prime ideal P. The GK-dimension of B/P is at least 1 because B/P is not simple. Hence the ring of fractions $Q := Q(B/P)$ has uncountable dimension over k. Let a be a regular element in B/P. The long exact sequence for Ext associated to the exact sequence

$$0 \longrightarrow B/P \xrightarrow{a} B/P \longrightarrow B/P + aB \longrightarrow 0$$

shows that the induced action of a on $\mathrm{Ext}^i_B(B/P, M)$ is bijective.

Hence $\mathrm{Ext}^i_B(B/P, M)$ is a module over Q, so is either zero or has uncountable dimension over k. By Lemma 8.3, it must be zero.

Thus $\mathrm{Ext}^i_B(L, M) = 0$ for all noetherian modules L. But this is absurd because $\mathrm{Hom}_B(B, M) \cong M \neq 0$. □

Roughly speaking, the next result says that every subspace of X meets some fiber, or that the fibers cover X.

Proposition 8.5 *Suppose that k is uncountable. If M is a non-zero finitely generated right A-module, then $\mathrm{Ext}^i_A(F_p, M)$ is nonzero for some i and some closed point $p \in \mathrm{Spec}\, R$.*

Proof. Since $\mathrm{Ext}^i_A(\xi^*\mathcal{O}_p, M) = \mathrm{Ext}^i_R(\mathcal{O}_p, M)$ and since M has countable dimension over k, the result follows from the previous lemma. □

Combining Lemma 8.2 and Proposition 8.5, we say that X is a disjoint union of fibers X_p. This is the second part of Theorem 1.1.

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