ON THE GLOBAL DIMENSION OF CERTAIN PRIMITIVE FACTORS OF THE ENVELOPING ALGEBRA OF A SEMI-SIMPLE LIE ALGEBRA

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1. *Introduction*

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over an algebraically closed field k of characteristic zero and let $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$ be a triangular decomposition of $\mathfrak{g}$. Denote by $U = U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$, denote by D_λ the primitive factor ring $U/\operatorname{ann}_U M(\lambda)$, where $M(\lambda)$ is the Verma module of highest weight $\lambda - \rho$ (where ρ is the half-sum of the positive roots). The main aim of this paper is to prove Theorem 3.9, which states that if λ is regular, then gldim $D_\lambda \leqslant \dim_k \mathfrak{n}^+ + n(\lambda)$, where $n(\lambda)$ is a non-negative integer less than or equal to $\dim_k \mathfrak{n}^+$.

In **[12]** Levasseur computed the injective dimension of D_λ in terms of the Gelfand–Kirillov dimension of $L(\lambda)$, the unique simple quotient of $M(\lambda)$. For λ regular, Theorem 3.9 implies that the global dimension of D_λ is finite and hence must be equal to the injective dimension of D_λ. It can be shown that this figure coincides with the bound given in Theorem 3.9 (the authors would like to thank Levasseur for pointing this out). On the other hand, if λ is not regular, Joseph and Stafford **[11]** have shown that gldim $D_\lambda = \infty$.

Special cases of this result have already appeared in the literature. In particular Stafford **[17]** computed the global dimension of D_λ for all $\lambda \in \mathfrak{h}^*$ in the case when $\mathfrak{g} = \mathrm{sl}(2, \mathbb{C})$. On the other hand Roos **[15]** computed gldim D_λ for general $\mathfrak{g}$ when λ satisfies some transcendental but generic conditions.

This paper is divided into two distinct parts. In Section 2 we prove a result analogous to a theorem of Roos relating the weak global dimension of a ring R to that of certain sets of torsion-theoretic localisations. If R is a commutative ring and $\{S_i\}_{i \in I}$ is a set of localisations of R such that $\oplus S_i$ is faithfully flat as an R-module, then it is well known that wgldim $R = \sup\{\text{wgldim } S_i\}$. To what extent this result is true for non-commutative rings is unknown. Roos and others have proved various results under the assumption that wgldim R is finite (see for instance **[3, 7, 13, 14]**). We prove here a slightly more general sort of result than Roos's which does not require this assumption. However, it is necessary to impose certain additional assumptions on the localisations S_i. The special case of this result needed for the second part is the following. Let R be a prime Noetherian ring and let S_i, $i = 1, \ldots, n$ be a finite collection of rings lying between R and its quotient ring. If $\oplus S_i$ is a faithfully flat right R-module and $S_i \otimes_R S_j \cong S_j \otimes_R S_i$ as R-R-bimodules for all pairs (i, j), then

$$\text{gldim } R = \text{wgldim } R \leqslant \sup\{\text{wgldim } S_i + \operatorname{lfd}_R(S_i)\}$$

(where $\operatorname{lfd}_R(S_i)$ denotes the flat dimension of S_i as a left R-module). An example is given where the theorem fails if the condition that $S_i \otimes_R S_j \cong S_j \otimes_R S_i$ is relaxed.

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In Section 3, we apply the results from the previous section to the ring D_λ. Briefly the idea is as follows. Let G/B denote the flag variety of the connected semi-simple algebraic group G associated with $\mathfrak{g}$. Denote by $\mathscr{D}_\lambda$ the sheaf of twisted differential operators on G/B constructed in [2]. By translating the large Bruhat cell and taking local sections of $\mathscr{D}_\lambda$ we obtain a finite collection of rings $\{S_i\}$ contained between D_λ and its ring of fractions. Each S_i is isomorphic to the n-th Weyl algebra (where $n = \dim_k \mathfrak{n}^+$) and the construction of $\mathscr{D}_\lambda$ ensures that $S_i \otimes_{D_\lambda} S_j \cong S_j \otimes_{D_\lambda} S_i$. When λ is dominant regular the equivalence of categories established by Beilinson and Bernstein implies that $S_1 \oplus \ldots \oplus S_m$ is faithfully flat as a right D_λ-module. One may now apply the results of Section 2, together with a result of Joseph and Stafford, to obtain the theorem described above.

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2. *Weak global dimension and localisation*

For the applications in part 3, it is necessary to consider torsion-theoretic localisation, a slightly more general type of localisation than the usual elementwise Ore localisation. Let R be a (not necessarily commutative) ring. An hereditary torsion class $\mathscr{T}$ for R is a collection of left R-modules closed under quotients, submodules, direct sums and extensions. The ring of quotients of R with respect to $\mathscr{T}$ is defined to be $R_{\mathscr{T}} = \lim_{\rightarrow} \operatorname{Hom}_R(I, R)$, where the direct limit is taken over the filter of all left ideals I of R such that R/I belongs to $\mathscr{T}$. There is a natural map $\phi: R \rightarrow R_{\mathscr{T}}$, and $R_{\mathscr{T}}$ can then be considered via ϕ as a left or right R-module. If $R_{\mathscr{T}}$ is flat as a *right* R-module $R_{\mathscr{T}}$ is said to be a *perfect left localisation* of R, and in this situation $\mathscr{T}$ consists precisely of those R-modules M such that $R_{\mathscr{T}} \otimes_R M = 0$. The torsion submodule $\tau(M)$ of an R-module M is defined to be the largest submodule of M belonging to $\mathscr{T}$. If $R_{\mathscr{T}}$ is a perfect localisation, then $\tau(M)$ is precisely the kernel of the natural map $\theta: M \rightarrow R_{\mathscr{T}} \otimes_R M$. The reader is referred to [18] for a more complete explanation.

If M is an R-bimodule its flat dimension as left module is denoted $\operatorname{lfd}_R(M)$. When there is no danger of ambiguity or when M is a left R-module only, we write $\operatorname{fd}_R(M)$.

THEOREM 2.1. *Let R be a ring and let $B_1, \ldots, B_n$ be a finite collection of perfect left localisations of R with associated torsion classes $\mathscr{T}_1, \ldots, \mathscr{T}_n$, respectively. Suppose that $B_1 \oplus \ldots \oplus B_n$ is faithfully flat as a right R-module and that each $\mathscr{T}_i$ is closed under $B_j \otimes_R -$ for $j = 1, \ldots, n$. Then*

$$\operatorname{wgldim} R \leqslant \max_i \{\operatorname{wgldim} B_i + \operatorname{lfd}_R(B_i)\}.$$

Proof. Clearly we may assume that $\max\{\operatorname{wgldim} B_i + \operatorname{lfd}_R(B_i)\} = m < \infty$. For each left R-module M let $\eta(M)$ be the number of distinct torsion classes amongst $\mathscr{T}_1, \ldots, \mathscr{T}_n$ to which M belongs. Suppose the theorem is false, and pick amongst all R-modules with flat dimension greater than m a module M such that $\eta(M)$ is greatest possible. Since $B_1 \oplus \ldots \oplus B_n$ is faithfully flat as a right R-module, $\mathscr{T}_1 \cap \ldots \cap \mathscr{T}_n = 0$, so that $\eta(M) < n$. Choose an index i such that M does not belong to $\mathscr{T}_i$ and let $\tau_i(M)$ be the torsion submodule of M with respect to $\mathscr{T}_i$. Then, if $N = M/\tau_i(M)$, there exist exact sequences

$$0 \longrightarrow \tau_i(M) \longrightarrow M \longrightarrow N \longrightarrow 0$$

and

$$0 \longrightarrow N \longrightarrow B_i \otimes N \longrightarrow B_i \otimes N/N \longrightarrow 0.$$

Now, by hypothesis, $B_i \otimes N$ belongs to all the torsion classes to which M belongs, and hence so does $B_i \otimes N/N$. Because each B_i is a perfect left localisation of R, $B_i \otimes_R B_i$ is isomorphic to B_i as an R-R-bimodule [**18**, X1.1.2]. The flatness of B_i then implies that $B_i \otimes (B_i \otimes N/N) = 0$. Hence, $\eta(B_i \otimes N/N) > \eta(M)$ and so the choice of $\eta(M)$ implies that $\mathrm{fd}_R(B_i \otimes N/N) \leqslant m$. Using the change-of-rings theorem for flat dimension, $\mathrm{fd}_R(B_i \otimes N) \leqslant \mathrm{fd}_{B_i}(B_i \otimes N) + \mathrm{lfd}_R(B_i) \leqslant m$ [**4**, p. 360, Example 5]. Thus, the exact-sequence lemma for flat dimension implies that $\mathrm{fd}_R(N) \leqslant m$.

Similarly, $\tau_i(M)$ belongs to $\mathscr{T}_i$, so that $\eta(\tau_i(M)) > \eta(M)$. Hence, $\mathrm{fd}_R(\tau_i(M)) \leqslant m$ and the top exact sequence implies that $\mathrm{fd}_R(M) \leqslant m$, a contradiction.

COROLLARY 2.2. *Let $R, B_1, \ldots, B_n$ be as in Theorem* 2.1. *If the B_i are also flat as left R-modules, then*

$$\mathrm{wgldim}\, R = \max_i \{\mathrm{wgldim}\, B_i\} = \mathrm{wgldim}\,(B_1 \oplus \ldots \oplus B_n).$$

The form in which we shall use this result in part 3 is as follows.

COROLLARY 2.3. *Let R be a two-sided Noetherian prime ring with simple quotient ring Q. Let $B_1, \ldots, B_n$ be rings lying between R and Q such that $B_1 \oplus \ldots \oplus B_n$ is a faithfully flat right R-module. Suppose further that $B_i \otimes_R B_j \cong B_j \otimes_R B_i$ as R-R-bimodules for all i and j. Then*

$$\mathrm{gldim}\, R \leqslant \max_i \{\mathrm{lgldim}\, B_i + \mathrm{lfd}_R(B_i)\}.$$

Proof. Each B_i lies between R and Q and is flat as a right R-module. Since Q is a right Ore localisation of R, B_i must be a perfect left localisation of R by [**18**, X1.3.4]. Also, each B_i is left Noetherian [**18**, X1.3.9], so that $\mathrm{lgldim}\, B_i = \mathrm{wgldim}\, B_i$. Of course, since R is Noetherian on both the left and right, $\mathrm{gldim}\, R = \mathrm{lgldim}\, R = \mathrm{wgldim}\, R$.

Hence, by Theorem 2.1, it suffices to check that the torsion class $\mathscr{T}_i$ associated with the localisation B_i is closed under $B_j \otimes_R -$ for each $j = 1, \ldots, n$. Suppose that M belongs to $\mathscr{T}_i$. Then $B_i \otimes_R B_j \otimes_R M \cong B_j \otimes_R B_i \otimes_R M = 0$. Hence, $B_j \otimes_R M$ belongs to $\mathscr{T}_i$, as required.

The following is a condition which is useful in checking the hypothesis of Corollary 2.3. It will be used in the next section.

PROPOSITION 2.4. *Let R be a prime Noetherian ring and let S_1 and S_2 be two perfect left localisations of R. Suppose that each S_i is generated as a left and as a right R-module by a subset C_i and that the elements of C_1 commute with those of C_2. Then $S_1 \otimes_R S_2 \cong S_2 \otimes_R S_1$ as* R-R-bimodules.

Proof. Let $\mathscr{F}_i$ be the Gabriel topology associated with the localisation S_i of R, that is, the set of all left ideals I of R such that $S_i \otimes (R/I) = 0$. If Q is the classical quotient ring of R then S_i can be identified as the set $\{x \in Q : Ix \subseteq R \text{ for some } I \in \mathscr{F}_i\}$. Let B be the subring of Q generated by S_1 and S_2. If suffices to show that $S_1 \otimes_R S_2 \cong B$ as R-R-bimodules.

The flatness of Q as a left R-module implies that $S_1 \otimes_R Q$ embeds in $Q \otimes_R Q$, which is naturally isomorphic to Q. The composite embedding sends $s \otimes q$ to sq. Further, the flatness of S_1 as a right R-module implies that $S_1 \otimes_R S_2$ embeds naturally in $S_1 \otimes_R Q$. Composing with the above map gives an isomorphism of R-R-bimodules from $S_1 \otimes_R S_2$ to $S_1 S_2$. But since C_1 and C_2 commute and $S_i = RC_i = C_i R$, it is clear that $B = S_1 S_2$. Hence we have an R-R-bimodule isomorphism from $S_1 \otimes_R S_2$ to B.

By analogy with Roos's result [**14**], there are two directions in which one might hope to weaken the hypotheses of Theorem 2.1. First, one might conjecture that the result is true for infinite sets of localisations $\{B_i\}_{i \in I}$, rather than just finite collections. Secondly, one might think that the conditions on the torsion-classes can be omitted completely. The former is an open question but the latter is false, as the following example shows.

EXAMPLE 2.5. Let k be an algebraically closed field of characteristic zero and let A be the Weyl algebra $k[p, q]$, where $pq - qp = 1$. Let R be the subalgebra of A generated by the elements q, qp, pqp. Then R is isomorphic to $U(\mathrm{sl}(2, k))/(\Omega+1)$, where Ω is the Casimir element [**16**]. Now [**16**, Corollary 1] states that the sets $k[q]-\{0\}$ and $k[pqp]-\{0\}$ are left and right Ore subsets of R, and that if S_1 and S_2 are the respective localisations, then $S_1 \oplus S_2$ is faithfully flat as both a right and left R-module. However, both S_1 and S_2 are isomorphic to the ring $k(q)[p]$ which has global dimension one, whilst R has infinite global dimension [**17**]. Of course, in this case, the two Ore sets do not commute and $S_1 \otimes_R S_2 \not\cong S_2 \otimes_R S_1$.

3. *Global dimension of primitive factors of* $U(\mathfrak{g})$

Let G be a connected semi-simple algebraic group over an algebraically closed field k of characteristic zero. Fix a Borel subgroup B and a maximal torus H inside B, let W be the Weyl group, put $X = G/B$ and let $\dim X = n$. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$ be the Lie algebras of G, B, H, respectively and put $U = U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$. Let R be the root system in $\mathfrak{h}^*$ with positive roots R^+ and let ρ be the half-sum of the positive roots. For $\alpha \in R$, let s_α be the corresponding reflection. A weight λ is called dominant if $\lambda(h_\gamma) \notin \{-1, -2, -3, \ldots\}$ for any $\gamma \in R^+$. A weight λ in $\mathfrak{h}^*$ is regular if $\{w \in W \mid w\lambda = \lambda\} = \{e\}$.

For each λ in $\mathfrak{h}^*$, Bernstein and Beilinson construct a sheaf $\mathscr{D}_\lambda$ of twisted differential operators on X. The construction is briefly as follows. Let $\mathcal{O}$ be the sheaf of regular functions on X. Form the sheaf of algebras $U^\circ = \mathcal{O} \otimes_k U$ with multiplication given by $(f \otimes Z)(g \otimes Y) = fZ(g) \otimes Y + fg \otimes ZY$ for $f, g \in \mathcal{O}$, $Z, Y \in \mathfrak{g}$. Since $\mathfrak{g}$ acts as global vector fields on X there is a map $\alpha\colon \mathcal{O} \otimes \mathfrak{g} \to \operatorname{Der} \mathcal{O}$, where $\operatorname{Der} \mathcal{O}$ is the sheaf of k-linear derivations. The kernel of α is denoted $\mathfrak{b}^\circ$ and there is a surjection $\mathfrak{b}^\circ \to \mathcal{O} \otimes \mathfrak{h}$. Thus, each λ in $\mathfrak{h}^*$ induces a map $\lambda^\circ\colon \mathfrak{b}^\circ \to \mathcal{O}$. Let $\mathscr{I}_\lambda$ denote the sheaf of ideals of $\mathcal{O} \otimes U$ generated by $\xi - (\lambda-\rho)^\circ(\xi)$ for $\xi \in \mathfrak{b}^\circ$. Then $\mathscr{D}_\lambda$ is defined to be $\mathcal{O} \otimes U/\mathscr{I}_\lambda$. For λ in $\mathfrak{h}^*$, let $M(\lambda)$ be the Verma-module as defined in [**6**, 17.1]. Define D_λ to be $U(\mathfrak{g})/\operatorname{ann}_U M(\lambda)$.

THEOREM 3.1 [**2**]. (1) $\Gamma(X, \mathscr{D}_\lambda) = D_\lambda$;

(2) *If λ is dominant and regular there is an equivalence of categories between the category of left D_λ-modules and the category of quasi-coherent left $\mathscr{D}_\lambda$-modules given by the mutually inverse functors $\mathscr{M} \to \Gamma(X, \mathscr{M})$ and $M \to \mathscr{D}_\lambda \otimes_{D_\lambda} M$.*

COROLLARY 3.2. *Suppose that λ is dominant and regular.*

(i) *If U is an affine open subset of X, then $\Gamma(U, \mathscr{D}_\lambda)$ is flat as a right D_λ-module.*

(ii) *Let $V_1, \ldots, V_n$ be an affine open cover of X and let $S_i = \Gamma(V_i, \mathscr{D}_\lambda)$. Then $\bigoplus_{i=1}^n S_i$ is faithfully flat as a right D_λ-module.*

Proof. A left $\mathscr{D}_\lambda$-module is quasi-coherent if and only if it is quasi-coherent as an $\mathcal{O}$-module. The corollary then follows from the vanishing of cohomology on affine algebraic varieties. The details may be found in [**8**].

PROPOSITION 3.3. *Let V be an affine open subset of X and let λ be dominant regular. Then $\Gamma(V, \mathscr{D}_\lambda)$ is contained between D_λ and its quotient division ring.*

Proof. Let $S = \Gamma(V, \mathscr{D}_\lambda)$. Since V is affine, the maps between S and the local rings at points are just Ore localisations. Hence the quasi-coherence of $\mathscr{D}_\lambda$ implies that the rings of local sections on affine open subsets of X can be viewed as subrings of a common quotient division ring, say E. Define $\mathscr{E}$ to be the constant sheaf whose module of local sections at any affine open set is just E. The module of global sections is then clearly also E. By Theorem 3.1, $\mathscr{E} \cong E \otimes_{D_\lambda} \mathscr{D}_\lambda$.

Now since V is affine there is an equivalence of categories between S-modules and $\mathscr{D}_\lambda|_V$-modules, given by the mutually inverse functors $\Gamma(V, -)$ and $-\otimes_S(\mathscr{D}_\lambda|_V)$. Since $\mathscr{D}_\lambda|_V \cong S \otimes_S (\mathscr{D}_\lambda|_V)$, it is clear that $\mathscr{E}|_V \cong E \otimes_{D_\lambda} S \otimes_S (\mathscr{D}_\lambda|_V)$. But equally clearly (from the definition), $\Gamma(V, \mathscr{E}) = E$. Thus

$$E = \Gamma(V, E \otimes_{D_\lambda} S \otimes_S (\mathscr{D}_\lambda|_V)) \cong E \otimes_{D_\lambda} S.$$

Tensoring on the right over S with E yields that $E \cong E \otimes_S E \cong E \otimes_{D_\lambda} E$. This implies that the embedding of D_λ in E must be that of D_λ into its quotient division ring.

COROLLARY 3.4. *Let V be an open affine subset of X and let $S = \Gamma(V, \mathscr{D}_\lambda)$. Then if λ is dominant regular, S is a perfect left localisation of D_λ.*

Proof. By Corollary 3.2, S is flat as a right D_λ-module, and by Proposition 3.3, S is contained between D_λ and its quotient ring. The result then follows from [**18**, X1.2.4].

THEOREM 3.5. *Let $g_1, \ldots, g_l$ be representatives in G of the Weyl group. Let V be the large Bruhat cell in X and let $V_i = g_i V$. If $\lambda \in \mathfrak{h}^*$ is dominant and regular, then*

$$\operatorname{gldim} D_\lambda \leqslant \dim X + \max\{\operatorname{lfd}_{D_\lambda} \Gamma(V_i, \mathscr{D}_\lambda)\}.$$

Proof. Let $S_i = \Gamma(V_i, \mathscr{D}_\lambda)$ and let $n = \dim X$. Now the V_i are translates of V, and so must be isomorphic to affine space of dimension n. Furthermore, it follows from the definition of $\mathscr{D}_\lambda$ that the ring of local sections on V_i is just the ring of untwisted differential operators on V_i. Hence, S_i is isomorphic to A_n, the n-th Weyl algebra. Thus, by [**14**], gldim $S_i = \operatorname{gldim} A_n = n$. Since λ is dominant and regular, Corollary 3.4 and Proposition 3.3 imply that S_i is a perfect left localisation of D_λ. Moreover, it follows from the definition of D_λ that S_i is generated as a left and right D_λ-module by $C_i = \Gamma(V_i, \mathcal{O})$. Since $\{g_1, \ldots, g_l\}$ is a full set of representatives of the

Weyl group, the V_i form an open affine cover for X. Hence, noting that the C_i commute, the theorem follows from Corollary 2.3 and Proposition 2.4.

In order to obtain a more precise bound on the global dimension, we first reduce the problem to that of computing $\operatorname{lfd}_{D_\lambda} \Gamma(V, \mathscr{D}_\lambda)$. The following lemma will be used twice. Its proof is straightforward and is left to the reader. We denote the right flat dimension of an R-R-bimodule M by $\operatorname{rfd}_R M$.

LEMMA 3.6. *Let R, S_1, S_2 be rings and, for $i = 1, 2$, let $\phi_i: R \to S_i$ be monomorphisms. If $\alpha: S_1 \to S_2$ and $\beta: R \to R$ are ring isomorphisms such that $\alpha\phi_1 = \phi_2\beta$, then* $\operatorname{lfd}_R S_1 = \operatorname{lfd}_R S_2$ *and* $\operatorname{rfd}_R S_1 = \operatorname{rfd}_R S_2$.

LEMMA 3.7. *For any open set U in X and any $g \in G$,*

$$\operatorname{lfd}_{D_\lambda} \Gamma(U, \mathscr{D}_\lambda) = \operatorname{lfd}_{D_\lambda} \Gamma(gU, \mathscr{D}_\lambda).$$

Proof. The map $l_g: X \to X$ defined by $x \to gx$ is a morphism of algebraic varieties and therefore induces a map $(l_g)_*: \mathcal{O} \to \mathcal{O}$ such that $(l_g)_*(\Gamma(U, \mathcal{O})) = \Gamma(g^{-1}U, \mathcal{O})$. Extend $(l_g)_*$ to a map $\pi_g: U^\circ \to U^\circ$ by letting π_g act via $\operatorname{Ad}(g)$ on elements of $U(\mathfrak{g})$. Then π_g preserves the multiplication described above, so π_g is a well-defined isomorphism of sheaves of algebras. Let $\rho_g: \operatorname{Der} \mathcal{O} \to \operatorname{Der} \mathcal{O}$ be the induced map on the tangent sheaf, and let $\alpha: \mathcal{O} \otimes \mathfrak{g} \to \operatorname{Der} \mathcal{O}$ be the natural map described above. Then it follows from the definition that $\alpha\pi_g = \rho_g \alpha$ and hence that $\mathfrak{b}^\circ = \ker \alpha$ is invariant under π_g. Moreover, since $\lambda^\circ: \mathfrak{b}^\circ \to \mathcal{O}$ is a fibre map on the quotient $\mathcal{O} \otimes \mathfrak{h}$, for any $\xi \in \mathfrak{b}^\circ$, we have that $\pi_g \lambda^\circ(\xi) = \lambda^\circ \pi_g(\xi)$. Hence the ideals $\mathscr{I}_\lambda = (\xi - (\lambda-\rho)^\circ(\xi))\, U^\circ$ are invariant under π_g. Thus, for each $\lambda \in \mathfrak{h}^*$, π_g induces a map $\pi_{g,\lambda}: D_\lambda \to D_\lambda$ sending $\Gamma(U, \mathscr{D}_\lambda)$ to $\Gamma(g^{-1}U, \mathscr{D}_\lambda)$ and inducing an automorphism on the global sections D_λ. The result then follows from Lemma 3.6.

DEFINITION 3.8. For $\lambda \in \mathfrak{h}^*$, write R_λ for the root system $R_\lambda = \{\alpha \in R: \alpha^*(\lambda) \in \mathbb{Z}\}$, and W_λ for the Weyl group of R_λ. Set $R_\lambda^+ = R_\lambda \cap R^+$. Let Δ_λ be the corresponding set of simple roots and S_λ the set of reflections corresponding to Δ_λ. Let $l_\lambda: W_\lambda \to \mathbb{Z}^+$ be the corresponding length function. Define $n: \mathfrak{h}^* \to \mathbb{Z}^+$ by $n(\lambda) = \max\{l_\lambda(w): w \in W_\lambda\}$. (See [9] for further details.)

A few remarks on $n(\lambda)$ may be made here. First, $n(\lambda)$ satisfies $0 \leqslant n(\lambda) \leqslant \dim X$ for all $\lambda \in \mathfrak{h}^*$. If both λ and $-\lambda$ are dominant regular (that is, for all $\alpha \in R$, $\alpha^*(\lambda) \notin \mathbb{Z}$), then $n(\lambda) = 0$, whilst if λ is dominant regular and integral (that is, $\alpha^*(\lambda) \in \mathbb{N}$ for all $\alpha \in R$), then $n(\lambda) = \dim X$.

Secondly, for all $w \in W$, a routine calculation reveals that $W_{w\lambda} = wW_\lambda w^{-1}$ and $R_{w\lambda} = wR_\lambda w^{-1}$. Thus, $n(w\lambda) = n(\lambda)$ and hence $n(\lambda)$ depends only on the Weyl group orbit of λ.

THEOREM 3.9. *If λ is regular, then* $\operatorname{gldim} D_\lambda \leqslant \dim X + n(\lambda)$.

Proof. For λ regular, there exists $w \in W$ such that $w\lambda$ is dominant regular. Since $D_{w\lambda} \cong D_\lambda$ and $n(w\lambda) = n(\lambda)$, we may therefore assume that λ is dominant. Let V be the large Bruhat cell as before. Then, by Theorem 3.5 and Lemma 3.7, it suffices to show that $\operatorname{lfd}_{D_\lambda} \Gamma(V, \mathscr{D}_\lambda) \leqslant n(\lambda)$. Since by [1, p. 44] $\mathscr{D}_\lambda$ is isomorphic to $\mathscr{D}^{\circ}_{-\lambda}$, the opposite sheaf of $\mathscr{D}_{-\lambda}$, this is equivalent to showing that $\operatorname{rfd}_{D_{-\lambda}} \Gamma(V, \mathscr{D}_{-\lambda}) \leqslant n(\lambda)$.

Now let w_0 be the longest element of the Weyl group, let $\mu = w_0\lambda$ and let $i: D_\mu \to A_n$ be the embedding, described by Conze [5], of $D_{-w_0\lambda}$ into the n-th Weyl algebra. Let $j: D_{-\lambda} \to \Gamma(V, \mathscr{D}_{-\lambda})$ be the restriction map. Then in [8] it is shown that there exist isomorphisms $\tau: D_{-\lambda} \to D_\mu$ and $\psi: \Gamma(V, \mathscr{D}_{-\lambda}) \to A_n$ such that $i\tau = \psi j$. But then it follows from Lemma 3.6 that

$$\mathrm{rfd}_{D_\mu}(A_n) = \mathrm{rfd}_{D_{-\lambda}}(\Gamma(V, \mathscr{D}_{-\lambda})).$$

Now, in [11, Theorem 5.8] it is shown that $\mathrm{rfd}_{D_\mu}(A_n) \leqslant n(\mu)$. Moreover, $n(\mu) = n(-w_0\lambda) = n(w_0\lambda) = n(\lambda)$. Hence, $\mathrm{rfd}_{D_{-\lambda}}(\Gamma(V, \mathscr{D}_{-\lambda})) \leqslant n(\lambda)$, as required.

COROLLARY 3.10. *If λ is regular, then* $\operatorname{gldim} D_\lambda = \dim X + n(\lambda)$.

Proof. Let $n = \dim X$. As above, we may assume that λ is antidominant. It is shown in [12] that $\operatorname{inj\,dim}(D_\lambda) = 2n - \min_{w\in W}\{d(L(w\lambda))\}$, where $L(w\lambda)$ is the unique simple factor of the Verma module $M(w\lambda)$ and $d(-)$ denotes the Gelfand–Kirillov dimension. If w_0 is the longest element of the Weyl group, then certainly $n(\lambda) = l_\lambda(w_0)$. On the other hand, it is shown in [10, §8.18] that (for λ antidominant and regular), $d(L(w_0\lambda)) = n - l_\lambda(w_0)$. Hence

$$n + n(\lambda) = n + l_\lambda(w_0) = 2n - d(L(w_0\lambda)) \leqslant \operatorname{inj\,dim}(D_\lambda).$$

Thus $\operatorname{gldim} D_\lambda = n + n(\lambda)$, as required.

Note. Bernstein has recently announced that Theorem 3.9 also follows from a more general theorem on the vanishing of cohomology (similar to that in [2]) together with a spectral sequence of Grothendieck.

References

1. A. BEILINSON and I. N. BERNSTEIN, 'A generalisation of Casselman's submodule theorem', *Representation theory of Lie groups* (Birkhauser, Boston 1983).
2. A. BEILINSON and I. N. BERNSTEIN, 'Localisation de g-modules', *C.R. Acad. Sci. Paris Sér. I* 292 (1981) 15–18.
3. J.-E. BJORK, 'The global homological dimension of some algebras of differential operators', *Invent. Math.* 17 (1972) 67–78.
4. H. CARTAN and S. EILENBERG, *Homological algebra* (University Press, Princeton 1956).
5. N. CONZE, 'Algèbres d'operateurs différentiels et quotients des algèbres enveloppantes', *Bull. Soc. Math. France* 102 (1974) 379–415.
6. J. DIXMIER, *Enveloping algebras* (North-Holland, Amsterdam 1977).
7. K. R. GOODEARL, 'Global dimension of differential operator rings', *Trans. Amer. Math. Soc.* 209 (1975) 65–85.
8. T. J. HODGES and S. P. SMITH, 'Differential operators on the flag variety and the Conze embedding', preprint, University of Warwick 1984.
9. J. C. JANTZEN, *Moduln mit einem hochsten Gewicht*, Lecture Notes in Mathematics 750 (Springer, Berlin 1979).
10. A. JOSEPH, 'On the variety of a highest weight module, *J. Algebra* 88 (1984) 238–278.
11. A. JOSEPH and J. T. STAFFORD, 'Modules of k-finite vectors over semi-simple Lie algebras,' *Proc. London Math. Soc.* (3) 49 (1984) 361–384.
12. T. LEVASSEUR, 'Dimension injective des quotients primitifs minimaux de l'algèbre enveloppante d'une algèbre de Lie semi-simple', *C.R. Acad. Sci. Paris Sér. I* 292 (1981) 385–387.
13. J. C. MCCONNELL, 'On the global dimension of some rings,' *Math. Z.* 153 (1977) 253–265.
14. J.-E. ROOS, 'Détermination de la dimension homologique globale des algèbres de Weyl', *C.R. Acad. Sci. Paris Sér. A.* 274 (1972) 23–26.
15. J.-E. ROOS, 'Propriétés homologiques des quotients primitifs des algèbres enveloppantes des algèbres de Lie semi-simple', *C.R. Acad. Sci. Paris Sér. A.* 276 (1973) 351–354.

16. J.-E. Roos, 'Compléments à l'étude des quotients primitifs des algèbres enveloppantes des algèbres de Lie semi-simple', *C.R. Acad. Sci. Paris Sér. A.* 276 (1973) 447–450.

17. J. T. Stafford, 'Homological properties of the enveloping algebra $U(\mathrm{sl}_2)$', *Math. Proc. Cambridge Philos. Soc.* 91 (1982) 29–37.

18. B. Stenstrom, *Rings of quotients* (Springer, New York 1975).

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