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Primitive Ideals and Nilpotent Orbits in Type G_2

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1. INTRODUCTION

1.1. Throughout this paper all Lie algebras will be finite dimensional and defined over $\mathbb{C}$. Denote by $\mathfrak{g}_2$ the simple Lie algebra of type G_2. This paper examines the 8-dimensional nilpotent orbit in $\mathfrak{g}_2$ and the two completely prime primitive ideals associated to it. The main technique is to embed $\mathfrak{g}_2$ in $so(7)$ and to use information about the minimal nilpotent orbit in $so(7)$ and the Joseph ideal in $U(so(7))$ to obtain the required information about $\mathfrak{g}_2$.

Most of the questions we consider arise explicitly in two papers of Vogan [26, 27].

1.2. If $\mathfrak{g}$ is any semi-simple Lie algebra, and J a primitive ideal of $U(\mathfrak{g})$, then the associated variety $\mathscr{V}(J) \subseteq \mathfrak{g}^*$ is defined as the zeroes of the associated graded ideal, gr J, in $S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$. Through the non-degeneracy of the Killing form, $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified, and $\mathscr{V}(J)$ is considered as a subvariety of $\mathfrak{g}$. Let G denote the adjoint algebraic group of $\mathfrak{g}$. If $X \in \mathfrak{g}$, and ad X acts nilpotently on $\mathfrak{g}$, we refer to $G \cdot X$ as a nilpotent orbit. By Joseph [20], $\mathscr{V}(J)$ is the closure of a single nilpotent G-orbit, which we denote by $\mathbf{O}_J$. We say that $\mathbf{O}_J$ is associated to J, and conversely, that J is associated to $\mathbf{O}_J$.

An ideal J of $U(\mathfrak{g})$ is said to be completely prime if $U(\mathfrak{g})/J$ contains no zero-divisors.

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Let $\mathfrak{g}$ be simple and not of type A_n. Let $\mathbf{O}_{\min}$ denote the (unique) minimal non-zero nilpotent orbit. There is a unique completely prime primitive ideal, denoted J_0 and called the Joseph ideal, associated to $\mathbf{O}_{\min}$ [13]. It is an important problem to be able to determine "many" completely prime ideals (see, for example, [16, 17]).

1.3. For $\mathfrak{g}_2$ the nilpotent orbits are of dimension 0, 6, 8, 10, 12. We denote by $\mathbf{O}_d$ the unique nilpotent orbit of dimension d. In [16, 18] Joseph shows that there are exactly two completely prime primitive ideals associated with $\mathbf{O}_8$. Let α_1 and α_2 be simple roots for $\mathfrak{g}_2$ with α_1 short and α_2 long. Let $\bar{\omega}_1$ and $\bar{\omega}_2$ be the corresponding fundamental weights. With the notation of 2.4, the two completely prime primitive ideals associated with $\mathbf{O}_8$ are $J_1 = J(\frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2))$ and $J_2 = J(\frac{1}{2}(5\bar{\omega}_1 - \bar{\omega}_2))$.

The following are the main results obtained in this paper:

(a) a new proof that J_1 is a completely prime ideal by showing that $J_1 = J_0 \cap U(\mathfrak{g}_2)$, where J_0 is the Joseph ideal for $so(7)$;

(b) the embedding $U(\mathfrak{g}_2)/J_1 \subsetneq U(so(7))/J_0$ obtained from (a) is an equality;

(c) $\bar{\mathbf{O}}_8$ (the Zariski closure of O_8 in $\mathfrak{g}_2$) is not a normal variety; there is a natural map $\pi: so(7) \to \mathfrak{g}_2$ (see 2.5) such that $\pi: \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ is bijective and $\bar{\mathbf{O}}_{\min}$ is the normalisation of $\bar{\mathbf{O}}_8$ (here $\bar{\mathbf{O}}_{\min}$ is the Zariski closure of the minimal orbit $\mathbf{O}_{\min}$ in $so(7)$);

(d) $\bar{\mathbf{O}}_8 \cap \mathfrak{n}^+$ (where $\mathfrak{n}^+$ = the span of the positive root vectors in $\mathfrak{g}_2$) has two irreducible components, $\mathscr{V}_1$ and $\mathscr{V}_2$ (both are singular varieties), and there is an algebra embedding $U(\mathfrak{g}_2)/J_1 \subsetneq \mathscr{D}(\mathscr{V}_1)$, the ring of differential operators on $\mathscr{V}_1$, such that $\mathscr{O}(\mathscr{V}_1)$ becomes a simple highest weight module for $\mathfrak{g}_2$;

(e) the graded ideal $\operatorname{gr} J_1$ is not prime. There is an isomorphism of $\mathfrak{g}_2$-modules $S(\mathfrak{g}_2)/\operatorname{gr} J_1 \cong \mathscr{O}(\mathbf{O}_8)$ and $\mathscr{O}(\mathbf{O}_8)$ is the ring of regular functions on the normalisation of $\mathbf{O}_8$ (by [2, Lemma 3.7]).

1.4. The results given above answer a number of questions raised in [26, 27]. One other question of Vogan which we answer is the following. Consider $\mathfrak{g}_2$ as a subalgebra of $so(8)$. Let $G_2 \subseteq SO(8)$ be the closed connected subgroup of $SO(8)$ with Lie algebra $\mathfrak{g}_2$. Does the minimal nilpotent $SO(8)$-orbit in $so(8)$ (which is 10-dimensional) contain a dense open G_2-orbit? We answer this in the affirmative in 2.7.

1.5. Each section of the paper begins with an extensive introduction, so we only briefly indicate here the format of the paper. In Section 2 we give details of the inclusions $\mathfrak{g}_2 \subseteq so(7) \subseteq so(8)$ which will be used throughout. The inclusion $\mathfrak{g}_2 \subseteq so(7)$ gives rise to a linear map $\pi: so(7) \to \mathfrak{g}_2$ (see 2.5). We examine the relationship between nilpotent orbits in $so(7)$ and $\mathfrak{g}_2$ under the action of π. In Section 3 we prove (a), (b), that $\bar{\mathbf{O}}_8$ is not normal, and part of (d). Sections 4 and 5 are devoted to a more detailed examination of $\bar{\mathbf{O}}_8, \bar{\mathbf{O}}_{\min}$ and the components of $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+$ and $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}_1^+$, where $\mathfrak{n}_1^+$ (respectively $\mathfrak{n}_2^+$) is the span of the positive root vectors in $so(7)$ (respectively $\mathfrak{g}_2$).

2. Embedding $\mathfrak{g}_2$ in $so(7)$

2.1. In this section we describe the inclusions $\mathfrak{g}_2 \subseteq so(7) \subseteq so(8)$ which will be used later. Apart from notation and terminology a key point is the introduction in 2.5 of a $\mathfrak{g}_2$-module map $\pi: so(7) \to \mathfrak{g}_2$. This map is obtained from the restriction $so(7)^* \to \mathfrak{g}_2^*$ by identifying each Lie algebra with its dual via the Killing form. There is a similar map $\pi_2: so(8) \to \mathfrak{g}_2$. Two results concerning π are proved in 2.6 and 2.7, respectively. First some notation: let $\mathbf{O}_{\min} \subseteq SO(7)$ denote the minimal nilpotent $SO(7)$ orbit, and $\mathbf{O} \subseteq so(8)$ the minimal nilpotent $SO(8)$-orbit (these varieties are of dimensions 10 and 12, respectively). We show that $\pi(\bar{\mathbf{O}}_{\min}) = \bar{\mathbf{O}}_8$, and $\pi_2(\bar{\mathbf{O}}) = \bar{\mathbf{O}}_{10}$. An immediate consequence of the second fact is that $\mathbf{O}$ contains a dense G_2-orbit, where $G_2 \subseteq SO(8)$ is the connected simple subgroup with Lie algebra $\mathfrak{g}_2$.

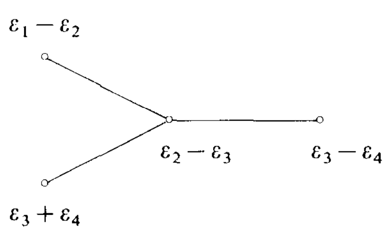
2.2. It is well known that $\mathfrak{g}_2$ embeds in $so(7)$, but for the convenience of the reader one such embedding is described below.

We shall consider inclusions $\mathfrak{g}_2 \subseteq so(7) \subseteq so(8)$. It will be notationally convenient to write $\mathfrak{g}_1 = so(7)$ and $\mathfrak{g}_0 = so(8)$. The subscripts 0, 1, 2 will be used in an obvious way to distinguish root systems R_0, R_1, R_2, systems of simple roots $\Delta_0, \Delta_1, \Delta_2$, and other objects associated to these three Lie algebras.

Let $\{e_i, e_{-i} \mid 1 \leqslant i \leqslant 4\}$ be a basis for $\mathbb{C}^8$. Let $E_{ij} \in gl(8)$ for $i, j \in \pm\{1, 2, 3, 4\}$ be the usual matrix units. Define a Cartan subalgebra $\mathfrak{h}_0$ for so(8) with basis $\{H_i = E_{i,i} - E_{-i,-i} \mid 1 \leqslant i \leqslant 4\}$.

Take a dual basis to the H_i in $\mathfrak{h}_0^*$, $\{\varepsilon_i \mid 1 \leqslant i \leqslant 4\}$. A root system for $so(8)$ is given by $R_0 = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant 4\}$ and a system of simple roots is given by $\Delta_0 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_3 + \varepsilon_4\}$.

The Dynkin diagram D_4 is labelled



The symmetric group S_3 acts as diagram automorphisms, and hence as automorphisms of $so(8)$. Write

$$S_3 = \langle \sigma, \tau \mid \sigma(\varepsilon_3 - \varepsilon_4) = \varepsilon_3 + \varepsilon_4, \tau(\varepsilon_1 - \varepsilon_2) = \varepsilon_3 + \varepsilon_4, \tau(\varepsilon_3 + \varepsilon_4) = \varepsilon_3 - \varepsilon_4, \sigma^2 = \tau^3 = 1 \rangle.$$

Define $\mathfrak{g}_1 \subseteq so(8)$ to be the subalgebra of σ-invariants. Then $\mathfrak{g}_1 \cong so(7)$. Write $\mathfrak{h}_1 = \mathfrak{h}_0 \cap \mathfrak{g}_1$. This is a Cartan subalgebra for $\mathfrak{g}_1$, and has a basis $\{H_i \mid 1 \leqslant i \leqslant 3\}$. The root system for $so(7)$ is given by $R_1 = \{\pm\eta_i, \pm\eta_i \pm \eta_j \mid 1 \leqslant i < j \leqslant 3\}$ and we take as simple roots $\Delta_1 = \{\eta_1 - \eta_2, \eta_2 - \eta_3, \eta_3\}$. The inclusion $\mathfrak{h}_1 \subseteq \mathfrak{h}_0$ gives a restriction map $j: \mathfrak{h}_0^* \to \mathfrak{h}_1^*$ such that $\ker j = \mathbb{C}\varepsilon_4$ and for $1 \leqslant i \leqslant 3, j(\varepsilon_i) = \eta_i$. Note in particular that $j(R_0) = R_1$, $j(\Delta_0) = \Delta_1$, and (when R^+ denotes the positive roots) $j(R_0^+) = R_1^+$.

Write $\beta_1 = \eta_1 - \eta_2, \beta_2 = \eta_2 - \eta_3, \beta_3 = \eta_3$ for the simple roots. The corresponding fundamental weights are $\bar{\omega}_1 = \eta_1, \bar{\omega}_2 = \eta_1 + \eta_2$, $\bar{\omega}_3 = \frac{1}{2}(\eta_2 + \eta_3)$.

2.3. Define $\mathfrak{g}_2 = so(8)^{S_3}$ to be the space of S_3 invariants. This subalgebra is simple of type G_2[3, Ex. 5.13, p. 238]. This gives the required inclusions $\mathfrak{g}_2 \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}_0$.

Define $\mathfrak{h}_2 = \mathfrak{h}_0 \cap \mathfrak{g}_2$. Since $H_{\varepsilon_i - \varepsilon_j} = H_i - H_j$, a basis for $\mathfrak{h}_2$ is given by $\{H_1 - H_2 + 2H_3, H_2 - H_3\}$. This is a Cartan subalgebra for $\mathfrak{g}_2$. Fix simple roots $\Delta_2 = \{\alpha_1, \alpha_2\}$ for $\mathfrak{g}_2$, with α_1 short and α_2 long. We give below the Chevalley basis for $\mathfrak{g}_2$ in terms of that for $so(7)$.

$$X_{\alpha_2} = X_{\eta_2 - \eta_3} \qquad X_{-\alpha_2} = X_{\eta_3 - \eta_2}$$
$$X_{3\alpha_1 + \alpha_2} = -X_{\eta_1 + \eta_3} \qquad X_{-(3\alpha_1 + \alpha_2)} = X_{-(\eta_1 + \eta_3)}$$
$$X_{3\alpha_1 + 2\alpha_2} = -X_{\eta_1 + \eta_2} \qquad X_{-(3\alpha_1 + 2\alpha_2)} = X_{-(\eta_1 + \eta_2)}$$
$$X_{\alpha_1} = X_{\eta_1 - \eta_2} + X_{\eta_3} \qquad X_{-\alpha_1} = X_{-(\eta_1 - \eta_2)} + X_{-\eta_3}$$
$$X_{\alpha_1 + \alpha_2} = -X_{\eta_1 - \eta_3} + X_{\eta_2} \qquad X_{-(\alpha_1 + \alpha_2)} = -X_{-(\eta_1 - \eta_3)} + X_{\eta_2}$$
$$X_{2\alpha_1 + \alpha_2} = -X_{\eta_2 + \eta_3} - X_{\eta_1} \qquad X_{-(2\alpha_1 + \alpha_2)} = -X_{-(\eta_2 + \eta_3)} - X_{-\eta_1}$$
$$H_{\alpha_1} = H_{\eta_1 - \eta_2} + H_{\eta_3} = H_1 - H_2 + 2H_3$$
$$H_{\alpha_2} = H_{\eta_2 - \eta_3} = H_2 - H_3.$$

The fundamental weights of $\mathfrak{g}_2$ are denoted $\bar{\omega}_1$ and $\bar{\omega}_2$, where $\bar{\omega}_i(H_{\alpha_j}) = \delta_{ij}$ for $i, j \in \{1, 2\}$. Hence $\bar{\omega}_1 = 2\alpha_1 + \alpha_2$, and $\bar{\omega}_2 = 3\alpha_1 + 2\alpha_2$, while $\alpha_1 = 2\bar{\omega}_1 - \bar{\omega}_2$, and $\alpha_2 = -3\bar{\omega}_1 + 2\bar{\omega}_2$.

Remarks. (1) Because the S_3-action on $\mathfrak{g}_0 = so(8)$ is such that the triangular decomposition $\mathfrak{g}_0=\mathfrak{n}_0^+\oplus\mathfrak{h}_0\oplus\mathfrak{n}_0^-$ (with respect to R_0^+) is a decomposition into S_3-modules we have

$$\mathfrak{n}_2^+=(\mathfrak{n}_0^+)^{S_3}\subseteq\mathfrak{n}_1^+=(\mathfrak{n}_0^+)^{\sigma}\subseteq\mathfrak{n}_0^+$$

and

$$\mathfrak{h}_2=(\mathfrak{h}_0)^{S_3}\subseteq\mathfrak{h}_1=(\mathfrak{h}_0)^{\sigma}\subseteq\mathfrak{h}_0.$$

This is extremely convenient. It means that a highest weight module for $so(8)$ will contain a highest weight module for $so(7)$, which will contain a highest weight module for $\mathfrak{g}_2$. To understand how the weight of a highhest weight vector changes when considered as an $so(8)$, $so(7)$, or $\mathfrak{g}_2$ weight vector is a matter of understanding the restriction maps $\mathfrak{h}_0^*\to\mathfrak{h}_1^*$ and $\mathfrak{h}_1^*\to\mathfrak{h}_2^*$.

(2) The restriction $j\colon\mathfrak{h}_0^*\to\mathfrak{h}_1^*$ is given in 2.2. The restriction $j\colon\mathfrak{h}_1^*\to\mathfrak{h}_2^*$ is given by $j(\eta_1-\eta_2)=j(\eta_3)=\alpha_1$, and $j(\eta_2-\eta_3)=\alpha_2$, and $\ker j=\mathbb{C}(\eta_1-\eta_2-\eta_3)$. In particular, $j(\eta_1)=2\alpha_1+\alpha_2$ and $j(\eta_2)=\alpha_1+\alpha_2$. Thus the restriction $j\colon\mathfrak{h}_0^*\to\mathfrak{h}_2^*$ is given by $j(\varepsilon_1-\varepsilon_2)=\alpha_1$, $j(\varepsilon_2-\varepsilon_3)=\alpha_2$, $j(\varepsilon_3-\varepsilon_4)=\alpha_1$, $j(\varepsilon_3+\varepsilon_4)=\alpha_1$ and $\ker j=\mathbb{C}(\varepsilon_1-\varepsilon_2-\varepsilon_3)\oplus\mathbb{C}\varepsilon_4$.

(3) An important observation to make concerning the expressions above for the root vectors of $\mathfrak{g}_2$ in terms of the root vectors of $so(7)$ is the following. If $\beta\in R_1$ then there exists a unique $\alpha\in R_2$ such that X_β appears in the expression for X_α with a non-zero coefficient. See 2.6, where this observation is applied. It is not really a coincidence and would also hold for any pair of simple Lie algebras $\mathfrak{g}'\subseteq\mathfrak{g}$, where $\mathfrak{g}'$ is the invariant in $\mathfrak{g}$ under a group of diagram automorphisms for $\mathfrak{g}$ (see [3, Ex. 5.13, p. 238]). In 2.7, this observation is applied to $so(7)\subseteq so(8)$.

2.4. Let $G_2\subseteq SO(7)$ be the connected algebraic subgroup of $SO(7)$ with Lie algebra $\mathfrak{g}_2$. By [5, Théorème 1, p. 21.07] all connected simple algebraic groups over $\mathbb{C}$ of type G_2 are isomorphic. In particular G_2 has centre $\{1\}$, is simply connected, and is of adjoint type.

Let $\mathscr{U}\subseteq G_2$ denote the set of unipotent elements, and $\mathscr{N}\subseteq\mathfrak{g}_2$ the set of nilpotent elements. Write $\phi\colon\mathscr{U}\to\mathscr{N}$ for the natural isomorphism (of G_2-varieties). Then $C_{G_2}(u)=C_{G_2}(\phi(u))$ for all $u\in\mathscr{U}$, where $C_{G_2}(\cdot)$ denotes the *centraliser* (see [4, p. 30]). In [4, p. 401] it is stated that if $u\in\mathscr{U}$ and $\dim G_2\cdot u=8$, then $C_{G_2}(u)$ is connected. Hence if $X\in\mathbf{O}_8$, then $C_{G_2}(X)$ is connected.

Notation. Let $\mathfrak{g}$ be semi-simple with simple roots Δ and roots R. For each subset $S\subseteq\Delta$ we write $\mathfrak{p}_S$ for the parabolic subalgebra of $\mathfrak{g}$ generated by its Borel subalgebra $\mathfrak{b}$ and $\{X_{-\alpha}\mid\alpha\in S\}$. We write $\mathfrak{m}_S$ for the nilpotent radical of $\mathfrak{p}_S$, $\mathfrak{q}_S$ for the reductive part of $\mathfrak{p}_S$ and $\mathfrak{l}_S$ for the semi-simple part of $\mathfrak{q}_S$.

If $\alpha \in R$, then $\mathfrak{s}_\alpha \subseteq \mathfrak{g}$ denotes the $sl(2)$-subalgebra with basis X_α, H_α, $X_{-\alpha}$.

If G is a connected algebraic group with Lie algebra $\mathfrak{g}$, the connected subgroups corresponding to $\mathfrak{p}_S$, $\mathfrak{m}_S$, etc., will be denoted by P_S, M_S, etc. In general, subalgebras of $\mathfrak{g}$ are denoted by lowercase letters, and the corresponding connected subgroup is denoted by the corresponding uppercase letter. For example, the decomposition $so(7) = \mathfrak{g}_1 = \mathfrak{n}_1^+ \oplus \mathfrak{h}_1 \oplus \mathfrak{n}_1^-$ gives connected subgroups N_1^+, H_1 N_1^- of $SO(7)$.

If $X \in \mathfrak{g}$, the *stabiliser* of X in $\mathfrak{g}$ is $\operatorname{stab}_{\mathfrak{g}}(X) = \{X \in \mathfrak{g} \mid [X, Y] = 0\}$.

Let $\rho \in \mathfrak{h}^*$ denote the half-sum of the positive roots; thus ρ_1 denotes ρ for $so(7)$, ρ_2 denotes ρ for $\mathfrak{g}_2$. For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma-module of highest weight $\lambda - \rho$; $L(\lambda)$ denotes the unique simple factor module of $M(\lambda)$ and set $J(\lambda) = \operatorname{Ann} L(\lambda)$, the annihilator of $L(\lambda)$.

The Weyl groups for $so(7)$ and $\mathfrak{g}_2$ will be denoted W_1 and W_2, respectively. If α is a root then s_α denotes the corresponding simple reflection.

Given an affine algebraic variety X, we denote by $\tilde{X}$ the normalisation, and by $\mathcal{O}(X)$ the ring of regular functions on X. Sometimes the integral closure of $\mathcal{O}(X)$ in the field of rational functions on X will be denoted $\mathcal{O}(X)$. If Y is a closed subvariety of X, then the ideal of functions in $\mathcal{O}(X)$ vanishing on Y is denoted $\mathcal{I}(Y)$. If I is an ideal in $\mathcal{O}(X)$, then $\mathcal{V}(I)$ denotes the zero variety of I.

The Gelfand–Kirillov dimension of a $\mathfrak{g}$-module M is denoted by $d(M)$.

2.5. The embedding $\mathfrak{g}_2 \to \mathfrak{g}_1 = so(7)$ induces a linear map $\pi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ by first taking the dual $p\colon \mathfrak{g}_1^* \to \mathfrak{g}_2^*$, where p is the restriction, and then $\mathfrak{g}_1$ is identified with $\mathfrak{g}_1^*$ via the Killing form on $\mathfrak{g}_1$ (denoted B_1), and $\mathfrak{g}_2$ is identified with $\mathfrak{g}_2^*$ via the Killing form on $\mathfrak{g}_2$ (denoted B_2). Thus π is defined such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{g}_1^* & \xrightarrow{\;p\;} & \mathfrak{g}_2^* \\ {\scriptstyle B_1}\updownarrow & & \updownarrow{\scriptstyle B_2} \\ \mathfrak{g}_1 & \xrightarrow{\;\pi\;} & \mathfrak{g}_2 \end{array}$$

More specifically, π is defined as follows: given $X \in \mathfrak{g}_1$ then $\pi(X) \in \mathfrak{g}_2$ is the unique element of $\mathfrak{g}_2$ such that

$$B_2(\pi(X), Y) = B_1(X, Y) \qquad \text{for all } Y \in \mathfrak{g}_2.$$

LEMMA. *There exists* $0 \neq q \in \mathbb{C}$ *such that* $B_1|_{\mathfrak{g}_2 \times \mathfrak{g}_2} = qB_2$.

Proof. Recall that the Killing form on a complex simple Lie algebra is the unique (up to a scalar multiple) non-degenerate contravariant bilinear form, so we just have to check that the restriction of B_1 to $\mathfrak{g}_2 \times \mathfrak{g}_2$ is non-degenerate. If it is not, then the radical is a non-zero ideal of $\mathfrak{g}_2$, hence equal to $\mathfrak{g}_2$, and we conclude that $B_1|_{\mathfrak{g}_2 \times \mathfrak{g}_2} = 0$. However, the restriction of B_1 to $\mathfrak{h}_1 \times \mathfrak{h}_1$ is an inner product, and so B_1 is non-zero on $\mathfrak{h}_2 \times \mathfrak{h}_2$, since $\mathfrak{h}_2 \subseteq \mathfrak{h}_1$. ∎

Remark. The precise value of q is 5/4.

LEMMA. *π is G_2-equivariant (or equivalently, a $\mathfrak{g}_2$-module map).*

Proof. Immediate, since B_1 is $SO(7)$-contravariant (hence G_2-contravariant), and B_2 is G_2-contravariant. ∎

Remark. (1) Thus we may write $\mathfrak{g}_1 = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp$, where $\mathfrak{g}_2^\perp$ is the orthogonal to $\mathfrak{g}_2$, under the form B_1, or equivalently $\mathfrak{g}_2^\perp = \ker \pi$; furthermore, $\mathfrak{g}_2^\perp$ is a $\mathfrak{g}_2$-submodule of $so(7)$ under the adjoint action of $\mathfrak{g}_2$ on $so(7)$. Returning to the definition of π, one has an alternative definition of π, namely that for $X \in \mathfrak{g}_1$, $\pi(X) \in \mathfrak{g}_2$ is the unique element such that $\pi(X) - qX \in \mathfrak{g}_2^\perp$ (where q is as in the above Lemma). In particular $\pi|_{\mathfrak{g}_2}: \mathfrak{g}_2 \to \mathfrak{g}_2$ is scalar multiplication by q.

(2) Observe that $\dim \mathfrak{g}_2^\perp = 7$, and since $[\mathfrak{g}_2, \mathfrak{g}_2^\perp] \neq 0$ (else $\mathfrak{g}_2$ becomes an ideal of $so(7)$!), the only possibility is that $\mathfrak{g}_2^\perp \cong E(\bar{\omega}_1)$, the unique 7-dimensional irreducible representation of $\mathfrak{g}_2$, of highest weight $\bar{\omega}_1$. Let $E(\bar{\omega}_1)_\lambda$ denote the λ-weight space. Recall that $\dim E(\bar{\omega}_1)_0 = 1$, and for each long root $\alpha \in R_2$, $X_\alpha \cdot E(\bar{\omega}_1)_0 = 0$. So $E(\bar{\omega}_1)_0 = \mathbb{C}H$, where $H \in \mathfrak{h}_1$ satisfies $[H, X_\alpha] = 0$ for all long roots $\alpha \in R_2$. Such an H is given by $H = H_2 + H_3 - H_1 = H_{\eta_2 + \eta_3} - \frac{1}{2} H_{\eta_1}$ (notation 2.2).

We need to know later a highest weight vector for $\mathfrak{g}_2^\perp$. For each short root $\alpha \in R_2$, $0 \neq [H, X_\alpha] \in E(\bar{\omega}_1)_\alpha$. Hence a highest weight vector is given by $X_{\eta_1} - 2X_{\eta_2 + \eta_3}$.

2.6. Write $\mathcal{N}_i \subseteq g_i$ $(i = 1, 2)$ for the cone of nilpotent elements. Because π is a $\mathfrak{g}_2$-module homomorphism, and is just multiplication by a non-zero scalar on $\mathfrak{g}_2$, it is an easy exercise to check that $\pi(\mathcal{N}_1) \subseteq \mathcal{N}_2$.

Recall that $\mathbf{O}_{\min}$ denotes the minimal non-zero nilpotent orbit in $so(7)$, and $\mathbf{O}_8$ denotes the 8-dimensional nilpotent orbit in $\mathfrak{g}_2$. Our goal is the proposition below, that $\pi(\bar{\mathbf{O}}_{\min}) = \bar{\mathbf{O}}_8$.

It is well known that

(i) $\mathbf{O}_{\min} = SO(7) \cdot X_\beta$ for any long root $\beta \in R_1$,

(ii) $\mathbf{O}_8 = G_2 \cdot X_\alpha$ for any short root $\alpha \in R_2$,

(iii) $\mathbf{O}_6 = G_2 \cdot X_\alpha$ for any long root $\alpha \in R_2$.

Recall Remark (3) of 2.3.

LEMMA. *Let $\beta \in R_1$, and let $\alpha \in R_2$ be the unique element such that X_β*

appears in the expression for X_α *with non-zero coefficient. Then there exists* $0 \neq c \in \mathbb{C}$ *such that* $\pi(X_\beta) = cX_\alpha$.

Proof. From the definition of π in 2.5, we must show for some $0 \neq c \in \mathbb{C}$ that $cB_2(X_\alpha, Y) - B_1(X_\beta, Y)$ is identically zero for all $Y \in \mathfrak{g}_2$. Thus we require for all $\gamma \in R_2$, that $cB_2(X_\alpha, X_\gamma) - B_1(X_\beta, X_\gamma)$ is zero. If $\alpha \neq -\gamma$ then both terms of this expression are zero. If $\alpha = -\gamma$ then both terms are non-zero and so there is a unique $0 \neq c \in \mathbb{C}$ such that the expression is zero. ∎

Remark. To be specific, c is given by $cB_2(X_\alpha, X_{-\alpha}) = rB_1(X_\beta, X_{-\beta})$, where r is the coefficient of $X_{-\beta}$ in the expression for $X_{-\alpha}$.

PROPOSITION. $\pi(\bar{\mathbf{O}}_{\min}) = \bar{\mathbf{O}}_8$.

Proof. Since $\dim \bar{\mathbf{O}}_{\min} = 8$, and π is a morphism of varieties, $\dim \pi(\bar{\mathbf{O}}_{\min}) \leqslant 8$, whence $\pi(\bar{\mathbf{O}}_{\min}) \subseteq \bar{\mathbf{O}}_8$ because $\pi(\mathcal{N}_2) \subseteq \mathcal{N}_1$. Notice that $X_{\eta_2 - \eta_3} = X_{\alpha_2} \in \mathbf{O}_{\min} \cap \mathfrak{g}_2$. By 2.5, Remark (1), $\pi(X_{\eta_2 - \eta_3}) = qX_{\alpha_2} \in \mathbf{O}_6$, since $q \neq 0$. By the previous lemma, $\pi(X_{\eta_1 - \eta_2}) = cX_{\alpha_1} \in \mathbf{O}_8$, since $c \neq 0$. Hence, by G_2-equivariance of π, $\mathbf{O}_6 \cup \mathbf{O}_8 \subseteq \pi(\mathbf{O}_{\min})$. But $\bar{\mathbf{O}}_8 = \mathbf{O}_8 \cup \mathbf{O}_6 \cup \{0\}$. Hence $\bar{\mathbf{O}}_8 \subseteq \pi(\bar{\mathbf{O}}_{\min})$, and there is equality. ∎

2.7. We now consider and give a positive answer to the following question of Vogan [26]. Let $\mathbf{O}$ denote the minimal nilpotent orbit in $so(8)$ (it is of dimension 10); does $\mathbf{O}$ contain a dense G_2-orbit?

We consider, as before, $\mathfrak{g}_2 \subseteq so(7) \subseteq so(8)$. The preceding analysis gives linear maps

$$\begin{array}{ccc} so(8) & \xrightarrow{\pi_1} & so(7) \\ & \searrow^{\pi_2} & \downarrow \pi \\ & & \mathfrak{g}_2 \end{array}$$

such that $\pi_2 = \pi \circ \pi_1$, where π is the map introduced in 2.5, and π_1 and π_2 are defined in an analogous way. The fact that $\pi_2 = \pi \circ \pi_1$ is verified by carefully considering the definitions in terms of the three Killing forms. Since π_2 is G_2-equivariant, it is sufficient to show that for some $X \in \mathbf{O}$, $\pi_2(X) \in \mathbf{O}_{10}$. Because, in that case, $\dim G_2 \cdot \pi_2(X) = 10$ and thus, $\overline{G_2 \cdot X} \subseteq \bar{\mathbf{O}}$ is a 10-dimensional closed subvariety of a 10-dimensional irreducible variety, hence equal.

Write $\mathbf{O}'$ for the 10-dimensional nilpotent orbit in $so(7)$. We show that $\pi_1(\bar{\mathbf{O}}) = \bar{\mathbf{O}}'$ and that $\pi(\bar{\mathbf{O}}') = \bar{\mathbf{O}}_{10}$. This will give the result.

LEMMA. $\pi_1(\bar{\mathbf{O}}) = \bar{\mathbf{O}}'$.

Proof. Recall that $\mathbf{O} = SO(8) \cdot X_\gamma$, where $\gamma \in R_0$ is any root, and that $\mathbf{O}' = SO(7) \cdot X_\beta$, where $\beta \in R_1$ is any short root. Given the embedding of

$so(7)$ in $so(8)$, we have $X_{\eta_1} = X_{\varepsilon_1+\varepsilon_4} + X_{\varepsilon_1-\varepsilon_4}$. Remark (3) of 2.3 applies also to $so(7) \subseteq so(8)$. Thus by the lemma above, we have $\pi_1(X_{\varepsilon_1+\varepsilon_4}) = cX_{\eta_1}$ for some $0 \neq c \in \mathbb{C}$. The proof is completed along the lines of the proposition of 2.6. ∎

LEMMA. $\pi(\bar{\mathbf{O}}') = \bar{\mathbf{O}}_{10}$.

Proof. Observe that $X_{\eta_1-\eta_2} + X_{\eta_1+\eta_2} \in \mathbf{O}'$. To see this simply compute the stabiliser of this element in $so(7)$, and check that it is of codimension 10 in $so(7)$. However, $\pi(X_{\eta_1-\eta_2} + X_{\eta_1+\eta_2}) = cX_{\alpha_1} + dX_{3\alpha_1+2\alpha_2}$ for some $0 \neq c, d \in \mathbb{C}$. This element belongs to $\mathbf{O}_{10}$: again compute the stabiliser. The proof is completed along the lines of the proposition of 2.6. ∎

COROLLARY. $\pi(\bar{\mathbf{O}}) = \bar{\mathbf{O}}_{10}$, *whence the minimal nilpotent orbit in* $so(8)$ *contains a dense* G_2*-orbit.*

3. THE JOSEPH IDEAL FOR $so(7)$ AND ITS INTERSECTION WITH $U(\mathfrak{g}_2)$

3.1. In [16] Joseph showed that there were either exactly two completely prime primitive ideals of $U(\mathfrak{g}_2)$ associated to the orbit $\mathbf{O}_8$, or there were no completely prime primitives associated to $\mathbf{O}_8$. The candidates were $J(\frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2))$ and $J(\frac{1}{2}(5\bar{\omega}_1 - \bar{\omega}_2))$. These ideals will be denoted J_1 and J_2, respectively. Subsequently, Joseph [18] was able to explicitly construct a homomorphism from $U(\mathfrak{g}_2)$ into a domain, having kernel precisely J_1. Hence J_1 is completely prime, and thus both J_1 and J_2 are completely prime. A key result in this section is a new proof that J_1 is completely prime. This is done by showing that $J_1 = U(\mathfrak{g}_2) \cap J_0$, where J_0 is the Joseph ideal in $U(so(7))$. See 3.2.

The Joseph ideal for $so(7)$ may be realised as the kernel of an algebra homomorphism $\varphi: U(so(7)) \to \mathscr{D}(X)$, where $\mathscr{D}(X)$ denotes the ring of differential operators on the affine variety X defined by $\mathcal{O}(X) = \mathbb{C}[x, u_1, u_2, y_1, y_2]$ with relation $x^2 + u_1 y_1 + u_2 y_2 = 0$. This construction is made in [8], and the connection with the Joseph ideal is made explicit in [21]. There are, of course, many different choices for the map φ, but we choose one (given explicitly in 3.3) such that $\mathcal{O}(X)$ becomes a simple highest weight module for $so(7)$. A suprising fact is that even when $\mathcal{O}(X)$ is considered as a $U(g_2)$-module it remains simple. This has the further surprising consequence that the embedding $U(\mathfrak{g}_2)/J_1 \subsetneq U(so(7))/J_0$ is in fact an equality (see 3.9). The variety X is isomorphic to an irreducible component of $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}_1^+$ (and also isomorphic to an irreducible component of $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+$). These components are studied in more detail in Section 5.

The equality between $U(\mathfrak{g}_2)/J_1$ and $U(so(7))/J_0$ gives an equality as $\mathfrak{g}_2$-modules between $S(\mathfrak{g}_2)/\mathrm{gr}\, J_1$ and $S(so(7))/\mathrm{gr}\, J_0$. But this last is isomorphic to $\mathcal{O}(\bar{\mathrm{O}}_{\min})$, whence as $\mathfrak{g}_2$-modules $S(\mathfrak{g}_2)/\mathrm{gr}\, J_1 \cong \mathcal{O}(\bar{\mathbf{O}}_{\min})$. In Section 4 we show that $\mathcal{O}(\bar{\mathbf{O}}_{\min})$ coincides with the regular functions on the normalisation of $\bar{\mathbf{O}}_8$. These $\mathfrak{g}_2$-module isomorphisms are not algebra isomorphisms because (as is shown in 3.12), $\mathrm{gr}\, J_1$ is not a prime ideal of $S(\mathfrak{g}_2)$. A further consequence of this is that $\bar{\mathbf{O}}_8$ cannot be a normal variety (see 3.13). Finally in 3.14 we show that J_1 and J_2 are related by the translation principle.

In a forthcoming paper with J. T. Stafford we shall show that the homomorphism $U(so(7))/J_0 \to \mathscr{D}(X)$ is an isomorphism.

3.2. Recall [13, Table 1] that the Joseph ideal J_0 in $U(so(7))$ is given by $J_0 = J(\lambda)$, where $\lambda = \frac{1}{2}\bar{\omega}_1 + \frac{1}{2}\bar{\omega}_2 + \bar{\omega}_3$ (here $\bar{\omega}_i$ is the fundamental weight for $so(7)$ corresponding to the simple root β_i). The highest weight vector e_μ of $L(\lambda)$ is of weight $\mu = \lambda - \rho_1 = -\eta_1 - \frac{1}{2}\eta_2$. After Remark (1) of 2.3, $U(g_2)\cdot e_\mu$ is a highest weight module for g_2, of highest weight $j(\mu)$, where $j\colon \mathfrak{h}_1^* \to \mathfrak{h}_2^*$ is the restriction. As in 2.3, let $\bar{\omega}_1 = 2\alpha_1 + \alpha_2$, $\bar{\omega}_2 = 3\alpha_1 + 2\alpha_2$ be the fundamental weights for $\mathfrak{g}_2$. Thus $j(\mu) + \rho_2 = j(-\eta_1 - \frac{1}{2}\eta_2) + 5\alpha_1 + 3\alpha_2 = -(2\alpha_1 + \alpha_2) - \frac{1}{2}(\alpha_1 + \alpha_2) + (5\alpha_1 + 3\alpha_2) = \frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2)$. Therefore $U(\mathfrak{g}_2)\cdot e_\mu$ is a non-zero quotient of $M(\frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2))$, annihilated by $J_0 \cap U(\mathfrak{g}_2)$. Hence, $J_0 \cap U(\mathfrak{g}_2) \subseteq J(\frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2))$.

However, $J_0 \cap U(\mathfrak{g}_2)$ is completely prime, and (using [1, Sect. 4])

$$\begin{aligned} 8 = d(U(\mathfrak{g}_2)/J(\tfrac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2)) &\leqslant d(U(\mathfrak{g}_2)/J_0 \cap (\mathfrak{g}_2)) \\ &\leqslant d(U(so(7))/J_0) = \dim V(\mathrm{gr}\, J_0) = \dim \bar{\mathbf{O}}_{\min} = 8, \end{aligned}$$

so we have equality throughout. Hence the result:

THEOREM. *$J(\frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2))$ is a completely prime ideal of $U(g_2)$. Furthermore, $J(\frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2)) = J_0 \cap U(g_2)$.*

3.3. The homomorphism $\varphi\colon U(so(7)) \to \mathscr{D}(X)$ is defined below, where we have identified $so(7)$ with its image in $\mathscr{D}(X)$ under φ. These expressions may be found in [21], although the reader is warned that we have chosen a φ different (by composing with an automorphism of $so(7)$) from that given in [21]. Recall that $\mathcal{O}(X) \cong \mathbb{C}[x, u_1, u_2, y_1, y_2]/(x^2 + u_1 y_1 + u_2 y_2)$.

Write

$$I = x\partial/\partial x + \sum_{i=1}^{2} (u_i\, \partial/\partial u_i + y_i\, \partial/\partial y_i)$$

$$\Lambda = \tfrac{1}{2}\partial^2/\partial x^2 + 2 \sum_{i=1}^{2} \partial^2/\partial u_i \partial y_i.$$

Define φ as follows:

$$X_{\eta_1-\eta_2}=\tfrac{1}{2}u_1\Delta-\partial/\partial y_1(I+\tfrac{1}{2}) \qquad X_{-(\eta_1-\eta_2)}=y_1$$
$$X_{\eta_2-\eta_3}=y_1\partial/\partial y_2-u_2\partial/\partial u_1 \qquad X_{-(\eta_2-\eta_3)}=y_2\partial/\partial y_1-u_1\partial/\partial u_2$$
$$X_{\eta_3}=y_2\partial/\partial x-2x\partial/\partial u_2 \qquad X_{-\eta_3}=2x\partial/\partial y_2-u_2\partial/\partial x$$
$$X_{\eta_1-\eta_3}=\tfrac{1}{2}u_2\Delta-\partial/\partial y_2(I+\tfrac{1}{2}) \qquad X_{-(\eta_1-\eta_3)}=y_2$$
$$X_{\eta_2}=y_1\partial/\partial x-2x\partial/\partial u_1 \qquad X_{-\eta_2}=2x\partial/\partial y_1-u_1\partial/\partial x$$
$$X_{\eta_1}=x\Delta-\partial/\partial x(I+\tfrac{1}{2}) \qquad X_{-\eta_1}=2x$$
$$X_{\eta_2+\eta_3}=y_1\partial/\partial u_2-y_2\partial/\partial u_1 \qquad X_{-(\eta_2+\eta_3)}=u_2\partial/\partial y_1-u_1\partial/\partial y_2$$
$$X_{\eta_1+\eta_3}=\tfrac{1}{2}y_2\Delta-\partial/\partial u_2(I+\tfrac{1}{2}) \qquad X_{-(\eta_1+\eta_3)}=u_2$$
$$X_{\eta_1+\eta_2}=\tfrac{1}{2}y_1\Delta-\partial/\partial u_1(I+\tfrac{1}{2}) \qquad X_{-(\eta_1+\eta_2)}=u_1$$
$$H_{\eta_1-\eta_2}=(u_1\partial/\partial u_1-y_1\partial/\partial y_1)-(I+\tfrac{3}{2})$$
$$H_{\eta_2-\eta_3}=y_1\partial/\partial y_1-y_2\partial/\partial y_2+u_2\partial/\partial u_2-u_1\partial/\partial u_1$$
$$H_{\eta_3}=2(y_2\partial/\partial y_2-u_2\partial/\partial u_2).$$

3.4. PROPOSITION. *As an $so(7)$-module $\mathcal{O}(X)\cong L(\eta_1+\tfrac{3}{2}\eta_2+\tfrac{1}{2}\eta_3)$.*

Proof. This is implicit in [21, Sect. 3]. First $\mathcal{O}(X)$ is generated by 1 as an $so(7)$-module because $\mathcal{O}(X)\subseteq\varphi(U(so(7)))$. The elements of $\mathfrak{n}_1^+$ all annihilate $1\in\mathcal{O}(X)$, so $\mathcal{O}(X)$ is a highest weight module. The action of $\mathfrak{h}_1$ on 1 shows that the highest weight of $\mathcal{O}(X)$ is $-\tfrac{3}{2}\eta_1$, and thus $\mathcal{O}(X)$ is a homomorphic image of $M(\eta_1+\tfrac{3}{2}\eta_2+\tfrac{1}{2}\eta_3)$. It remains to show that $\mathcal{O}(X)$ is a simple $so(7)$-module. This we will not do, since it is established in 3.8 that $\mathcal{O}(X)$ is simple even as a $\mathfrak{g}_2$-module (and hence as an $so(7)$-module). ▮

3.5. Using the embedding $\mathfrak{g}_2\subseteq so(7)$, explicit formulae for the basis elements of $\mathfrak{g}_2$ in $\mathcal{D}(X)$ are given below:

$$X_{\alpha_2}=y_1\partial/\partial y_2-u_2\partial/\partial u_1$$
$$X_{3\alpha_1+\alpha_2}=-\tfrac{1}{2}y_2\Delta+\partial/\partial u_2(I+\tfrac{1}{2})$$
$$X_{3\alpha_1+2\alpha_2}=-\tfrac{1}{2}y_1\Delta+\partial/\partial u_1(I+\tfrac{1}{2})$$
$$X_{\alpha_1}=\tfrac{1}{2}u_1\Delta-\partial/\partial y_1(I+\tfrac{1}{2})+y_2\partial/\partial x-2x\partial/\partial u_2$$
$$X_{\alpha_1+\alpha_2}=-\tfrac{1}{2}u_2\Delta+\partial/\partial y_2(I+\tfrac{1}{2})+y_1\partial/\partial x-2x\partial/\partial u_1$$
$$X_{2\alpha_1+\alpha_2}=-x\Delta+\partial/\partial x(I+\tfrac{1}{2})-y_1\partial/\partial u_2+y_2\partial/\partial u_1$$
$$X_{-\alpha_2}=y_2\partial/\partial y_1-u_1\partial/\partial u_2$$

$$X_{-(3\alpha_1+\alpha_2)} = -u_2$$
$$X_{-(3\alpha_1+2\alpha_2)} = -u_1$$
$$X_{-\alpha_1} = y_1 + 2x\partial/\partial y_2 - u_2\partial/\partial x$$
$$X_{-(\alpha_1+\alpha_2)} = -y_2 + 2x\partial/\partial y_1 - u_1\partial/\partial x$$
$$X_{-(2\alpha_1+\alpha_2)} = -2x - u_2\partial/\partial y_1 + u_1\partial/\partial y_2.$$

3.6. The first step in showing that $A = \mathcal{O}(X)$ is a simple highest weight module for $\mathfrak{g}_2$ is to establish that $A = U(\mathfrak{n}_2^-)\cdot 1$. This is done in 3.7.

Notation. Consider A as the factor of the polynomial ring in indeterminates x, u_1, u_2, y_1, y_2 by the ideal generated by $x^2 + u_1y_1 + u_2y_2$. Since A is a factor by a homogeneous ideal, the usual filtration by degree on the polynomial ring induces a filtration on A by degree. Write $A_n = \{a \in A \mid \deg(a) \leqslant n\}$. Thus A_n is spanned by monomials of the form $u_1^{i_1}u_2^{i_2}y_1^{j_1}y_2^{j_2}x^k$ with $i_1 + i_2 + j_1 + j_2 + k \leqslant n$. Write $|a| = n$, for the least integer n such that $a \in A_n$. Order the monomials of degree n lexicographically through $u_1 < u_2 < y_1 < y_2 < x$. Give $U(g_2)$ its usual filtration (so $\mathbb{C} + \mathfrak{g}_2$ are all elements of degree $\leqslant 1$). From the expressions in 3.5 it is clear that $U_n(\mathfrak{g}_2)$. $A_m \subseteq A_{n+m}$ for all n, m.

3.7. LEMMA. *For all n, $U_n(\mathfrak{n}_2^-)\cdot 1 = A_n$, and hence $U(\mathfrak{n}_2^-)\cdot 1 = A$.*

Proof. It is clear that $U_1(\mathfrak{n}_2^-)\cdot 1 = A_1$. Now argue by induction on n. Let $a = u_1^{i_1}u_2^{i_2}y_1^{j_1}y_2^{j_2}x^k$, with $|a| = n$. To show $a \in U_n(\mathfrak{n}_2^-)\cdot 1$ use induction with respect to the lexicographic ordering. For example, if $i_1 = i_2 = j_1 = j_2 = 0$, $k \neq 0$, then $a = -X_{-(2\alpha_1+\alpha_2)}\cdot x^{k-1} \in U_n(\mathfrak{n}_2^-)\cdot 1$. Using the expressions in 3.5 the details of the induction are straightforward. ▮

3.8. LEMMA. *As a $U(\mathfrak{g}_2)$-module, $\mathcal{O}(X) \cong L(\frac{1}{2}(-\bar{\omega}_1 + 2\bar{\omega}_2))$.*

Proof. Let $j\colon \mathfrak{h}_1^* \to \mathfrak{h}_2^*$ be the restriction (see 3.4). After the proof of 3.4, using 3.5, the weight of $1 \in \mathcal{O}(X)$ is $j(-\frac{3}{2}\eta_1) = -\frac{3}{2}(2\alpha_1 + \alpha_2)$. Hence, after 3.7, $\mathcal{O}(X)$ as a $U(\mathfrak{g}_2)$-module is a factor of $M(\frac{1}{2}(-\bar{\omega}_1 + 2\bar{\omega}_2))$.

In order to show that $\mathcal{O}(X)$ is a simple $U(\mathfrak{g}_2)$-module, it is enough to show that given $0 \neq a \in \mathcal{O}(X)$ there exists $u \in U(\mathfrak{g}_2)$ such that $u\cdot a = 1$. The existence of such an element u will be obtained by an inductive argument through studying how the basis elements of $\mathfrak{n}_2^+$ given in 3.5 act on various subalgebras of $\mathcal{O}(X)$. The reader will be able to supply the detailed proofs we do not give, but one must keep in mind the fact that $I + \frac{1}{2}$ acts by (non-zero) scalar multiplication on each monomial $u_1^{i_1}u_2^{i_2}y_1^{j_1}y_2^{j_2}x^k$, and that Δ acts trivially on $\mathbb{C}[y_1, y_2] \subseteq \mathcal{O}(X)$.

If $a \in \mathbb{C}[y_2]\backslash\{0\}$, $a \in \mathbb{C}[y_1, y_2]\backslash\{0\}$, then for some $n \in \mathbb{N}$,

$X^n_{\alpha_1} \cdot a \in \mathbb{C}[y_2]\backslash\{0\}$. Note that the action of $X_{2\alpha_1+\alpha_2}$ on $\mathbb{C}[x, y_1, y_2]$ coincides with the action of $(y_1\partial/\partial y_1 + y_2\partial/\partial y_2 + 1)\,\partial/\partial x$. Hence, if $a \in \mathbb{C}[x, y_1, y_2]\backslash\{0\}$ then for some $n \in \mathbb{N}$, $X^n_{2\alpha_1+\alpha_2} \cdot a \in \mathbb{C}[y_1, y_2]\backslash\{0\}$. Note that the action of $X_{3\alpha_1+\alpha_2}$ on $\mathbb{C}[x, y_1, y_2, u_2]$ coincides with the action of $-\frac{1}{4}y_2\partial^2/\partial x^2 + (I - y_2\partial/\partial y_2 + \frac{3}{2})\,\partial/\partial u_2$. Hence, if $a \in \mathbb{C}[x, y_1, y_2, u_2]\backslash\{0\}$ then for some $n \in \mathbb{N}$, $X^n_{3\alpha_1+\alpha_2} \cdot a \in \mathbb{C}[x, y_1, y_2]\backslash\{0\}$. Finally, if $0 \neq a \in \mathcal{O}(X)$ then for some $n \in \mathbb{N}$, $X^n_{3\alpha_1+2\alpha_2} \cdot a \in \mathbb{C}[x, y_1, y_2, u_2]\backslash\{0\}$. The result follows. ▮

Remark. Notice that $s_{\alpha_1} \cdot \frac{1}{2}(\bar{\omega}_1 + \bar{\omega}_2) = \frac{1}{2}(-\bar{\omega}_1 + 2\bar{\omega}_2)$.

3.9. THEOREM. *The embedding $U(\mathfrak{g}_2)/J_1 \hookrightarrow U(so(7))/J_0$ is an equality.*

Proof. Put $\lambda = \eta_1 + \frac{3}{2}\eta_2 + \frac{1}{2}\eta_3$; by 3.4, as an $so(7)$-module, $\mathcal{O}(X) \cong L(\lambda)$. Set $\lambda_1 = \frac{1}{2}(-\bar{\omega}_1 + 2\bar{\omega}_2)$; by 3.8, as a $\mathfrak{g}_2$-module, $\mathcal{O}(X) \cong L(\lambda_1)$. Note in particular that λ_1 is dominant regular (as is λ).

If M is a $\mathfrak{g}_1$-module (respectively, $\mathfrak{g}_2$-module), we write $L_1(M, M)$ (respectively, $L_2(M, M)$) for the space of $\mathfrak{g}_1$-finite (respectively, $\mathfrak{g}_2$-finite) linear maps from M to itself. There are natural maps $U(\mathfrak{g}_1)/J_0 \to L_1(L(\lambda), L(\lambda))$ and $U(\mathfrak{g}_2)/J_1 \to L_2(L(\lambda_1), L(\lambda_1))$, both of which are injective algebra homomorphisms. Because λ_1 is dominant regular the last is an isomorphism by [14, Theorem 5.7]. Through the inclusion $\mathfrak{g}_2 \subseteq \mathfrak{g}_1$ and the identifications $L(\lambda) = \mathcal{O}(X) = L(\lambda_1)$, any $\mathfrak{g}_1$-finite map $L(\lambda) \to L(\lambda)$ is automatically a $\mathfrak{g}_2$-finite map $L(\lambda_1) \to L(\lambda_1)$. Thus we have the following commutative diagram of algebra homomorphisms:

$$\begin{array}{ccc} U(g_2)/J_1 & \xhookrightarrow{\ j\ } & U(g_1)/J_0 \\ \downarrow \wr & \overset{\theta}{\swarrow} & \cap\!\!\downarrow \\ L_2(L(\lambda_1), L(\lambda_1)) & \hookleftarrow & L_1(L(\lambda), L(\lambda)) \end{array}$$

Here θ is the obvious composition. The diagram commutes, so θj is an isomorphism. In particular, θ is surjective. But it is also injective, being the composition of two injective maps. Thus θ is an isomorphism and we conclude that j must be an isomorphism. ▮

3.10. We now consider $U(\mathfrak{g}_2)/J_1 = U((so(7))/J_0$. This ring is naturally endowed with two (distinct) filtrations, one coming from the natural filtration on $U(\mathfrak{g}_2)$, the other coming from the natural filtration on $U(so(7))$. Recall that as $\mathfrak{g}_2$-modules $U(\mathfrak{g}_2)/J_1 \cong S(\mathfrak{g}_2)/\mathrm{gr}\,J_1$, and as $so(7)$-modules $U(so(7))/J_0 \cong S(so(7))/\mathrm{gr}\,J_0 \cong \mathcal{O}(\bar{\mathbf{O}}_{\min})$, where this last isomorphism is a consequence of the fact that $\mathrm{gr}\,J_0$ is prime, which is established in [7, Chap. IV]. The natural $\mathfrak{g}_2$-module structure on $U(\mathfrak{g}_2)/J_1$ coincides with that induced from the $so(7)$-module structure on $U(so(7))/J_0 = U(\mathfrak{g}_2)/J_1$ and the inclusion $\mathfrak{g}_2 \subseteq so(7)$. Hence we have:

PROPOSITION. *As* $\mathfrak{g}_2$*-modules* $S(\mathfrak{g}_2)/\operatorname{gr} J_1 \cong S(so(7))/\operatorname{gr} J_0 \cong \mathcal{O}(\bar{\mathbf{O}}_{\min})$.

3.11. Two frequently used results are the following:

LEMMA [2, Lemma 3.7]. *Let G be a reductive group, and V a finite dimensional representation. Let* $v \in V$, *and suppose that* $\operatorname{codim}_{\overline{G \cdot v}} G \cdot v \geqslant 2$. *Then* $\mathcal{O}(G \cdot v)$ *is the integral closure of* $\mathcal{O}(\overline{G \cdot v})$.

THEOREM [25, Theorem 3]. *Let G be a reductive algebraic group, and V a finite dimensional representation. If* $v \in V$ *is a highest weight vector, then* $\overline{G \cdot v}$ *is a normal variety.*

Remark. In particular, since $\mathbf{O}_{\min}$ is the orbit of the highest weight vector, $\bar{\mathbf{O}}_{\min}$ is normal. Furthermore, combining these two results with Proposition 3.10, and the fact (to be established in 4.5) that $\bar{\mathbf{O}}_{\min}$ is the normalisation of $\bar{\mathbf{O}}_8$, it follows that $S(\mathfrak{g}_2)/\operatorname{gr} J_1 \cong \mathcal{O}(\mathbf{O}_8)$ as $\mathfrak{g}_2$-modules.

3.12. PROPOSITION. *The ideal* $\operatorname{gr} J_1$ *of* $S(\mathfrak{g}_2)$, *is not a prime ideal.*

Proof. This is an easy consequence of computations in [27, Sect. 5]. Let E denote the 7-dimensional irreducible $\mathfrak{g}_2$-module. The multiplicity of E in a $\mathfrak{g}_2$-module M is denoted $[M: E]$. In [27, Sect. 5] Vogan shows that $[U(\mathfrak{g}_2)/J_1 : E] = 1$ and $[U(\mathfrak{g}_2)/J_2 : E] = 0$.

Observe that one therefore has

$$0 = [S(\mathfrak{g}_2)/\operatorname{gr} J_2 : E] \geqslant [S(\mathfrak{g}_2)/\sqrt{\operatorname{gr} J_2} : E] = [S(\mathfrak{g}_2)/\sqrt{\operatorname{gr} J_1} : E].$$

Hence, if $\operatorname{gr} J_1$ were prime then $\sqrt{\operatorname{gr} J_1} = \operatorname{gr} J_1$, and $0 = [S(\mathfrak{g}_2)/\operatorname{gr} J_1 : E] = [U(\mathfrak{g}_2)/J_1 : E] = 1$. This contradiction ensures that $\operatorname{gr} J_1$ is not prime. ∎

Remarks. (1) The isomorphism of $\mathfrak{g}_2$-modules in 3.10 cannot be an algebra isomorphism because $\mathcal{O}(\bar{\mathbf{O}}_{\min})$ is prime, but $\operatorname{gr} J_1$ is not a prime ideal.

(2) Consider the following filtrations on $U(\mathfrak{g}_2)/J_1$ giving a commutative associated graded algebra. The natural one induced from $\mathfrak{g}_2$ does not give a prime ring (by the above Proposition). The filtration induced by $so(7)$ and the equality $U(\mathfrak{g}_2)/J_1 = U(so(7))/J_0$ does give a prime ring since $\operatorname{gr} J_0$ is prime [8]. The embedding $U(\mathfrak{g}_2)/J_1 \subsetneq \mathscr{D}(X)$ allows one to filter $U(\mathfrak{g}_2)/J_1$ by the order of the differential operators (here the filtration subspaces are not finite dimensional) and the associated graded algebra is a subalgebra of $\operatorname{gr} \mathscr{D}(X)$, which is a prime ring. Hence with the differential operator filtration $U(\mathfrak{g}_2)/J_1$ gives a prime associated graded algebra.

3.13. THEOREM. $\bar{\mathbf{O}}_8$ *is not a normal variety.*

Proof. Recall that $\bar{\mathbf{O}}_8 = \mathbf{O}_8 \cup \mathbf{O}_6 \cup \{0\}$, hence the codimension of $\mathbf{O}_6$ in $\bar{\mathbf{O}}_8$ is 2. Thus by Lemma 3.11 $\mathcal{O}(\tilde{\bar{\mathbf{O}}}_8) = \mathcal{O}(\mathbf{O}_8)$, where $\tilde{\bar{\mathbf{O}}}_8$ denotes the normalisation of $\bar{\mathbf{O}}_8$. Suppose that $\bar{\mathbf{O}}_8$ is normal. Let E be the 7-dimensional irreducible representation of $\bar{\mathbf{O}}$. Then

$$0 = [S(\mathfrak{g}_2)/\sqrt{\operatorname{gr} J_2} : E] = [\mathcal{O}(\bar{\mathbf{O}}_8) : E] = [\mathcal{O}(\mathbf{O}_8) : E],$$

where the first equality comes as in the proof of 3.12. However, $[\mathcal{O}(\mathbf{O}_8) : E] = \dim_{\mathbb{C}}(E^*)^S$, where $S = C_{G_2}(X_{\alpha_1})$, and E^* is the dual of E (actually isomorphic to E), and $(E^*)^S$ denotes the space of S-invariants. By 2.4, S is connected, so $(E^*)^S = (E^*)^{\mathfrak{s}}$ the space of invariants under $\mathfrak{s}$, the Lie algebra of S. But $\mathfrak{s} = \{X \in \mathfrak{g}_2 \mid [X, X_{\alpha_1}] = 0\}$. This is easily calculated and so is the space $(E^*)^{\mathfrak{s}}$. One finds $\dim(E^*)^S = 1$. This contradiction ensures that $\bar{\mathbf{O}}_8$ is not normal. ∎

3.14. We now show that $J_1$and J_2 are related by the translation principle.

Let $\mathfrak{g}$ be an arbitrary semi-simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$, and Weyl group W. If $\mu \in \mathfrak{h}^*$, set $W_\mu = \{w \in W \mid w(\mu) - \mu \in Q(R)\}$, where $Q(R) = \sum_{\alpha \in R} \mathbb{Z}\alpha$, and R is the set of roots. If M is a $\mathfrak{g}$-module, and $\mu \in \mathfrak{h}^*$ write $M_\mu = \{m \in M \mid$ for each $z \in Z(\mathfrak{g})$, $\exists n \in \mathbb{N}$ such that $(z - \chi_\mu(z))^n m = 0\}$, where $Z(\mathfrak{g})$ denotes the centre of $U(\mathfrak{g})$ and $\chi_\mu : Z(\mathfrak{g}) \to \mathbb{C}$ is the central character with $\ker \chi_\mu \subseteq \operatorname{Ann} M(\mu)$.

THEOREM [11]. *Let $\lambda \in \mathfrak{h}^*$, and let E be a finite dimensional simple* $\mathfrak{g}$-*module with extreme weight v. Suppose, for all weights $v' \neq v$ of E, that $\lambda + v' \notin W_{\lambda+v}(\lambda + v)$. Then*

$$(L(\lambda) \otimes E)_{\lambda+v} \cong L(\lambda + v) \qquad \text{(or zero)}.$$

Denote by $\mathcal{V}(L(\lambda))$ the associated variety of $L(\lambda)$ determined by $\operatorname{ann}(\operatorname{gr} L(\lambda))$. Then $\mathcal{V}(L(\lambda)) \subseteq \mathfrak{n}^+$, and in the situation of the above theorem $\mathcal{V}(L(\lambda + v)) = \mathcal{V}(L((\lambda))$.

3.15. The variety X introduced in 3.1 and 3.3 is isomorphic to an irreducible component of $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+$. In Section 5, this irreducible component is denoted $\mathcal{V}_1$, and for the rest of this section we shall refer to X as $\mathcal{V}_1$.

We now look for weights $\lambda_1, \lambda_2 \in \mathfrak{h}_2^*$ such that

(a) $J_1 = J(\lambda_1)$, $J_2 = J(\lambda_2)$;

(b) $\mathcal{V}(L(\lambda_1)) = \mathcal{V}(L(\lambda_2)) = \mathcal{V}_1$;

(c) there exists a finite dimensional irreducible E, and $(L(\lambda_1) \otimes E)_\lambda \cong L(\lambda_2)$.

The λ_1 which will be successful is that given in 3.9, namely

$\lambda_1 = \frac{1}{2}(-\omega_1 + 2\omega_2)$. The arguments in 3.8 and 3.9 guarantee that $J(\lambda_1) = J_1$. Furthermore, it is shown in [19, Example 10.1] that $\mathscr{V}(L(\lambda_1)) = \mathscr{V}_1$ (see the description of $\mathscr{V}_1$ given in 5.3).

Set $\lambda_2 = \frac{1}{2}(4\omega_2 - 5\omega_1) = s_{\alpha_1}(\frac{1}{2}(5\omega_1 - \omega_2))$. One may check that λ_1 and λ_2 are both dominant regular. Define $\nu = \lambda_2 - \lambda_1 = -\alpha_1$, and note that ν is an extreme weight of E, the 7-dimensional irreducible representation of $\mathfrak{g}_2$. A simple exercise ensures that the condition $\lambda_1 + \nu' \notin W_\lambda(\lambda_2)$ is satisfied for all weights $\nu' \neq \nu$ of E. Hence one obtains $(L(\lambda_1) \otimes E)_\lambda \cong L(\lambda_2)$. Thus conditions (b) and (c) are satisfied. Finally, to see that $J(\lambda_2) = J(\frac{1}{2}(5\omega_1 - \omega_2))$, recall [12], that if α is simple and $(\alpha^\vee, \mu) \notin \mathbb{Z}$, then $J(s_\alpha \mu) = J(\mu)$.

4. Normalisation of $\bar{\mathbf{O}}_8$

4.1. The main result in this section is Theorem 4.5, which says that the normalisation of $\bar{\mathbf{O}}_8$, denoted by $\tilde{\bar{\mathbf{O}}}_8$, is isomorphic to $\bar{\mathbf{O}}_{\min}$, with $\pi: \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ being the natural projection from the normalisation. One of the main steps is to show that $\pi: \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ is bijective (and hence birational). This involves decomposing $\bar{\mathbf{O}}_{\min}$ as a union of G_2-orbits, to obtain in Corollary 4.4 that $\bar{\mathbf{O}}_{\min} = \{0\} \cup \mathbf{O}_6 \cup G_2 X_{\eta_1 - \eta_3}$. Since $\pi: \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ is bijective, and $\operatorname{codim}_{\bar{\mathbf{O}}_{\min}}(\bar{\mathbf{O}}_{\min} \backslash G_2 X_{\eta_1 - \eta_3}) = 2$, the results of 3.11 imply that $\bar{\mathbf{O}}_{\min}$ is the normalisation of $\bar{\mathbf{O}}_8$. Recall that we have already given a proof that $\bar{\mathbf{O}}_8$ is not normal in 3.13; we give, in 4.6, another proof of this result using the isomorphism, $\bar{\mathbf{O}}_{\min} \cong \tilde{\bar{\mathbf{O}}}_8$ and the result of 3.9 which says that $U(\mathfrak{g}_2)/J_1 = U(\mathrm{so}(7))/J_0$.

4.2. Proposition. *$\bar{\mathbf{O}}_{\min}$ contains a unique dense (open) G_2-orbit, $\mathscr{V}_8 := G_2 \cdot X_{\eta_1 - \eta_3}$. Furthermore π: $\mathscr{V}_8 \to \mathbf{O}_8$ is an isomorphism of varieties.*

Proof. As observed in 2.6, $\pi(X_{\eta_2 - \eta_3}) = c \cdot X_{\alpha_1 + \alpha_2}$ for some $c \in \mathbb{C}^*$. Hence, as π is G_2-equivariant, $\pi(G_2 \cdot X_{\eta_1 - \eta_3}) = G_2 \cdot X_{\alpha_1 + \alpha_2} = \mathbf{O}_8$. Thus $\overline{G_2 \cdot X_{\eta_1 - \eta_3}} \subseteq \bar{\mathbf{O}}_{\min}$ is a closed irreducible subvariety of $\bar{\mathbf{O}}_{\min}$, of dimension at least 8, so we must have equality.

For the second assertion, write $T = C_{G_2}(X_{\eta_1 - \eta_3})$ and $S = C_{G_2}(X_{\alpha_1 + \alpha_2})$. By the first part of the proof $T \subseteq S$, and thus we may consider π: $G_2/T \to G_2/S$. Both these are smooth varieties so it is sufficient to show that π is bijective to get the isomorphism (see Zariski's Main Theorem [6, Chap. 5]). Both S and T are subgroups of G_2 of dimension 6, so $\dim S/T = 0$. But by 2.4, S is connected. Thus $S/T = \{1\}$ and π must be bijective. ∎

Remark. It follows from the proposition that $\pi: \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ is birational, being bijective on a dense open subset [10, Theorem 4.6].

4.3. LEMMA. *The G_2-orbits in $\mathfrak{g}_2^\perp$ are*

(a) *the zero orbit* $\{0\}$;

(b) *a* 6-*dimensional orbit* $G_2 \cdot Y$, *where* $Y \in \mathfrak{g}_2^\perp$ *is the highest weight vector*;

(c) *a* 1-*parameter family of* 6-*dimensional orbits generated by the non-zero multiples of the zero weight vector* $H \in \mathfrak{g}_2^\perp$.

Proof. The annihilator in $\mathfrak{g}_2$ of an element of $\mathfrak{g}_2^\perp$ is a subalgebra of codimension at most 7. Using [3, Chap. VIII, Sect. 10, Corollaire 1] the only such subalgebras are conjugate to one of the following: $\mathfrak{g}_2$ itself, the commutator $\mathfrak{p}'_{\alpha_2} = [\mathfrak{p}_{\alpha_2}, \mathfrak{p}_{\alpha_2}]$, and the subalgebra $\mathfrak{s} \cong sl(3)$ generated by the long root vectors. Looking for elements of $\mathfrak{g}_2^\perp$ annihilated by these subalgebras immediately gives the orbits listed above. ∎

COROLLARY. $\mathbf{O}_{\min} \cap \mathfrak{g}_2^\perp = \varnothing$.

Proof. Let $X \in \mathbf{O}_{\min} \cap \mathfrak{g}_2^\perp$. Conjugate by a suitable element of G_2, and apply the lemma. Case (c) cannot occur since H is ad-semi-simple, and X is ad-nilpotent. Hence we may assume (after 2.5, Remark (2)) that $X = X_{\eta_1} - 2X_{\eta_2+\eta_3}$. However, this element does not belong to $\mathbf{O}_{\min}$, since the codimension of its stabiliser in $so(7)$ is greater than 8. ∎

4.4. PROPOSITION. *If* $X \in \bar{\mathbf{O}}_{\min} \backslash \mathscr{V}_8$, *then* $X \in \mathfrak{g}_2$.

Proof. Let $X \in \bar{\mathbf{O}}_{\min} \backslash \mathscr{V}_8$. Write $X = X'' + X'$ with $X'' \in \mathfrak{g}_2, X' \in \mathfrak{g}_2^\perp$. Assume $X' \neq 0$, and we obtain a contradiction. After 2.5, $\pi(X)$ is a non-zero scalar multiple of X''. By (4.2) $7 \geqslant \dim \pi(G_2 X) = \dim G_2 X''$. But X'' is nilpotent by 2.6, and by Corollary 4.3, $X'' \neq 0$. Hence $X'' \in \mathbf{O}_6$. Conjugating by an element of G_2, we may assume that X'' is the highest root vector in $\mathfrak{g}_2$. Hence $\mathrm{Stab}_{\mathfrak{g}_2} X'' = \mathfrak{p}'_{\alpha_1} = [\mathfrak{p}_{\alpha_1}, \mathfrak{p}_{\alpha_1}]$.

As $X' \neq 0$, the description of the G_2-orbits in $\mathfrak{g}_2^\perp$ ensures that $\mathrm{Stab}_{\mathfrak{g}_2} X'$ is either a G_2-conjugate of $\mathfrak{s} \cong sl(3)$, or a G_2-conjugate of $\mathfrak{p}'_{\alpha_2} = [\mathfrak{p}_{\alpha_2}, \mathfrak{p}_{\alpha_2}]$. In particular, $\dim(\mathrm{Stab}_{\mathfrak{g}_2} X') = 8$.

Since X' and X'' belong to distinct $\mathfrak{g}_2$-modules, $\mathrm{Stab}_{\mathfrak{g}_2} X = (\mathrm{Stab}_{\mathfrak{g}_2} X') \cap (\mathrm{Stab}_{\mathfrak{g}_2} X'')$. But $\dim G_2 X \leqslant 7$, whence $\dim(\mathrm{Stab}_{\mathfrak{g}_2} X) \geqslant 7$. Since any proper subalgebra of $\mathfrak{s} \cong sl(3)$ has dimension at most 6, we conclude that $\mathrm{Stab}_{\mathfrak{g}_2} X'$ is conjugate to $\mathfrak{p}'_{\alpha_2}$.

As $\mathfrak{p}'_{\alpha_1}$ and $\mathfrak{p}'_{\alpha_2}$ are not conjugate, $\mathrm{Stab}_{\mathfrak{g}_2} X$ is a codimension 1 subalgebra of $\mathfrak{p}'_{\alpha_1} = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}X_{-\alpha_1} \oplus \mathfrak{n}_2^+$. The only possibility, up to G_2-conjugacy, is $\mathbb{C}H_{\alpha_1} \oplus \mathfrak{n}_2^+$. But this does not stabilise any (non-zero) vector in $\mathfrak{g}_2^\perp$. This contradiction proves $X' = 0$. ∎

COROLLARY. $\bar{\mathbf{O}}_{\min} = \{0\} \cup \mathbf{O}_6 \cup \mathscr{V}_8$, *and* $\pi\colon \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ *is bijective.*

Proof. It is an immediate consequence of the proposition that $\bar{\mathbf{O}}_{\min} = \{0\} \cup \mathbf{O}_6 \cup \mathscr{V}_8$. Recall that $\bar{\mathbf{O}}_8 = \{0\} \cup \mathbf{O}_6 \cup \mathbf{O}_8$. By (4.2), $\pi\colon \mathscr{V}_8 \to \mathbf{O}_8$ is bijective. As $\mathbf{O}_6 \subseteq \mathfrak{g}_2$, and $\pi|_{\mathfrak{g}_2}$ is multiplication by a non-zero scalar, $\pi\colon \mathbf{O}_6 \to \mathbf{O}_6$ is bijective (in fact an isomorphism). ∎

4.5. THEOREM. *$\bar{\mathbf{O}}_{\min}$ is the normalisation of $\bar{\mathbf{O}}_8$, and $\pi\colon \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ is the natural projection from the normalisation.*

Proof. By 4.4 the codimension of $\overline{G_2 X_{\eta_1-\eta_3}} \backslash G_2 X_{\eta_1-\eta_3} = \bar{\mathbf{O}}_{\min} \backslash \mathscr{V}_8$ in $\bar{\mathbf{O}}_{\min} = \overline{G_2 X_{\eta_1-\eta_3}}$ is 2. Thus by 3.11 we have that the integral closure of $\mathscr{O}(\bar{\mathbf{O}}_{\min})$ is $\mathscr{O}(G_2 \cdot X_{\eta_1-\eta_3})$, that is, $\widetilde{\mathscr{O}(\bar{\mathbf{O}}_{\min})} = \mathscr{O}(G_2 \cdot X_{\eta_1-\eta_3}) \cong \mathscr{O}(G_2)^T$, where the $\tilde{}$ denotes the integral closure and $T = C_{G_2}(X_{\eta_1-\eta_3})$. Similarly we have $\widetilde{\mathscr{O}(\bar{\mathbf{O}}_8)} = \mathscr{O}(G_2 \cdot X_{\alpha_1+\alpha_2}) \cong \mathscr{O}(G_2)^S$, where $S = C_{G_2}(X_{\alpha_1+\alpha_2})$ as in 4.2. But as $S = T$ (see the proof of 4.2), we obtain

$$\widetilde{\mathscr{O}(\bar{\mathbf{O}}_8)} \cong \widetilde{\mathscr{O}(\bar{\mathbf{O}}_{\min})} = \mathscr{O}(\bar{\mathbf{O}}_{\min}).$$

The last equality is true because of the normality of $\bar{\mathbf{O}}_{\min}$ (see 3.11). Finally the birationality of $\pi\colon \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ gives the result. ∎

4.6. We can now offer a second proof of the non-normality of $\bar{\mathbf{O}}_8$ (see 3.13 for a first one). We know that $\mathscr{O}(\bar{\mathbf{O}}_{\min}) = S(so(7))/\operatorname{gr} J_0$ is the normalisation of $\mathscr{O}(\bar{\mathbf{O}}_8)$ by 4.5 and by 3.10, $S(so(7))/\operatorname{gr} J_0$ is isomorphic as a $\mathfrak{g}_2$-module to $S(\mathfrak{g}_2)/\operatorname{gr} J_1$. As $\mathscr{O}(\bar{\mathbf{O}}_8) = S(\mathfrak{g}_2)/\sqrt{\operatorname{gr} J_2}$, if we denote by E the irreducible $\mathfrak{g}_2$-module $E(\omega_1)$ we get as in 3.12: $[\mathscr{O}(\bar{\mathbf{O}}_{\min}) : E] = [S(\mathfrak{g}_2)/\operatorname{gr} J_1 : E] = 1$ and on the other hand,

$$[\mathscr{O}(\bar{\mathbf{O}}_8) : E] = [S(\mathfrak{g}_2)/\sqrt{\operatorname{gr} J_2} : E] \leqslant [S(\mathfrak{g}_2)/\operatorname{gr} J_2 : E] = 0.$$

Hence $\mathscr{O}(\bar{\mathbf{O}}_{\min}) \neq \mathscr{O}(\bar{\mathbf{O}}_8)$ and $\bar{\mathbf{O}}_8$ is not normal.

5. COMPONENTS OF $\bar{\mathbf{O}} \cap \mathfrak{n}^+$

5.1. In this section we show that $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}_1^+$ and $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+$ have two irreducible components. These components are explicitly described in terms of their defining equations. Writing $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+ = \mathscr{V}_1 \cup \mathscr{V}_2$ and $\bar{\mathbf{O}}_{\min} = \tilde{\mathscr{V}}_1 \cup \tilde{\mathscr{V}}_2$ for the two decompositions into irreducible components, each $\tilde{\mathscr{V}}_i$ is the normalisation of $\mathscr{V}_i$, and $\pi\colon \tilde{\mathscr{V}}_i \to \mathscr{V}_i$ is the normalisation map. In fact, $\mathscr{V}_1 \cong \tilde{\mathscr{V}}_1$ but $\mathscr{V}_2 \not\cong \tilde{\mathscr{V}}_2$. The failure of $\pi\colon \tilde{\mathscr{V}}_2 \to \mathscr{V}_2$ to be an isomorphism can eventually be used to give a third proof of the non-normality of $\bar{\mathbf{O}}_8$ (see 5.10). In 5.11 we describe the (unique) component of $\bar{\mathbf{O}}_6 \cap \mathfrak{n}_2^+$.

5.2. It is already shown in [19, Example 10.1] that $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+$ has two irreducible components. We recall some other facts presented in [19] which will be useful. For the purposes of 5.2, $\mathfrak{g}$ will denote an arbitrary semi-simple Lie algebra over $\mathbb{C}$, G the adjoint algebraic group of $\mathfrak{g}$, B the Borel subgroup, W the Weyl group of $\mathfrak{g}$, $\mathfrak{n}$ the upper triangular part of $\mathfrak{g}$, and for each $w \in W$, $w(\mathfrak{n})$ denotes the linear span of $\{X_{w\alpha} \mid X_\alpha \in \mathfrak{n}\}$. Each $G(\mathfrak{n} \cap w(\mathfrak{n}))$ contains a unique dense nilpotent orbit, denoted St(w) after Steinberg, who proved its existence [26]. Thus St: $W \to \mathcal{N}/G$, the space of nilpotent orbits, denotes the obvious map. This map is surjective but not injective in general. For $w \in W$, set $V(w) = \overline{B(\mathfrak{n} \cap w(\mathfrak{n})) \cap St(w)} \cap St(w)$. Let $\mathbf{O}$ be a nilpotent orbit.

Proposition [23]. *The irreducible components of* $\mathbf{O} \cap n$ *all have dimension equal to* $1/2 \dim \mathbf{O}$.

Proposition [19, (2.6), (9.6)]. *The irreducible components of* $\mathbf{O} \cap \mathfrak{n}$ *are precisely the* $V(w)$ *for* $w \neq \in \mathrm{St}^{-1}(\mathbf{O})$. *The number of irreducible components of* $\mathbf{O} \cap \mathfrak{n}$ *equals the number of distinct* $B(\mathfrak{n} \cap w(\mathfrak{n}))$ *such that* $w \in \mathrm{St}^{-1}(\mathbf{O})$.

Remark. The irreducible components of $\bar{\mathbf{O}} \cap \mathfrak{n}$ are the $V(w)$ for $w \in \mathrm{St}^{-1}(\mathbf{O})$.

5.3. Using 5.2, it is easy to prove the following.

Proposition. (a) *There are two irreducible components of* $\bar{\mathbf{O}}_8 \cap \mathfrak{n}_2^+$. *These are*

$$\mathscr{V}_1 := \overline{V(s_{3\alpha_1+2\alpha_2})} = \overline{P_{\alpha_2} \cdot X_{\alpha_1}} \subseteq \mathfrak{m}_{\alpha_2},$$
$$\mathscr{V}_2 := \overline{V(s_{3\alpha_1+\alpha_2} s_{\alpha_2})} = \overline{P_{\alpha_1} \cdot X_{\alpha_1+\alpha_2}} \subseteq \mathfrak{m}_{\alpha_1}.$$

(b) *There are two irreducible components of* $\bar{\mathbf{O}}_{\min} \cap \mathfrak{n}_1^+$. *These are*

$$\tilde{\mathscr{V}}_1 := \overline{P_{\eta_2-\eta_3,\eta_3} \cdot X_{\eta_1-\eta_2}} \subseteq \mathfrak{m}_{\eta_2-\eta_3,\eta_3},$$
$$\tilde{\mathscr{V}}_2 := \overline{P_{\eta_1-\eta_2,\eta_3} \cdot X_{\eta_1-\eta_3}} \subseteq \mathfrak{m}_{\eta_1-\eta_2,\eta_3}.$$

5.4. The main properties of the components $\mathscr{V}_i$ and $\tilde{\mathscr{V}}_i$ which will be proved in the rest of Section 5 are the following:

(a) $\tilde{\mathscr{V}}_1$ is the cone $t_1^2 + \cdots + t_5^2 = 0$ in $\mathbb{C}^5$, and is therefore normal. Furthermore, π: $\tilde{\mathscr{V}}_1 \to \mathscr{V}_1$ is an isomorphism of varieties.

(b) $\tilde{\mathscr{V}}_2$ is normal, and $\dim(\mathrm{Sing}\, \tilde{\mathscr{V}}_2) = 1$.

(c) $\mathscr{V}_2$ is not normal; the singular locus of $\mathscr{V}_2$ equals its non-normal locus and is of dimension 3.

(d) π: $\tilde{\mathscr{V}}_2 \to \mathscr{V}_2$ is the normalisation of $\mathscr{V}_2$.

5.5. *Proof that* $\pi(\tilde{\mathscr{V}}_i)=\mathscr{V}_i$. Note that $P_{\alpha_2}\subseteq G_2\cap P_{\eta_2-\eta_3,\eta_3}$ and $P_{\alpha_1}\subseteq G_2\cap P_{\eta_1-\eta_2,\eta_3}$ (see 2.2 and 2.3). Thus $\pi|_{\tilde{\mathscr{V}}_i}$ is P_{α_i}-equivariant ($i=1,2$). Because $\pi(\bar{\mathbf{O}}_{\min})=\bar{\mathbf{O}}_8$ by (2.6), and $\pi(\mathfrak{n}_1^+)=\mathfrak{n}_2^+$, we obtain $\pi(\tilde{\mathscr{V}}_i)\subseteq\mathbf{O}_8\cap\mathfrak{n}_2^+$. But $X_{\alpha_1}\in\pi(\mathbb{C}X_{\eta_1-\eta_2})\subseteq\pi(\tilde{\mathscr{V}}_1)$ and $X_{\alpha_1+\alpha_2}\in\pi(\mathbb{C}X_{\eta_1-\eta_3})\subseteq\pi(\tilde{\mathscr{V}}_2)$. Hence, by P_{α_i}-equivariance, irreducibility of $\pi(\tilde{\mathscr{V}}_i)$, and dimension reasons it follows that $\overline{\pi(\tilde{\mathscr{V}}_i)}=\mathscr{V}_i$ ($i=1,2$). Now observe that $\pi\colon\bar{\mathbf{O}}_{\min}\to\bar{\mathbf{O}}_8$, being a finite morphism (4.5), is closed and so $\pi(\tilde{\mathscr{V}}_i)=\overline{\pi(\tilde{\mathscr{V}}_i)}$. ∎

5.6. *Proof that* $\pi\colon\tilde{\mathscr{V}}_1\to\mathscr{V}_1$ is an isomorphism. Note that $\pi\colon\mathfrak{m}_{\eta_2-\eta_3,\eta_3}\to\mathfrak{m}_{\alpha_2}$ is an isomorphism of vector spaces. Hence π restricts to an isomorphism on any subvariety; in particular, $\pi\colon\tilde{\mathscr{V}}_1\to\pi(\tilde{\mathscr{V}}_1)=\mathscr{V}_1$ is an isomorphism. ∎

5.7. We now want to show that the $\tilde{\mathscr{V}}_i$ are normal varieties, and that $\pi\colon\tilde{\mathscr{V}}_i\to\mathscr{V}_i$ is the normalisation map. To do this we shall give an explicit description of $\mathcal{O}(\tilde{\mathscr{V}}_i)$ and $\mathcal{O}(\mathscr{V}_i)$.

LEMMA. *Let* $f=X^2_{-\eta_1}+4X_{(-\eta_1-\eta_2)}X_{-(\eta_1+\eta_2)}+4X_{-(\eta_1-\eta_3)}X_{-(\eta_1+\eta_3)}$. *Then the ideal of functions in* $S(\mathfrak{m}^-_{\eta_2-\eta_3,\eta_3})$ *vanishing on* $\tilde{\mathscr{V}}_1$ *is generated by* f.

Proof. After 5.3, $\tilde{\mathscr{V}}_1=\overline{P_{\eta_2-\eta_3,\eta_3}\cdot X_{\eta_1-\eta_2}}$. Because $\mathfrak{m}_{\eta_2-\eta_3,\eta_3}$ is abelian, and $Q_{\eta_2-\eta_3,\eta_3}=L_{\eta_2-\eta_3,\eta_3}\times\mathbb{C}^*$ we have $\tilde{\mathscr{V}}_1=\overline{L_{\eta_2-\eta_3,\eta_3}\cdot X_{\eta_1-\eta_2}}$. It is easily to see that $\mathfrak{l}_{\eta_2-\eta_3,\eta_3}$ is of type B_2 and $\mathfrak{m}_{\eta_2-\eta_3,\eta_3}$ is its natural 5-dimensional irreducible representation. Thus we may think of $L_{\eta_2-\eta_3,\eta_3}$ being $SO(5)$ embedded in $SO(7)$ and $\mathfrak{m}_{\eta_2-\eta_3,\eta_3}\equiv\mathbb{C}^5$. There are only three $SO(5)$-orbits in $\mathbb{C}^5$, namely $\{0\}$, the 4-dimensional orbit of isotropic vectors for the associated quadratic form, and the open orbit of non-isotropic vectors. Hence $\tilde{\mathscr{V}}_1$ must be the orbit of isotropic vectors, since dim $\mathscr{V}_1=4$. Note that f is invariant under $\mathfrak{l}_{\eta_2-\eta_3}$. Hence $V(f)$ is a 4-dimensional irreducible $SO(5)$-variety, and so equals the closure of the unique 4-dimensional $SO(5)$-orbit. ∎

COROLLARY. $\tilde{\mathscr{V}}_1$ *is a normal variety.*

Proof. By the lemma above, $\mathcal{O}(\tilde{\mathscr{V}}_1)$ is the ring of functions on a quadratic cone in $\mathbb{C}^5$, hence $\mathcal{O}(\tilde{\mathscr{V}}_1)$ is integrally closed (see [9, II, Ex. 6.4] for example). ∎

Remark. Using the comorphism of $\pi\colon\mathfrak{m}_{\eta_2-\eta_3,\eta_3}\xrightarrow{\sim}\mathfrak{m}_{\alpha_2}$, it is easily seen that the ideal of functions vanishing on $\mathscr{V}_1=\pi(\tilde{\mathscr{V}}_1)$ is generated in $S(\mathfrak{m}^-_{\alpha_2})$ by the quadratic polynomial

$$X^2_{-2\alpha_1-\alpha_2}-4X_{-\alpha_1}X_{-3\alpha_1-2\alpha_2}+4X_{-\alpha_1-\alpha_2}X_{-3\alpha_1-\alpha_2}.$$

This shows that the primitive factor rings $U(g_2)/J_1$ and $U(so(7))/J_0$ are realised as differential operators on the rings of functions $\mathcal{O}(\mathscr{V}_1)\cong\mathcal{O}(\tilde{\mathscr{V}}_1)$ (see 3.3 and 3.5).

5.8. LEMMA. *$\mathcal{O}(\tilde{\mathscr{V}}_2)\cong\mathbb{C}[u_0]\otimes\mathbb{C}[u_1, u_2, u_3, u_{-1}, u_{-2}, u_{-3}]/I$ where I is the ideal generated by*

$$4u_1u_3-u_2^2, \qquad 4u_{-1}u_3-u_{-2}^2, \qquad u_1u_{-2}-u_2u_{-3},$$
$$u_1u_{-1}-u_3u_{-3}, \qquad u_2u_{-1}-u_{-2}u_3.$$

Proof. By (5.3), $\tilde{\mathscr{V}}_2=\overline{P_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}}$. Let U denote the product of the $\exp(\operatorname{ad}\mathbb{C}X_\alpha)$ for $\alpha\neq\eta_2+\eta_3$, $X_\alpha\in\mathfrak{m}_{\eta_1-\eta_2,\eta_3}$. Note that $M_{\eta_1-\eta_2,\eta_3}=U\cdot\exp(\operatorname{ad}\mathbb{C}X_{\eta_2+\eta_3})$, and $P_{\eta_1-\eta_2,\eta_3}=Q_{\eta_1-\eta_2,\eta_3}\cdot M_{\eta_1-\eta_2,\eta_3}$. Thus $P_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}=Q_{\eta_1-\eta_2,\eta_3}\cdot(X_{\eta_1-\eta_3}+\mathbb{C}X_{\eta_1+\eta_3})$ and $\tilde{\mathscr{V}}_2=\mathbb{C}X_{\eta_1+\eta_2}\times\overline{Q_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}}$.

We need to describe $\overline{Q_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}}$. To do this we may think of $Q_{\eta_1-\eta_2,\eta_3}$ as $GL(2)\times SL(2)$, and then $\mathfrak{m}_{\eta_1-\eta_2,\eta_3}$ decomposes as a $GL(2)\times SL(2)$-module into $\mathbb{C}X_{\eta_1+\eta_2}\oplus F$, where F is the 6-dimensional irreducible representation of $GL(2)\times SL(2)$. Identify F with the space of 3×2 matrices with canonical basis $\{Y_{ij}\mid 1\leqslant i\leqslant 3, 1\leqslant j\leqslant 2\}$, where Y_{ij} is the matrix with 1 in the (i,j)-position and 0 elsewhere. The action of $GL(2)\times SL(2)$ is given by

$$(g_1, g_2)\,Y=\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & bc+ad & 2bd \\ c^2 & cd & d^2\end{pmatrix} Y\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix},$$

where $Y\in M_{3\times 2}(\mathbb{C})$, $g_1=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in GL(2)$, $g_2=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in SL(2)$. Under this identification, the element $X_{\eta_1-\eta_3}$ becomes $-1/2Y_{31}$ and we must compute $\overline{Q_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}}=\overline{GL(2)\times SL(2)\cdot Y_{31}}$. This equals

$$\left\{\begin{pmatrix}\alpha b^2 & \gamma b^2\\ 2\alpha bd & 2\gamma bd\\ \alpha d^2 & \gamma d^2\end{pmatrix}\ (\alpha, \gamma, b, d)\in\mathbb{C}^4\right\}.$$

It is now easy to obtain, $\dim\overline{Q_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}}=3$, and $\mathcal{O}(\overline{Q_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}})=\mathbb{C}[u_1, u_2, u_3, u_{-1}, u_{-2}, u_{-3}]/I$ with I as described in the statement of the lemma. Hence the result. ∎

COROLLARY. *$\tilde{\mathscr{V}}_2$ is a normal variety, whose singular locus is of dimension* 1.

Proof. Given the above description of $\mathcal{O}(\tilde{\mathscr{V}}_2)\cong\mathbb{C}[u_0]\otimes R$, the analysis of the singularities of $\tilde{\mathscr{V}}_2$ depends only on R. Looking at the Jacobian matrix of I, the only singular point of Spec R is $0\in F=M_{3\times 2}(\mathbb{C})$. To prove that R is normal recall that (with the notation of the previous proof), $\overline{Q_{\eta_1-\eta_2,\eta_3}\cdot X_{\eta_1-\eta_3}} \cong \overline{GL(2)\times SL(2)\cdot Y_{31}} = \overline{GL(2)\times SL(2)\cdot Y_{11}}$. As Y_{11} is the highest weight vector in the 6-dimensional irreducible representation of $GL(2)\times SL(2)$, this orbit closure is normal by Theorem 3.11. ∎

5.9. LEMMA. *$\mathcal{O}(\tilde{\mathscr{V}}_2)\cong\mathbb{C}[v_0]\otimes\mathbb{C}[v_1, v_2, v_3, v_4]/(d)$, where*

$$d = v_2^2v_3^2 + 18v_1v_2v_3v_4 - 4v_1v_3^3 + 4v_2^3v_4 - 27v_1^2v_4^2$$

is the discriminant of a cubic form.

Proof. Recall from 5.3 that $\mathscr{V}_2 = \overline{P_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}}$, and note that as a $\mathfrak{q}_{\alpha_1}$-module $\mathfrak{m}_{\alpha_1}$ decomposes as $\mathbb{C}X_{3\alpha_1+2\alpha_2}\oplus S^3(\mathbb{C}^2)$, and $\mathfrak{q}_{\alpha_1}=\mathbb{C}H\oplus\mathfrak{s}_{\alpha_1}$ for some $H\in\mathfrak{h}_2$. Write $P_{\alpha_1}=Q_{\alpha_1}M_{\alpha_1}$, $Q_{\alpha_1}\cong\mathbb{C}^*\times SL(2)$, $M_{\alpha_1}=U\cdot\exp(\operatorname{ad}\mathbb{C}X_{3\alpha_1+\alpha_2})$, where U is the product of the $\exp(\operatorname{ad}\mathbb{C}X_\alpha)$ for $X_\alpha\in\mathfrak{m}_{\alpha_1}$, $\alpha\neq 3\alpha_1+\alpha_2$. We obtain $\overline{P_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}}=\mathbb{C}X_{3\alpha_1+2\alpha_2}\times\overline{Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}}$. Consider $S^3(\mathbb{C}^2)$ as having basis $t_1^3=X_{3\alpha_1+\alpha_2}$, $3t_1^2t_2=X_{2\alpha_1+\alpha_2}$, $3t_1t_2^2=-X_{\alpha_1+\alpha_2}$, $t_2^3=X_{\alpha_2}$ and give $SL(2)$ its natural action. Hence to compute $\overline{Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}}$ we must determine the orbit of $t_1t_2^2$ under $SL(2)$. This is done in [22, pp. 827–828]. We find $\mathcal{O}(\overline{SL(2)\cdot t_1t_2^2})=\mathbb{C}[v_1, v_2, v_3, v_4]/(d)$, where the v_i are the coordinate functions with respect to the basis $\{t_1^3, t_1^2t_2, t_1t_2^2, t_2^3\}$. The result follows. ∎

COROLLARY. (i) *$\mathscr{V}_2$ is not a normal variety: the singular locus of $\mathscr{V}_2$ is equal to its non-normal locus, and is of dimension* 3.

(ii) *$\pi\colon\tilde{\mathscr{V}}_2\to\mathscr{V}_2$ is the normalisation of $\mathscr{V}_2$.*

Proof. By the above lemma it is sufficient to analyse the variety $\overline{Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}}\cong\overline{SL(2)\cdot t_1t_2^2}$. By [22, pp. 827–828] we have

$$\overline{Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}} = Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}\cup Q_{\alpha_1}\cdot X_{\alpha_2}\cup\{0\},$$

$$\operatorname{Reg}\overline{Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}} = \operatorname{Nor}\overline{Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}} = Q_{\alpha_1}\cdot X_{\alpha_1+\alpha_2}$$

(where Reg and Nor denote the regular, and normal locus). Thus the singular locus $\operatorname{Sing}\mathscr{V}_2=\mathbb{C}X_{3\alpha_1+2\alpha_2}\times\overline{Q_{\alpha_1}\cdot X_{\alpha_2}}$ is of dimension 3 and is also the non-normal locus of $\mathscr{V}_2$.

In 5.5, we showed that $\pi\colon\tilde{\mathscr{V}}_2\to\mathscr{V}_2$ is a bijective morphism and thus birational [10, Theorem 4.6]. The diagram of morphisms and varieties

$$\begin{array}{ccc} \bar{\mathbf{O}}_{\min} & \hookleftarrow & \tilde{\mathscr{V}}_2 \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \bar{\mathbf{O}}_8 & \hookleftarrow & \mathscr{V}_2 \end{array}$$

gives for the coordinate rings

$$\begin{array}{ccc} \mathcal{O}(\bar{\mathbf{O}}_{\min}) & \longrightarrow & \mathcal{O}(\tilde{\mathscr{V}}_2) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ \mathcal{O}(\bar{\mathbf{O}}_8) & \longrightarrow & \mathcal{O}(\mathscr{V}_2) \end{array}$$

Since $\mathcal{O}(\bar{\mathbf{O}}_{\min})$ is the integral closure of $\mathcal{O}(\bar{\mathbf{O}}_8)$ by 4.5, it follows that $\mathcal{O}(\tilde{\mathscr{V}}_2)$ is integral over $\mathcal{O}(\mathscr{V}_2)$ (by means of π^*). However, $\mathcal{O}(\tilde{\mathscr{V}}_2)$ is integrally closed by 5.8. Thus $\mathcal{O}(\tilde{\mathscr{V}}_2)$ is the integral closure of $\mathcal{O}(\mathscr{V}_2)$. ∎

5.10. We can now give a more geometrical proof of the non-normality of $\bar{\mathbf{O}}_8$. If $\bar{\mathbf{O}}_8$ were normal, then $\pi: \bar{\mathbf{O}}_{\min} \to \bar{\mathbf{O}}_8$ would be an isomorphism by 4.5. This would restrict to give an isomorphism of the closed subvariety $\tilde{\mathscr{V}}_2$ with its image $\pi(\tilde{\mathscr{V}}_2) = \mathscr{V}_2$. However, by (5.8) and (5.9), $\tilde{\mathscr{V}}_2$ is normal and $\mathscr{V}_2$ is not, so $\tilde{\mathscr{V}}_2$ and $\mathscr{V}_2$ are not isomorphic.

5.11. To complete the picture of the components of the nilpotent orbits in $\mathfrak{g}_2$ we consider $\bar{\mathbf{O}}_6 \cap \mathfrak{n}_2^+$. We use the notation of 5.2.

LEMMA. *$\bar{\mathbf{O}}_6 \cap \mathfrak{n}_2^+$ is irreducible and equal to* $\overline{V(s_{2\alpha_1+\alpha_2})} = \overline{P_{\alpha_1} \cdot X_{\alpha_2}}$.

Proof. The only possibility for $\mathfrak{n}_2^+ \cap w(\mathfrak{n}_2^+)$ to contain no short root vectors is $w = s_{2\alpha_1+\alpha_2}$. In that case, $\mathfrak{n}_2^+ \cap w(\mathfrak{n}_2^+) = \mathbb{C}X_{\alpha_2}$. It is easy to see that $\dim \overline{P_{\alpha_1} \cdot X_{\alpha_2}} = 3$, and the lemma follows. ∎

COROLLARY. $\mathcal{O}(\bar{\mathbf{O}}_6 \cap \mathfrak{n}_2^+) = \mathbb{C}[v_0] \otimes \mathbb{C}[v_1, v_2, v_3, v_4]/J$, *where J is the ideal generated by*

$$9v_1v_4 - v_2v_3, \qquad v_3^2 - 3v_2v_4, \qquad v_2^2 - 3v_1v_3.$$

Proof. We adopt the notation of the proof of Lemma 5.9. We have $\overline{P_{\alpha_1} \cdot X_{\alpha_2}} \subseteq \mathbb{C}X_{3\alpha_1+2\alpha_2} \times \overline{Q_{\alpha_1} \cdot X_{\alpha_2}}$, and $\overline{Q_{\alpha_1} \cdot X_{\alpha_2}} \cong \overline{GL(2)\, t_2^3} = \{bt_1^3 + 3b^2dt_1^2t_2 + 3bd^2t_1t_2^2 + d^3t_2^3 \mid (b, d) \in \mathbb{C}^2\}$. From this we deduce easily that $\mathcal{O}(\overline{Q_{\alpha_1} \cdot X_{\alpha_2}}) \cong \mathbb{C}\{v_1, v_2, v_3, v_4]/J$, where J is as described above. For dimension reasons,

$$\overline{P_{\alpha_1} \cdot X_{\alpha_2}} = \bar{\mathbf{O}}_6 \cap \mathfrak{n}_2^+ = \mathbb{C}X_{3\alpha_1+2\alpha_2} \times \overline{Q_{\alpha_1} \cdot X_{\alpha_2}}. \quad ∎$$

Remark. The variety $\bar{\mathbf{O}}_6 \cap \mathfrak{n}_2^+$ is normal, by Theorem 3.11, because $\overline{Q_{\alpha_1} \cdot X_{\alpha_2}}$ is the orbit of the lowest weight vector in the 4-dimensional irreducible representation of $GL(2) \cong Q_{\alpha_1}$. Notice that the singular locus of $\bar{\mathbf{O}}_6 \cap \mathfrak{n}_2^+$ is $\mathbb{C}X_{3\alpha_1+2\alpha_2} \times \{0\}$.

An alternative proof of the normality of $\overline{Q_{\alpha_1} \cdot X_{\alpha_2}}$ would be to remark that $\mathcal{O}(\overline{Q_{\alpha_1} \cdot X_{\alpha_2}})$ is the invariant ring of a polynomial ring in two variables under the action of the finite group $\mathbb{Z}/3\mathbb{Z}$.

NOMENCLATURE

We give a list of frequently used notation and the section in which it is introduced.

1.1 $\mathfrak{g}_2$;
1.2 $\mathbf{O}_{\min}$;
1.3 $\mathbf{O}_d, J_0, J_1, J_2$;
2.2 $\mathfrak{g}_0, \mathfrak{g}_1, R_i, \Delta_i\ (i=0, 1, 2)$;
2.3 $\mathfrak{n}_i^+, \mathfrak{h}_i, \mathfrak{n}_i^-$;
2.4 $C_G(\ ), \mathfrak{p}_S, \mathfrak{q}_S, \mathfrak{m}_S, \mathfrak{l}_S, \mathfrak{s}_\alpha, M(\lambda), L(\lambda), J(\lambda), \rho_i, W_i\ (i=0, 1, 2), s_\alpha$, $\mathcal{O}(X), \widetilde{\mathcal{O}(X)}=\mathcal{O}(\tilde{X}), P_S, Q_S, M_S, L_S, \mathcal{I}(\mathbf{O}_{\min}), V(I), d(M)$;
2.5 $\pi, \mathfrak{g}_2^\perp$;
2.7 $\mathbf{O}, \mathbf{O}'$;
3.12 $[M: E]$;
4.2 $\mathscr{V}_8$;
5.2 $\mathrm{St}(w), w(n), V(w)$;
5.3 $\mathscr{V}_1, \mathscr{V}_2, \tilde{\mathscr{V}}_1, \tilde{\mathscr{V}}_2$.

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