REGULARITY OF THE FOUR DIMENSIONAL SKLYANIN ALGEBRA.

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ABSTRACT. *The notion of a (non-commutative) regular, graded algebra is introduced in* [**AS**]. *The results of that paper, combined with those in* [**ATV1, 2**], *give a complete description of the regular graded rings of global dimension three. This paper considers certain algebras that were defined by Sklyanin* [**Sk1, 2**] *in conection with his work on the Quantum Inverse Scattering Method. We prove that these Sklyanin algebras are regular graded algebras of global dimension four, and are Noetherian domains. Moreover, we show that much of the machinery developed in* [**ATV**] *to deal with* 3*-dimensional algebras has an analogue in dimension* 4.

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INTRODUCTION.

0.1. Fix once and for all an algebraically closed field k of characteristic not equal to 2. Throughout this paper, a graded algebra A will mean a (connected) $\mathbb{N}$-graded algebra, generated in degree one; thus $A = \bigoplus_{i\geq 0} A_i$, where $A_0 = k$ is central, $\dim_k A_i < \infty$ for all i, and A is generated as an algebra by A_1. We will be interested in rings that satisfy the following notion of regularity.

Definition. A graded algebra A is *regular of dimension* d provided that

(i) A has finite global homological dimension, $\operatorname{gldim} A = d$;

(ii) A has finite Gelfand-Kirillov dimension, $\operatorname{GKdim} A < \infty$, in the sense of, say, [KL]; that is, there exists $\rho \in \mathbb{R}$ such that $\dim A_n \leq n^\rho$ for all n.

(iii) A is *Gorenstein*; that is, $\operatorname{Ext}^q_A(k, A) = \delta_{d,q} k$. Equivalently, there exists a projective resolution
$$0 \to P_d \to \cdots \to P_0 \to {}_A k \to 0$$
of the trivial left A-module k, such that the dual complex $0 \to P_0^* \to \cdots \to P_d^*$ is a (deleted) projective resolution of the right module k_A. Here, $P^* = \operatorname{Hom}_A(P, A)$.

The papers [AS] and **[ATV1, 2]** give a complete description of the regular graded algebras of dimension 3. In doing so, they provide some new and interesting examples of non-commutative Noetherian rings and develop interesting techniques for studying these rings.

0.2. In **[Sk1, 2]**, Sklyanin considers the following class of algebras. Let $\alpha, \beta, \gamma \in k$ satisfy
$$\alpha + \beta + \gamma + \alpha\beta\gamma \;=\; 0. \tag{0.2.1}$$
The *Sklyanin algebra* $S = S(\alpha, \beta, \gamma)$ is the graded k-algebra with generators x_0, x_1, x_2, x_3 of degree one, and relations $f_i = 0$, where

$$\begin{aligned} f_1 &= x_0x_1 - x_1x_0 - \alpha(x_2x_3 + x_3x_2), & f_2 &= x_0x_1 + x_1x_0 - (x_2x_3 - x_3x_2), \\ f_3 &= x_0x_2 - x_2x_0 - \beta(x_3x_1 + x_1x_3), & f_4 &= x_0x_2 + x_2x_0 - (x_3x_1 - x_1x_3), \\ f_5 &= x_0x_3 - x_3x_0 - \gamma(x_1x_2 + x_2x_1), & f_6 &= x_0x_3 + x_3x_0 - (x_1x_2 - x_2x_1). \end{aligned} \tag{0.2.2}$$

Sklyanin's interest in these algebras arises from his study of Yang-Baxter matrices and the related "Quantum Inverse Scattering Method" (called the Quantum Inverse Problem Method in [Sk]) as these algebras provide the general solution to this method corresponding to Baxter's simplest examples of Yang-Baxter matrices. Among other questions, Sklyanin raises the problem of describing the Hilbert series $H_S(t) = \sum_{i\geq 0}(\dim S_i)t^i$.

0.3. The main result of this paper is the following fairly complete description of the structure of $S(\alpha, \beta, \gamma)$.

Theorem. *Assume that* $\{\alpha,\ \beta,\ \gamma\}$ *is not equal to* $\{-1,\ +1,\ \gamma\}$, $\{\alpha,\ -1,\ +1\}$ *or* $\{+1,\ \beta,\ -1\}$. *Then:*

(i) *S is a regular graded algebra of dimension 4.*

(ii) *$H_S(t) = 1/(1-t)^4$ is the Hilbert series of a commutative polynomial ring in 4 variables.*

(iii) *S is a Noetherian domain.*

For the other values of α, β, γ, the ring S has many zero-divisors.

For generic values of $\{\alpha,\ \beta,\ \gamma\}$, part (ii) of the theorem has been proved by Cherednik [Ch, Theorem 14], by regarding S as a deformation of the graded analogue of $U(\mathfrak{so}_3)$ (see (1.6) for more details). Some of part (i) of the theorem, again for generic values of $\{\alpha,\ \beta,\ \gamma\}$, has been proved in **[OF1, OF2]**.

0.4. Assume that S satisfies the hypothesis of Theorem 0.3. In order to prove that S is regular we examine the *Koszul complex* $(K_\bullet(S),\ d)$, as defined in [Ma1] or [Ma2]. In the terminology of [AS], this is a *potential resolution* of ${}_Sk$. This Koszul complex is studied in Section 4 and we prove there that it is a complex of free S-modules of the following form:

$$0 \longrightarrow S \xrightarrow{\bullet \mathbf{t}} S^{(4)} \xrightarrow{\bullet N} S^{(6)} \xrightarrow{\bullet M} S^{(4)} \xrightarrow{\bullet \mathbf{x}} S \xrightarrow{\epsilon} {}_Sk \longrightarrow 0. \qquad (0.4.1)$$

The right hand end, $S^{(6)} \xrightarrow{\bullet M} S^{(4)} \xrightarrow{\bullet \mathbf{x}} S \xrightarrow{\epsilon} {}_Sk \longrightarrow 0$, of this sequence is the natural exact sequence obtained from the generators and relations for S. Thus the map $\bullet\mathbf{x}$ denotes right multiplication by the vector $\mathbf{x} = (x_0,\ x_1,\ x_2,\ x_3)^T$, while M is a matrix obtained from the defining relations f_i. Moreover, each map is graded of degree one. Once one has proved that (0.4.1) is exact, this gives a simple recursion formula for $\dim S_i$, and hence determines those dimensions. Unfortunately there seems to be no simple way to prove that the Koszul complex is exact, and no simple way to determine $\dim S_i$. Indeed, some explicit computations suggest that there is no easily described basis for the S_i and, in particular, that S will not have a PBW basis, in the sense of [Pr]. Thus there seems to be no direct method for proving that S is regular.

The same problem occurred in [AS]. For certain of the three dimensional algebras that they constructed, there was also no canonical basis, and they were unable to prove regularity. Thus, one of the main aims of [ATV1] was to complete the results of [AS], by showing that these rings were indeed regular. The method, however, was rather indirect. This involved using the defining relations of the given algebra A to construct a certain projective variety, and then to use the geometry of that variety to construct and describe a factor ring B of the algebra A. This factor ring was then exploited to give sufficient information about A to prove exactness of the Koszul complex, and hence establish regularity.

The approach of [ATV1] will be used in order to prove Theorem 0.3. This will also show that many of the basic results of [ATV1] have an analogue in dimension 4 (see (5.6) for more details).

0.5. Some of the results in this paper depend upon explicit computations that tend to obscure the ideas involved in the proof. Thus, we will devote the rest of this introduction to an outline of the strategy behind the proof of Theorem 0.3.

Consider the condition

$$\{\alpha, \beta, \gamma\} \cap \{0, 1, -1\} = \emptyset. \tag{0.5.1}$$

If $\{\alpha, \beta, \gamma\}$ do not satisfy this condition, then it is fairly easy to describe $S(\alpha, \beta, \gamma)$ directly (see Section 1) and so *for the rest of this introduction we assume that* $\{\alpha, \beta, \gamma\}$ *satisfy (0.5.1).* To each of the defining relations $f_i = \sum a_{ijk}x_jx_k$, one associates a multi-homogeneous form

$$\widetilde{f}_i = \sum a_{ijk}x_{j1}x_{k2}.$$

The set of zeroes of $\{\widetilde{f}_i : 1 \le i \le 6\}$ is a subvariety $\Gamma \subseteq \mathbb{P}^3 \times \mathbb{P}^3$. Let π_i, for $i = 1, 2$, denote the two projections $\mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ and write $E_i = \pi_i(\Gamma)$. In Section 2 we prove:

Lemma. *If* $\{\alpha, \beta, \gamma\}$ *satisfy (0.2.1) and (0.5.1), then:*

(i) E_1 *is the union of an (irreducible, non-singular) elliptic curve* E *and four points.*

(ii) The maps $\pi_i|_\Gamma$ *provide isomorphisms* $\Gamma \to E_i$.

(iii) The natural identification of $\pi_1(\mathbb{P}^3 \times \mathbb{P}^3)$ *with* $\pi_2(\mathbb{P}^3 \times \mathbb{P}^3)$ *also identifies* E_1 *with* E_2. *Thus* $\sigma = \pi_2 \circ \pi_1^{-1}$ *is an automorphism of* E.

Let $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^3}(1)$ be the ample line bundle associated to the embedding $i : E \hookrightarrow \mathbb{P}^3$. Thus, associated to the Sklyanin algebra $S(\alpha, \beta, \gamma)$ one has the *geometric data*

$$\mathcal{F} = (E, \sigma, \mathcal{L}).$$

0.6. The main technique of this paper is to exploit this geometric data. First, define $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ and set $B_n = H^0(E, \mathcal{L}_n)$, where $B_0 = k$. Next, define the *geometric ring* $B = B(\alpha, \beta, \gamma) = \bigoplus_{n=0}^\infty B_n$, where the multiplication map $B_n \times B_m \to B_{n+m}$ is obtained from the natural isomorphism $\mathcal{L}_n \otimes \mathcal{L}_m^{\sigma^n} \to \mathcal{L}_{n+m}$. Algebraic-geometric methods allow one to understand B and, in particular, to prove the following result (see Section 3).

Proposition. *If* $\{\alpha, \beta, \gamma\}$ *satisfy (0.2.1) and (0.5.1), then* B *satisfies:*

(i) $\dim B_0 = 1$ *and* $\dim B_n = 4n$ *for all* $n \ge 1$.

(ii) B *is a graded domain, generated by* B_1, *with defining relations in degree 2.*

(iii) The socle, $Soc(B/gB)$, *is zero for all* $g \in B$.

0.7. It is implicit in the construction of B that S_1 and B_1 are isomorphic and that this induces a ring homomorphism $\phi : S \to B$. By Proposition 0.6(ii), ϕ is surjective and hence, by part (i) of the proposition, $B \cong S/(\Omega_1 S + \Omega_2 S)$ for some $\Omega_1, \Omega_2 \in S_2$. In fact, the Ω_i are central elements of S. These observations, together with Proposition 0.6 and a careful analysis of the Koszul complex (0.4.1), are sufficient to prove parts (i) and (ii) of Theorem 0.3. Part (iii) of Theorem 0.3 then follows from the results in [**ATV1, 2**]. See Section 5 for the details.

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1. DEGENERATE CASES.

1.1. This section discusses certain degenerate values of $\{\alpha, \beta, \gamma\}$ for which the analysis of the rest of the paper does not hold. In particular, we find values of $\{\alpha, \beta, \gamma\}$ for which the Sklyanin algebra has zero-divisors and show that, for certain other values of these parameters, S is an iterated Ore extension. In the latter case, S maps onto the quantized enveloping algebra $U_q(\mathfrak{sl}_2(k))$.

For many computations it is useful to rewrite the condition $\alpha+\beta+\gamma+\alpha\beta\gamma=0$ as

$$(1+\alpha)(1+\beta)(1+\gamma) \;=\; (1-\alpha)(1-\beta)(1-\gamma). \tag{1.1.1}$$

1.2. Suppose that $\gamma = -\beta \neq 0$. Then the defining relations for S can be rewritten in a particularly simple form. Write $t = \sqrt{\beta}$ for a fixed square root of β and let $i = \sqrt{-1}$. Set

$$Y_\pm \;=\; x_2 \pm i x_3 \qquad \text{and} \qquad K_\pm \;=\; x_0 \pm t x_1.$$

Write $[a,\, b] = ab - ba$ and $[a,\, b]_+ = ab + ba$. Then

$$[x_0,\, Y_\pm] = \beta\big\{[x_1,\, x_3]_+ \mp i[x_1,\, x_2]_+\big\}$$

and

$$[x_1,\, Y_\pm] = [x_0,\, x_3]_+ \mp i[x_0,\, x_2]_+.$$

Therefore $[K_+,\, Y_\pm] = \mp it[K_+,\, Y_\pm]_+$. It is now easy to obtain the following result.

Lemma. *If $\gamma = -\beta \neq 0$, define $Y_\pm$ and $K_\pm$ as above. Then*

$$(1 \pm it)K_+Y_\pm \;=\; (1 \mp it)Y_\pm K_+$$

and

$$(1 \mp it)K_-Y_\pm \;=\; (1 \pm it)Y_\pm K_- \,.$$

1.3 Corollary. *Assume that $\beta = -1$ and $\gamma = +1$, and set $t = \sqrt{\beta} = i$. Then*

$$Y_+K_+ \;=\; K_+Y_- \;=\; K_-Y_+ \;=\; Y_-K_- \;=\; 0.$$

Similar results hold if $\alpha = -1$ and $\beta = +1$, or if $\gamma = -1$ and $\alpha = +1$.

The final part of the corollary follows from the first part by cyclically reindexing the $\{x_j : 1 \leq j \leq 3\}$. Similar comments will apply whenever a result is proved for particular values of $\{\alpha,\, \beta,\, \gamma\}$, and so we will usually ignore such comments in future.

One can presumably show that, as is the case for the ring $k\{X,\, Y\}/\,\langle XY\rangle$, the algebra $S(\alpha,\, -1,\, 1)$ is a non-Noetherian, non-regular ring.

1.4. The second degenerate case occurs when $\alpha = 0$. By (1.1.1) this forces $\beta = -\gamma$ so we may use the notation of (1.2). Since the case $\beta = -1$ is covered by (1.3), we will also assume that $\beta \neq -1$. For the moment, we will also assume that $\beta \neq 0$. Thus $[x_0, x_1] = 0$, and $[x_2, x_3] = 2x_0x_1$. Therefore,

$$[K_+, K_-] = 0 \quad \text{and} \quad [Y_+, Y_-] = it^{-1}(K_-^2 - K_+^2). \tag{1.4.1}$$

It is easy to check that these two relations, combined with those of Lemma 1.2, are equivalent to the defining relations of S. Thus one obtains:

Lemma. *Assume that $\alpha = 0$ and $\beta \neq 0, -1$. Then, in the notation of Cohn* [Co, §12.2], *$S = S(\alpha, \beta, \gamma)$ is an iterated Ore extension:*

$$S \cong k[K_+; 1, 0]\,[K_-; 1, 0]\,[Y_+; \sigma_1, 0]\,[Y_-; \sigma_2, \delta_2],$$

for appropriate automorphisms σ_i and σ_2-derivation δ_2.

Similar results hold if $\beta = 0$ and $\gamma \neq 0, -1$ or $\gamma = 0$ and $\alpha \neq 0, -1$.

Proof. This follows easily from the earlier comments, combined with [Co, Theorem 1, p.438].

1.5. As is apparently well-known to physicists, if $\beta \neq 0, -1$ then $S(0, \beta, -\beta)$ is a graded analogue of the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. More precisely, observe that K_+K_- is central in S and that

$$S(0, \beta, -\beta)/(K_+K_- - 1) \cong U_q(\mathfrak{sl}_2(k)),$$

in the notation of [Ji]. We would like to thank S. Majid for this observation.

This observation also removes one awkward (or at least tedious) point in the proof of Lemma 1.4. For, in order to apply [Co], one needs to prove that the elements $K_+^iK_-^jY_+^kY_-^\ell$ form a basis for S. The earlier observations certainly show that they span S, but the fact that they are linearly independent (or, as is equivalent, the fact that the σ_i are automorphisms and that δ_2 is a σ_2-derivation) is tedious to prove directly. However, since S is graded, any linear dependence between these monomials will be a sum of homogeneous relations. Moreover, any such relation will induce a non-trivial relation between elements of the natural basis $\{K_+^iK_-^jY_+^kY_-^\ell : ij = 0\}$ of $U_q(\mathfrak{sl}_2(k))$, giving the required contradiction.

1.6. The final degenerate case arises when $\alpha = \beta = \gamma = 0$ (this occurs whenever two of α, β, γ are zero). The following is easy to prove.

Lemma. *If $\alpha = \beta = \gamma = 0$, then x_0 is central, while $[x_i, x_{i+1}] = 2x_0x_{i+2}$, for $i = 1, 2, 3$ mod 3. Furthermore, $S = S(0, 0, 0)$ is an iterated Ore extension*

$$S \cong k[x_0; 1, 0]\,[x_1; 1, 0]\,[(x_2 + ix_3); \sigma_1, 0]\,[(x_2 - ix_3); \sigma_2, \delta_2],$$

for appropriate automorphisms σ_i and σ_2-derivation δ_2.

Remark. Observe that, for any $\lambda \in k \setminus \{0\}$, one has $S(0, 0, 0)/(x_0 + \lambda) \cong U(\mathfrak{so}_3)$, the enveloping algebra of the Lie algebra $\mathfrak{so}_3$. Indeed, it is reasonable to regard the ring $S(0, 0, 0)$ as a graded version of $U(\mathfrak{so}_3)$. As such, its structure is fairly easy to understand. One can also regard $S(\alpha, \beta, \gamma)$ as a deformation of $S(0, 0, 0)$, and for applications of this to the study of $S(\alpha, \beta, \gamma)$, for generic values of $\{\alpha, \beta, \gamma\}$, see [Ch].

Alternatively, one can regard $S(\alpha, \beta, \gamma)$ as a deformation of the commutative polynomial ring $C = k[z_0, z_1, z_2, z_3]$. To see this, use the elements

$$\{ x_0, (\beta\gamma)^{1/3} x_1, (\alpha\gamma)^{1/3} x_2, (\alpha\beta)^{1/3} x_3 \}$$

to generate S and adjust the relations accordingly. Then it is an easy exercise to show that $S(0, 0, 0) \cong C$.

1.7 Corollary. *If α, β, γ satisfy the hypotheses of Lemmas 1.4 or 1.6, then $S(\alpha, \beta, \gamma)$ is a Noetherian domain of global dimension* 4.

Proof. Since S is an iterated Ore extension, this is standard; use, for example, [MR, Corollary 9.18, p.273] or [Pr] to prove that $\operatorname{gldim} S = 4$.

1.8. Suppose that α, β, γ are as in (1.4) or (1.6); thus $\alpha = 0$ and $\beta \neq -1$ (or the cyclic permutations thereof) and $S = S(\alpha, \beta, \gamma)$ can be written as an iterated Ore extension. The analysis of the later sections will not include these cases, although it can probably be extended to incorporate them. The problem is that, just as happens when iterated Ore extensions are considered in [ATV1], the projective variety constructed from S will not be an elliptic curve. For example, if $\alpha = \beta = \gamma = 0$, then one obtains a variety $E \cong \mathbb{P}^2$ while, in the situation of Lemma 1.4, E will be singular and reducible. It is just such special cases in dimension three that lead to many of the complications in [ATV1]. However, it is easy to show directly that these special cases are indeed regular (see Corollary 4.13).

1.9. In summary, the cases not covered by the computations of this section are as follows:

$$\{\alpha, \beta, \gamma\} \cap \{0, \pm 1\} = \emptyset \tag{1.9.1}$$

$$\beta = 1, \ \gamma = -1, \ \alpha \neq \pm 1, 0 \tag{1.9.2}$$

$$\gamma = 1, \ \alpha = -1, \ \beta \neq \pm 1, 0 \tag{1.9.3}$$

$$\alpha = 1, \ \beta = -1, \ \gamma \neq \pm 1, 0 \tag{1.9.4}$$

By cyclically reindexing $\{x_1, x_2, x_3\}$, the last three cases are equivalent. Thus the remainder of the paper considers just the cases (1.9.1) and (1.9.4).

2. THE GEOMETRIC DATA.

2.1. Throughout this section, assume that $\alpha, \beta, \gamma \in k$ satisfy (0.2.1) and either (1.9.1) or (1.9.4). Let $S = S(\alpha, \beta, \gamma)$ be the Sklyanin algebra defined by the relations (0.2.2). This section constructs the geometric data $\{E, \sigma, \mathcal{L}\}$, which consists of an elliptic curve $E \subset \mathbb{P}^3$ (throughout this paper an elliptic curve is taken to be smooth and irreducible), an automorphism σ of E, and an invertible $\mathcal{O}_E$-module $\mathcal{L}$.

There are two ways to obtain E. First, following [ATV1], one multilinearises the defining relations $\{f_i\}$ of S and uses the resulting polynomials to define a variety $\Gamma \subset \mathbb{P}^3 \times \mathbb{P}^3$. Then E is the projection of Γ, minus four points, onto the first copy of $\mathbb{P}^3$, while Γ (less those four points) is the graph of the automorphism σ of E. This method, which is the one we will use, requires a considerable amount of computation and makes the fact that E is actually an elliptic curve seem surprising. The second method comes from Sklyanin's original construction of S, where the constants $\alpha, \beta, \gamma \in \mathbb{C}$ are given in terms of certain theta functions. So (at least when $k = \mathbb{C}$) the elliptic curve E is implicit in the construction of S. This is explained at the end of the section.

2.2. The method for constructing geometric data from a graded ring is given in considerable generality in [ATV1]. We begin by recalling the details required for the Sklyanin algebra.

Write $T = k\{x_0, x_1, x_2, x_3\}$ for the free k-algebra on four generators $\{x_i\}$ of degree one. This induces a graded structure $T = \bigoplus_{n \geq 0} T_n$ on T. Given a homogeneous element $f = \sum f_\alpha x_{\alpha_1} \cdots x_{\alpha_n} \in T_n$, then associate to f the multihomogeneous form $\widetilde{f} = \sum f_\alpha x_{\alpha_1,1} \cdots x_{\alpha_n,n}$. Since $\widetilde{f}$ is multi-homogeneous, $\widetilde{f}$ defines a hypersurface in the product space $\mathbb{P} \times \cdots \times \mathbb{P}$ of n copies of $\mathbb{P} = \mathbb{P}^3$.

Let I be the ideal in T generated by the defining relations f_i of S, given in (0.2.2). Therefore I is a graded ideal; $I = \bigoplus I_n$, where $I_n = I \cap T_n$. Set $\widetilde{I}_n = \{\widetilde{g} : g \in I_n\}$ and write Γ for the locus of $\widetilde{I}_2$ in $\mathbb{P} \times \mathbb{P}$. Thus Γ is defined by the six equations $\widetilde{f}_i = 0$ where, for example,

$$\widetilde{f}_1 = x_{01}x_{12} - x_{11}x_{02} - \alpha(x_{21}x_{32} + x_{31}x_{22}).$$

It will be useful to write these as a single matrix equation $M_1\mathbf{v} = 0$, where

$$M_1 = M_1(x_{01}\, x_{11},\, x_{21},\, x_{31}) = \begin{pmatrix} -x_{11} & x_{01} & -\alpha x_{31} & -\alpha x_{21} \\ -x_{21} & -\beta x_{31} & x_{01} & -\beta x_{11} \\ -x_{31} & -\gamma x_{21} & -\gamma x_{11} & x_{01} \\ -x_{31} & -x_{21} & x_{11} & -x_{01} \\ -x_{11} & -x_{01} & -x_{31} & x_{21} \\ -x_{21} & x_{31} & -x_{01} & -x_{11} \end{pmatrix} \qquad (2.2.1)$$

and $\mathbf{v} = (x_{02},\, x_{12},\, x_{22},\, x_{32})^T$. Equivalently, one may write the equations for Γ as $\mathbf{u}M_2 = 0$, where $\mathbf{u} = (x_{01},\, x_{11},\, x_{21},\, x_{31})$ and

$$M_2 \;=\; M_2(x_{02},\, x_{12},\, x_{22},\, x_{32}) \;=\; \begin{pmatrix} -x_{12} & -x_{22} & -x_{32} & -x_{32} & -x_{12} & -x_{22} \\ x_{02} & \beta x_{32} & \gamma x_{22} & x_{22} & -x_{02} & -x_{32} \\ \alpha x_{32} & x_{02} & \gamma x_{12} & -x_{12} & x_{32} & -x_{02} \\ \alpha x_{22} & \beta x_{12} & x_{02} & -x_{02} & -x_{22} & x_{12} \end{pmatrix}. \quad (2.2.2)$$

Thus $\Gamma = \big\{(\mathbf{u},\mathbf{v}) \in \mathbb{P}^3 \times \mathbb{P}^3 \quad | \quad M_1(\mathbf{u})\mathbf{v} = 0\big\} = \big\{(\mathbf{u},\mathbf{v}) \in \mathbb{P}^3 \times \mathbb{P}^3 \quad | \quad \mathbf{u}M_2(\mathbf{v}) = 0\big\}.$

Remark. The precise form of the matrices M_1 and M_2 will be needed later. Whenever it will cause no confusion, we will ignore the second subscript in the x_{ij}.

2.3. For $i = 1, 2$, let π_i denote the projection from $\mathbb{P} \times \mathbb{P}$ to its i^{th} component. We will identify $\mathrm{Im}(\pi_1) = \mathrm{Im}(\pi_2) = \mathbb{P}$ via $x_{i1} \equiv x_{i2}$ and set $E_i = \pi_1(\Gamma) \subset \mathbb{P}$. The rest of this section is devoted to proving that:

(a) $E_1 = E_2$.

(b) E_1 is the union of an elliptic curve E and four other points.

(c) For $i = 1, 2$, the map $\pi_i|_\Gamma$ is an isomorphism from Γ to E_i. Thus $\sigma = \pi_2 \circ \pi_1^{-1}$ is an automorphism of E_1 and hence of E.

Given that these assertions hold, let $i : E \hookrightarrow \mathbb{P}$ be the inclusion, and write $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}}(1)$ for the corresponding very ample, invertible sheaf. The geometric data of interest is

$$\mathcal{F} \;=\; \big\{E,\, \sigma,\, \mathcal{L}\big\}.$$

2.4. If $M_1 = M_1(x_i)$ is the matrix defined in (2.2.1), then E_1 is defined by

$$E_1 \;=\; \big\{\mathbf{u} = (x_i) \in \mathbb{P} : \mathrm{rk}\, M_1(x_i) \le 3\big\}.$$

Equivalently, E_1 is defined by the vanishing of the 4×4 minors of M_1. The minor obtained by deleting rows i and j is denoted by h_{ij}. Thus E_1 is defined by the following 15 quartic equations:

$$
\begin{aligned}
h_{12} &= 2(x_0x_3 - \gamma x_1x_2)(x_0^2 + x_1^2 + x_2^2 + x_3^2)\\
h_{13} &= 2(x_0x_2 - \beta x_1x_3)(x_0^2 + x_1^2 + x_2^2 + x_3^2)\\
h_{14} &= -2(x_0x_2 + x_1x_3)(x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2)\\
h_{15} &= 2x_0^2\big((\beta-1)x_3^2 - (1+\gamma)x_2^2\big) + 2x_1^2\big(\beta(\gamma+1)x_3^2 + \gamma(\beta-1)x_2^2\big)\\
h_{16} &= 2(x_0x_3 - x_1x_2)(x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2)\\
h_{23} &= 2(x_0x_1 - \alpha x_2x_3)(x_0^2 + x_1^2 + x_2^2 + x_3^2)\\
h_{24} &= -2(x_0x_1 - x_2x_3)(x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2)\\
h_{25} &= -2(x_0x_3 + x_1x_2)(x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2)\\
h_{26} &= 2x_1^2\big((\gamma-1)x_0^2 + \gamma(1+\alpha)x_2^2\big) - 2x_3^2\big((\alpha+1)x_0^2 - \alpha(\gamma-1)x_2^2\big)\\
h_{34} &= 2x_0^2\big((\beta+1)x_1^2 + (1-\alpha)x_2^2\big) + 2x_3^2\big(\beta(1-\alpha)x_1^2 - \alpha(\beta+1)x_2^2\big)\\
h_{35} &= -2(x_0x_2 - x_1x_3)(x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2)\\
h_{36} &= -2(x_0x_1 + x_2x_3)(x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2)\\
h_{45} &= 2(x_0x_2 + \beta x_1x_3)(x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2)\\
h_{46} &= 2(x_0x_1 + \alpha x_2x_3)(x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2)\\
h_{56} &= 2(x_0x_3 + \gamma x_1x_2)(x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha\beta x_3^2).
\end{aligned}
$$

Proposition. *Let $\{\alpha, \beta, \gamma\}$ satisfy (1.9.1) or (1.9.4) and let $E_1 = \pi_1(\Gamma)$ be defined as in (2.3). Write*

$$g_1 \;=\; x_0^2 \,+\, x_1^2 \,+\, x_2^2 \,+\, x_3^2 \qquad \text{and} \qquad g_2 \;=\; x_3^2 \,+\, \tfrac{1-\gamma}{1+\alpha}x_1^2 \,+\, \tfrac{1+\gamma}{1-\beta}x_2^2$$

and set

$$E \;=\; \mathcal{V}(g_1, g_2) \;=\; \big\{(x_i) \in \mathbb{P} : g_1(x_i) = 0 = g_2(x_i)\big\} \;\subset\; \mathbb{P} \;=\; \mathbb{P}^3.$$

Then $E_1 = E \cup \big\{(1,0,0,0),\, (0,1,0,0),\, (0,0,1,0),\, (0,0,0,1)\big\}$.

Proof. This is a staightforward computation, making frequent use of (0.2.1). The proof is therefore omitted.

2.5. Proposition. *The variety E is a smooth elliptic curve.*

Proof. As a scheme, E is defined by

$$g_1 \;=\; x_0^2 + x_1^2 + x_2^2 + x_3^2 \quad \text{and} \quad g_2 \;=\; \tfrac{1-\gamma}{1+\alpha}x_1^2 + \tfrac{1+\gamma}{1-\beta}x_2^2 + x_3^2.$$

We first show that the Jacobian $J(g_1,g_2)$ has rank 2 at all points $p \in E$. Given $p = (x_0,x_1,x_2,x_3) \in \mathcal{V}(g_1,g_2)$, then

$$J(g_1,g_2)(p) \;=\; \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & \frac{1-\gamma}{1+\alpha}x_1 & \frac{1+\gamma}{1-\beta}x_2 & x_3 \end{pmatrix}.$$

Thus, rank $J(g_1,g_2)(p) < 2$ if and only if

$$\begin{aligned}\left(\tfrac{1-\gamma}{1+\alpha}\right)x_0x_1 \;=\; \left(\tfrac{1+\gamma}{1-\beta}\right)x_0x_2 \;=\; x_0x_3 \;=\; \left(\tfrac{1+\gamma}{1-\beta} - \tfrac{1-\gamma}{1+\alpha}\right)x_1x_2 \\ =\; \left(1 - \tfrac{1-\gamma}{1+\alpha}\right)x_1x_3 \;=\; \left(1 - \tfrac{1+\gamma}{1-\beta}\right)x_2x_3 \;=\; 0.\end{aligned}$$

The restrictions on the scalars $\{\alpha,\beta,\gamma\}$ imposed by (1.9.1) or (1.9.4) ensure that each coefficient in the above equation is defined and non-zero. Hence rank $J(g_1,g_2)(p) < 2$ if and only if $x_ix_j = 0$ for all $i \neq j$. Clearly this cannot happen for any point $p \in \mathcal{V}(g_1,g_2)$. Thus each of the local rings $\left(k[x_0,x_1,x_2,x_3]/(g_1,g_2)\right)_{\mathbf{m}}$ is a regular ring, and hence a domain. This in turn implies that the ring $k[x_0,x_1,x_2,x_3]/(g_1,g_2)$ is reduced.

It remains to prove that E is irreducible. Since $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathcal{V}(g_1) \subseteq \mathbb{P}^3$, E is a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$. Recall that Pic $(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, with the hyperplane in $\mathbb{P}^3$ intersecting $\mathbb{P}^1 \times \mathbb{P}^1$ in the divisor (1, 1) (see for example [Ha, Chapter II, Example 6.6.1]). Thus $E = (2,\,2)$. If E is reducible, say $E = F_1 \cup F_2 \cup \dots$, then write (2, 2) as a corresponding sum of effective divisors. Since the intersection pairing in Pic $(\mathbb{P}^1 \times \mathbb{P}^1)$ is given by the formula $(a,\,b).(a',\,b') = ab' + a'b$, it is easy to check that, however one writes E as a sum of divisors, this forces $F_i \cap F_j \neq \emptyset$, for some i and j. But, if $p \in F_i \cap F_j$, then rank $J(g_1,g_2)(p) < 2$, a contradiction. Thus E is irreducible and, by the Jacobian criterion, E is smooth.

Finally, the Adjunction Formula ([Ha, Ex.1.5.2, p.362]) shows that E has genus one.

2.6 Lemma. *For $i = 1, 2$, the map π_i induces an isomorphism $\pi_i : \Gamma \to E_i$.*

Proof. We will only prove this for π_1, the other case being similar. Let $M_1 = M_1(x_i)$ be the matrix defined in (2.2.1). If π_1 is not injective then, for some $\mathbf{u} = (x_i)$, there exist two linearly independent solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ to the equation $M_1(\mathbf{u})\mathbf{v} = 0$. Thus, rank $M_1 \leq 2$.

Now, the appropriate row operations reduce M_1 to the matrix

$$N \;=\; \begin{pmatrix} -x_1 & \\ -x_2 & * \\ -x_3 & \\ 0 & N' \end{pmatrix} \quad \text{where } N' \;=\; \begin{pmatrix} (1-\gamma)x_2 & -(1+\gamma)x_1 & 2x_0 \\ 2x_0 & (1-\alpha)x_3 & -(1+\alpha)x_2 \\ -(1+\beta)x_3 & 2x_0 & (1-\beta)x_1 \end{pmatrix}.$$

There are two cases to consider. First, suppose that 3 of the x_i are zero. Then it is easy to see that $M\mathbf{v} = 0$ has a unique solution. Secondly, assume that at most two of the x_i are zero. Then rank $M \leq 2$ implies that rank $N' \leq 1$. It is now routine to check that this contradicts both (1.9.1) and (1.9.4).

2.7 Lemma. $E_1 = E_2$.

Proof. By Proposition 2.4, E_1 is invariant under the automorphism

$$(x_0, x_1, x_2, x_3) \longrightarrow (-x_0, x_1, x_2, x_3).$$

Thus $E_1 = \{\mathbf{u} = (x_i) \in \mathbb{P} : \operatorname{rk} N_1(\mathbf{u}) \leq 3\}$, where $N_1(\mathbf{u})$ is the matrix obtained from $M_1(\mathbf{u})$ by replacing every occurence of x_0 by $-x_0$. On the other hand

$$E_2 = \{\mathbf{v} = (x_i) \in \mathbb{P} : \operatorname{rk} M_2(\mathbf{v}) \leq 3\},$$

where M_2 is defined by (2.2.2). Let $N_2(\mathbf{v})$ be the matrix obtained from $M_2(\mathbf{v})$ by multiplying rows 2, 3 and 4 by (-1). Then $(N_2)^T = N_1$ whence $E_2 = E_1$.

2.8 Corollary. *The map $\sigma = \pi_2 \circ \pi_1^{-1}$ is an automorphism of E_1. Each of the four isolated points is fixed by the action of σ and σ restricts to an automorphism of E. As an automorphism of E, σ is defined (on a dense open set) by*

$$\sigma : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -2\alpha\beta\gamma x_1x_2x_3 - x_0\left(-x_0^2 + \beta\gamma x_1^2 + \alpha\gamma x_2^2 + \alpha\beta x_3^2\right) \\ 2\alpha x_0x_2x_3 + x_1\left(x_0^2 - \beta\gamma x_1^2 + \alpha\gamma x_2^2 + \alpha\beta x_3^2\right) \\ 2\beta x_0x_1x_3 + x_2\left(x_0^2 + \beta\gamma x_1^2 - \alpha\gamma x_2^2 + \alpha\beta x_3^2\right) \\ 2\gamma x_0x_1x_2 + x_3\left(x_0^2 + \beta\gamma x_1^2 + \alpha\gamma x_2^2 - \alpha\beta x_3^2\right) \end{pmatrix}.$$

Proof. By Lemma 2.6, $\pi_1^{-1} : E_1 \to \Gamma$ and $\pi_2 : \Gamma \to E_2$ are both isomorphisms. Thus σ is an automorphism. That $\sigma|_E$ is an automorphism then follows from Proposition 2.4. The precise form of σ is left as an exercise to the reader (use the fact that σ is defined by $\sigma(x_0, x_1, x_2, x_3) = \mathbf{v}$, where $\mathbf{v}$ is defined by $M_1(x_i).\mathbf{v} = 0$).

Remarks. (i) The corollary can be rephrased as saying that Γ is the graph of the automorphism σ of E_1. The explicit formula for σ is also valid at the four isolated points.

(ii) Following [ATV1], a graded right S-module $M = \bigoplus_{n\geq 0} M_n$ is called a *point module* if (a) M is generated as an S-module by M_0, and (b) $\dim_k M_n = 1$ for all $n \geq 0$. It follows from Corollary 2.8, combined with [ATV1, Corollary 3.13] and the remarks thereafter, that the point modules for S are in bijection with the points of E_1.

2.9. Sklyanin's original construction of S was motivated by his interest in the Quantum Inverse Problem Method corresponding to Baxter's original R-matrix. This matrix is defined in terms of certain theta functions and the addition formulae for those functions determine the multiplication rules for S (see [Sk1, §2] for more details). Since theta functions can be used to define an elliptic curve, this suggests that (at least when $k = \mathbb{C}$) the existence of the elliptic curve E is not a coincidence, but rather is implicit in the very construction of S. The aim of the rest of this section is to show that this is, indeed, the case.

2.10. We adopt the notation of [We] for theta functions. Fix $\omega \in \mathbb{C}$ with $\mathrm{Im}(\omega) > 0$ and write $\Lambda = \mathbb{Z} + \mathbb{Z}\omega$ for the associated lattice. As in [We, p.71], let $\{\theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}\}$ be the four theta functions of Jacobi corresponding to this lattice. Thus the θ_{ab} are holomorphic functions on $\mathbb{C}$ such that

$$\theta_{ab}(z+1) = (-1)^a \theta_{ab}(z), \qquad \theta_{ab}(z+\omega) = exp(-\pi i\omega - 2\pi i z - \pi i b)\theta_{ab}(z),$$

and θ_{ab} has zeroes at the points $(\frac{1-b}{2}) + (\frac{1+a}{2})\omega + \Lambda$.

To explain the connection with the Sklyanin algebra, we need to introduce some auxiliary functions. Fix a point $\tau \in \mathbb{C}$ such that τ is not of order 4 in $\mathbb{C}/\Lambda$. Thus $\theta_{ab}(\tau) \neq 0$ for all $ab \in \{00, 01, 10, 11\}$. For each $ab \in \{00, 01, 10, 11\}$, define

$$g_{ab}(z) \ = \ \theta_{ab}(2z)\theta_{ab}(\tau)\gamma_{ab} \qquad \text{for } z \in \mathbb{C},$$

and

$$\alpha_{ab} \ = \ (-1)^{a+b}\left(\frac{\theta_{11}(\tau)\,\theta_{ab}(\tau)}{\theta_{ij}(\tau)\,\theta_{k\ell}(\tau)}\right)^2 ,$$

where $\{ab, ij, k\ell\} = \{00, 01, 10\}$ and $\gamma_{00} = \gamma_{11} = i$, $\gamma_{01} = \gamma_{10} = 1$.

Define $\alpha = \alpha_{00}, \beta = \alpha_{01}, \gamma = \alpha_{10}$. It follows from [We, Equs. (13) and (14), p.74], that

$$\theta_{00}^4(\tau) + \theta_{11}^4(\tau) = \theta_{01}^4(\tau) + \theta_{10}^4(\tau),$$

and a simple calculation shows that $\alpha+\beta+\gamma+\alpha\beta\gamma = 0$. Conversely, given a point (α, β, γ) on the surface $\alpha+\beta+\gamma+\alpha\beta\gamma = 0$, and with the property that $\{\alpha, \beta, \gamma\} \cap \{0, 1, -1\} = \emptyset$, then one can prove that there is a choice of ω and τ such that $\alpha = \alpha_{00}, \beta = \alpha_{01}$, and $\gamma = \alpha_{10}$.

The addition formulae for the θ_{ab} (see [We, Eqns (5)–(10), pp. 77-78]) give rise to the following six identities:

$$g_{11}(z)g_{ab}(z+\tau) \ - \ g_{ab}(z)g_{11}(z+\tau) \ = \ \alpha_{ab}\big(g_{ij}(z)g_{k\ell}(z+\tau) \ + \ g_{k\ell}(z)g_{ij}(z+\tau)\big) \tag{2.10.1}$$

$$g_{11}(z)g_{ab}(z+\tau) \ + \ g_{ab}(z)g_{11}(z+\tau) \ = \ g_{ij}(z)g_{k\ell}(z+\tau) \ - \ g_{k\ell}(z)g_{ij}(z+\tau),$$

whenever $\{ab, ij, k\ell\}$ is a cyclic permutation of $\{00, 01, 10\}$. For example, to get the first of these equations when $ab = 00$, use [We, Equs.(5) and (8), p.77], twice, once as presented there and once with v replaced by $-v$, and then replace $-2v$ by $+\tau$.

2.11. These six identities for the g_{ab} can be used to obtain representations of the Sklyanin algebra in the following manner. Let $\mathcal{O}$ denote the ring of holomorphic functions on $\mathbb{C}$. Define an algebra homomorphism from the free algebra $\Psi : \mathbb{C}\langle X_{00}, X_{01}, X_{10}, X_{11}\rangle$ to $\mathrm{End}_{\mathbb{C}}\mathcal{O}$ by $\Psi(X_{ab})(f)(z) = g_{ab}(z-\tau)f(z+\tau)$ for all $f \in \mathcal{O}$. The six identities (2.10.1) yield six quadratic relations satisfied by the $\Psi(X_{ab})$, and a careful examination of these relations shows that Ψ factors through the Sklyanin algebra via the map $X_{11} \mapsto x_0$, $X_{00} \mapsto x_1$, $X_{01} \mapsto x_2$, and $X_{10} \mapsto x_3$. Thus $\mathcal{O}$ becomes an $S(\alpha, \beta, \gamma)$-module.

2.12. We now make explicit the relation between the Sklyanin algebra and the elliptic curve $\mathbb{C}/\Lambda$. It will turn out that $\mathbb{C}/\Lambda$ is isomorphic to E.

Each of the $g_{ab}(z)$ is a theta function of weight 4 with respect to Λ. Thus, as is well-known, the map

$$z \longrightarrow \big(g_{11}(z),\, g_{00}(z),\, g_{01}(z),\, g_{10}(z)\big)$$

gives an isomorphic embedding $j : \mathbb{C}/\Lambda \to \mathbb{P}^3$ (see, for example, [Hu, Proposition 3.2, p.189]). Write $E' = \mathrm{Im}(j)$. Define an automorphism σ' of E' by $\sigma'(j(z)) = j((z+\tau))$ and let

$$\Gamma' \;=\; \big\{(\,j(z),\, j(z+\tau)) : z \in \mathbb{C}\big\} \;\subseteq \mathbb{P}^3 \times \mathbb{P}^3$$

be the graph of the automorphism σ' .

Let $X_{00}, X_{01}, X_{10}, X_{11}$ be the homogeneous coordinate functions on $\mathbb{P}^3$ defined by $X_{ab}(j(z)) = g_{ab}(z)$. Consider $X_{ab} \otimes X_{ij}$ as the coordinate function on $\mathbb{P}^3 \times \mathbb{P}^3$ such that $X_{ab} \otimes X_{ij}(p,q) = X_{ab}(p).X_{ij}(q)$. Then the six identities (2.10.1) show that, as functions on Γ', the X_{uv} satisfy the following six equations:

$$X_{11} \otimes X_{ab} \,-\, X_{ab} \otimes X_{11} \,-\, \alpha_{ab}\big(X_{ij} \otimes X_{k\ell} \,+\, X_{k\ell} \otimes X_{ij})\big) = 0$$

$$X_{11} \otimes X_{ab} \,+\, X_{ab} \otimes X_{11} \,-\, X_{ij} \otimes X_{k\ell} \,-\, X_{k\ell} \otimes X_{ij} = 0,$$

whenever $\{ab,\, ij,\, k\ell\}$ is a cyclic permutation of $\{00,\, 01,\, 10\}$. Thus $\Gamma' \subset \Gamma$ and, because Γ' is irreducible, Γ' equals the graph of $\sigma|_{E'}$.

2.13. The defining equations for E' are most easily obtained by using the ideas from [We]. For example, the g_{ab} satisfy an equation of the form $g_{10}^2(z) + Ag_{00}^2(z) + Bg_{01}^2(z) = 0$. As in [We, p.76], in order to determine A and B as functions of the $g_{ab}(\tau)$, evaluate this equation at $z = \tau$ and $z = \tau + \omega$. Then it follows that $A = \frac{1-\gamma}{1+\alpha}$ and $B = \frac{1+\gamma}{1-\beta}$. Thus, if $(x_0,\, x_1,\, x_2,\, x_3) = (X_{11}, X_{00}, X_{01}, X_{10})$ are the coordinate functions on $\mathbb{P}^3$, then this implies that $g_2 = x_3^2 + \frac{1-\gamma}{1+\alpha}x_1^2 + \frac{1+\gamma}{1-\beta}x_2^2$ vanishes on E'. Similarly, $g_1 \;=\; x_0^2 + x_1^2 + x_2^2 + x_3^2$ vanishes on E' and it follows easily that $E' = \mathcal{V}(g_1,\, g_2)$. This is in agreement with (and, when $k = \mathbb{C}$, gives an alternative proof of) Proposition 2.4. Finally, it follows from (2.12) that $\Gamma' \subseteq \mathcal{V}(\widetilde{f_i} : 1 \leq i \leq 6)$. As Γ' is the graph of the automorphism σ', it is immediate that $\pi_1(\Gamma') = \pi_2(\Gamma') = E'$ and that the $\pi_i|_{\Gamma'}$ are isomorphisms.

Thus we have shown that, if $k = \mathbb{C}$, then the earlier results (2.4) – (2.8) of this section all follow from the construction of $S(\alpha,\, \beta,\, \gamma)$ in terms of theta functions. Standard specialisation arguments can be used to show that the same results will then follow for "most" fields k.

2.14. In conclusion, if $\{\alpha, \beta, \gamma\}$ satisfy one of the conditions (1.9.1) – (1.9.4), then this section proves that one may attach to the Sklyanin algebra $S = S(\alpha,\, \beta,\, \gamma)$ the geometric data

$$\mathcal{F} \;=\; \{E, \sigma, \mathcal{L}\}.$$

Here, E is the elliptic curve defined by Propositions 2.4 and 2.5, σ is the automorphism of E defined by Corollary 2.8 and $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^1}(1)$ is the very ample line bundle associated to the embedding $E \hookrightarrow \mathbb{P} = \mathbb{P}^3$.

3. THE GEOMETRIC RING.

3.1. Throughout this section, we assume that $\{\alpha, \beta, \gamma\} \in k$ satisfy either (1.9.1) or (1.9.4).

The geometric data $\{E, \sigma, \mathcal{L}\}$ obtained in Section 2 is used in this section to construct a graded ring B. As will be shown, this ring is actually a homomorphic image of S. The advantage of this approach is that geometric results can be used to prove detailed results about the structure of B. These results will then be used in Section 5 to understand the structure of S.

3.2. The general construction is as follows. Let E be an (irreducible, smooth) elliptic curve with an automorphism σ and a very ample, invertible sheaf $\mathcal{L}$. Fix $r = \deg \mathcal{L}$ (of course, $r \geq 3$, as $\mathcal{L}$ is very ample). For each integer n, define the line bundle $\mathcal{L}^{\sigma^n} = (\sigma^n)^*\mathcal{L}$, the inverse image of $\mathcal{L}$ along $\sigma^n : E \to E$. For $n \geq 0$, write $\mathcal{L}_n = \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$, where $\mathcal{L}_0 = \mathcal{O}_E$. Define $B_n = H^0(E, \mathcal{L}_n)$ and $B = \bigoplus B_n$. Since E is fixed, we will usually write $H^i(\mathcal{M}) = H^i(E, \mathcal{M})$ and $h^i(\mathcal{M}) = \dim_k H^i(\mathcal{M})$, whenever $\mathcal{M}$ is a sheaf of $\mathcal{O}_E$-modules, and $i \geq 0$. For all $n, m \geq 0$ there is a canonical isomorphism $\mathcal{L}_n \otimes \mathcal{L}_m^{\sigma^n} \to \mathcal{L}_{n+m}$ and hence a map

$$\mu_{n,m} : H^0(\mathcal{L}_n) \otimes H^0(\mathcal{L}_m^{\sigma^n}) \longrightarrow H^0(\mathcal{L}_{n+m}).$$

This provides B with the structure of a graded ring; if $u \in B_n$ and $v \in B_m$, then define $uv = \mu_{n,m}(u \otimes v^{\sigma^n}) \in B_{n+m}$. This construction is taken from [ATV1], and the reader is referred to that paper for further details.

3.3. Apply the construction of (3.2) to the geometric data $\mathcal{F} = \{E, \sigma, \mathcal{L}\}$ obtained in Section 2, and let $B = B(\alpha, \beta, \gamma)$ denote the resulting algebra. B will be called the *geometric ring* associated to S. An easy, but important, connection between B and S is provided by the following result.

Lemma. *There is a natural isomorphism of vector spaces $S_1 \to B_1$ and this induces a k-algebra homomorphism $\phi : S \to B$.*

Proof. This is the same as the proof of the analogous result in dimension three (see [ATV1, Propositions 3.20 and 6.5]). However since it is the whole point of this construction, we will give the proof here.

Recall that $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}}(1)$, where $\mathbb{P} = \mathbb{P}^3$. Since E is defined by quadratic relations,

$$H^0(E, \mathcal{L}) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) = k.x_0 \oplus k.x_1 \oplus k.x_2 \oplus k.x_3,$$

in the notation of (2.4). Thus the map $x_i \to x_i$ provides an isomorphism $S_1 \to B_1 = H^0(E, \mathcal{L})$. In order to prove that this induces a ring homomorphism $\phi : S \to B$, we need to show that the relations of S are mapped to zero. First, one needs to be a little careful about the conventions. Given $p \in E$ and $z \in H^0(E, \mathcal{L})$, regard z as a rational function on E

and define $z^\sigma \in H^0(E, \mathcal{L}^\sigma)$ by $z^\sigma(p) = z(p^\sigma)$. Thus, if $z_i \in B_1$, then the element $z_1 z_2 \in B_2$ is (by definition) the rational function on E defined by $z_1 z_2(p) = z_1(p) z_2^\sigma(p) = z_1(p) z_2(p^\sigma)$. In particular, if $f = \sum \lambda_{ij} x_i x_j$ is a relation in S and $p \in E$, then we need to prove that

$$0 = \phi(f)(p) = \sum \lambda_{ij} x_i(p) x_j(p^\sigma). \tag{3.3.1}$$

But, by construction, $\widetilde{f}$ is zero on Γ and, by Corollary 2.8, $\Gamma = \{(p, p^\sigma) : p \in E_1\}$. Hence $\widetilde{f}$ is zero on $\{(p, p^\sigma) : p \in E\}$, as is required to prove (3.3.1).

3.4. The results that we require about the ring B are all contained in the following theorem, the proof of which occupies the rest of this section. Since it requires no extra effort, we state and prove the results in the general situation described by (3.2), rather than for the special case where $B = B(\alpha, \beta, \gamma)$.

Notation. *If a finite dimensional k-vector space is denoted by a capital Roman letter, say B_n, then the corresponding lower case letter will be used to denote its dimension. Thus* $\dim B_n = b_n$.

Theorem. *Let E be an elliptic curve, and $\mathcal{L}$ an invertible $\mathcal{O}_E$-module with $\deg \mathcal{L} = r \geq 4$, and let B be defined as in (3.2). Then:*

(i) B is a domain. Thus the socle of B, $\mathrm{Soc}\, B$, as a left or right B-module is zero.

(ii) $\dim B_0 = 1$ while $\dim B_n = rn$ for $n \geq 1$.

(iii) $B = k\langle B_1\rangle$ is generated, as a k-algebra, by B_1. Thus one may write $B = T/J$, where $T = T(B_1)$ is the tensor algebra on B_1, graded in the obvious manner.

(iv) If $J_n = T_n \cap J$, then $J_0 = J_1 = 0$, while $J_n = J_{n-1}T_1 + T_1 J_{n-1}$ for all $n \geq 3$. Thus B has its defining relations in degree 2.

(v) If $0 \neq y \in B_n$ then $\mathrm{Soc}\, B/By = \mathrm{Soc}\, B/yB = 0$.

(vi) Assume that $r = 4$ and let $C = k[z_1, \ldots, z_4]$ be the commutative polynomial ring in 4 variables, graded by giving each z_i degree one. Then $b_n = c_n - 2c_{n-2} + c_{n-4}$ for all $n \geq 0$.

Remark. If $A = \bigoplus_{n \geq 0} A_n$ is a graded ring, then an observation that will be used frequently is that $A/\bigoplus_{n \geq 1} A_n$ is the only graded, simple A-module. Thus, if A is generated as an algebra by A_1 and M is a finitely generated graded (left) A-module, then $\mathrm{Soc}\, M = \{m \in M : A_1 m = 0\}$.

3.5. With the exception of part (v), Theorem 3.4 follows from a supplement to **[ATV1]**. However, since that supplement will apparently not appear in the published version of the paper, we will give a proof here. The present argument, while not as general as that in [ATV1], is particularly easy since all the hard work is done in [Mu2]. We first set up some notation.

Let $\mathcal{M}$ and $\mathcal{N}$ be coherent $\mathcal{O}_E$-modules, and let

$$\mu(\mathcal{M}, \mathcal{N}) : H^0(\mathcal{M}) \otimes H^0(\mathcal{N}) \longrightarrow H^0(\mathcal{M} \otimes \mathcal{N})$$

be the natural map. Following [Mu2], write

$$R(\mathcal{M},\,\mathcal{N}) \;=\; \operatorname{Ker}(\mu(\mathcal{M},\,\mathcal{N})) \qquad \text{and} \qquad S(\mathcal{M},\,\mathcal{N}) \;=\; \operatorname{Coker}(\mu(\mathcal{M},\,\mathcal{N})).$$

3.6 Lemma. *Under the hypotheses of Theorem 3.4, $B = k\langle B_1\rangle$.*

Proof. We need to prove that $B_{n+1} = B_n B_1$ for all $n \geq 1$. By the definition of multiplication in B, this is equivalent to showing that $S(\mathcal{L}_n,\,\mathcal{L}^{\sigma^n}) = 0$. This is a special case of [Mu2, Theorem 6].

3.7. Next we prove that the defining relations of B are in degree 2. To do this, we must reinterpret the problem in terms of the appropriate $R(\mathcal{M},\,\mathcal{N})$. By Lemma 3.6, we may write $B = T/J$, where $J = \bigoplus J_n$ is a graded ideal of the tensor algebra $T = T(B_1)$ and $J_n = J \cap T_n$. Consider the vector spaces

$$X \;=\; H^0(\mathcal{L}) \otimes H^0(\mathcal{L}^\sigma_m) \otimes H^0(\mathcal{L}^{\sigma^{m+1}}),$$

$$Y \;=\; H^0(\mathcal{L} \otimes \mathcal{L}^\sigma_m) \otimes H^0(\mathcal{L}^{\sigma^{m+1}}),$$

and

$$Z \;=\; H^0(\mathcal{L} \otimes \mathcal{L}^\sigma_m \otimes \mathcal{L}^{\sigma^{m+1}}).$$

Clearly, $H^0(\mathcal{L}^\sigma_m) \cong H^0(\mathcal{L}_m) = B_m$. Thus, $X \cong B_1 \otimes B_m \otimes B_1$, while $Y \cong B_{m+1} \otimes B_1$ and $Z \cong B_{m+2}$. Moreover, there exist natural maps

$$X \xrightarrow{\theta_1} Y \xrightarrow{\theta_2} Z,$$

where $\theta_1 = \mu(\mathcal{L}, \mathcal{L}^\sigma_m) \otimes H^0(\mathcal{L}^{\sigma^{m+1}})$ and $\theta_2 = \mu(\mathcal{L}_{m+1}, \mathcal{L}^{\sigma^{m+1}})$. By Lemma 3.6, these maps are surjective.

Lemma. *For all $m \geq 1$ one has $J_{m+2} = T_1 J_{m+1} + J_{m+1} T_1$.*

Proof. Observe that θ_1 restricts to give a natural map

$$\begin{aligned}\psi : H^0(\mathcal{L}) \otimes R(\mathcal{L}^\sigma_m,\,\mathcal{L}^{\sigma^{m+1}}) \;&\longrightarrow\; R(\mathcal{L} \otimes \mathcal{L}^\sigma_m,\,\mathcal{L}^{\sigma^{m+1}}) \\ &=\; R(\mathcal{L}_{m+1},\,\mathcal{L}^{\sigma^{m+1}}) \;=\; \operatorname{Ker}(\theta_2).\end{aligned}$$

By [Mu, Theorem 2], ψ is surjective.

Now reinterpret these objects in terms of the tensor algebra T. Thus, $H^0(\mathcal{L}^\sigma_m) \cong B_m = T_m/J_m$ and so

$$X \;\cong\; T_1 \otimes (T_m/J_m) \otimes T_1 \;=\; T_{m+2}/T_1 J_m T_1.$$

Similarly, $Y \cong T_{m+2}/J_{m+1}T_1$ and $Z \cong T_{m+2}/J_{m+2}$. Moreover, under these identifications, the maps $\theta_1 : X \to Y$ and $\theta_2 : Y \to Z$ are just the natural projections. In particular, $\operatorname{Ker}\theta_2 = J_{m+2}/J_{m+1}T_1$, while

$$H^0(\mathcal{L}) \otimes R(\mathcal{L}^\sigma_m,\,\mathcal{L}^{\sigma^{m+1}}) \;=\; T_1 \otimes (J_{m+1}/J_m T_1) \;=\; \frac{T_1 J_{m+1}}{T_1 J_m T_1}.$$

Since ψ is surjective, this implies that

$$\frac{J_{m+2}}{J_{m+1}T_1} = \operatorname{Im}\psi = \theta_1\left(\frac{T_1J_{m+1}}{T_1J_mT_1}\right) = \frac{T_1J_{m+1}+J_{m+1}T_1}{J_{m+1}T_1}.$$

Thus $J_{m+2} = T_1J_{m+1} + J_{m+1}T_1$, as required.

3.8 Proof of Theorem 3.4. Since E is an irreducible variety, elements of each B_n may be regarded as elements of the field of rational functions $k(E)$. Of course, given $z \in B_n$ and $w \in B_m$, then the element $zw \in B_{n+m}$ is the rational function $zw^{\sigma^n} \in k(E)$. Since $k(E)$ is a field, this implies that B must be a domain and so part (i) follows. Part (ii) follows from the fact that, for all t, $H^0(\mathcal{L}^{\sigma^t}) \cong H^0(\mathcal{L})$ whence, by Riemann-Roch, $\dim_k B_n = n \cdot \dim H^0(\mathcal{L}) = nr$. Part (iii) of the theorem is just Lemma 3.6, while part (iv) follows from Lemma 3.7, combined with the definitions of B_0 and B_1. Part (vi) follows from part (ii), combined with the standard recursive formula for the c_n.

It remains to prove part (v). Suppose that the result is false. That is, suppose that $\operatorname{Soc} B/By \neq 0$ (we will only prove the result for the left socle; the proof for the right socle is similar). Now, the only graded, simple left B-module is $B/\langle B_1\rangle$. Hence there exists a second homogeneous element, say $z \in B_r$, such that $z \notin B_{n-r}y$ but $B_1z \subseteq B_{n-r+1}y$. Thus, $zy^{-1} \notin B_{n-r}$ and so there exists a point $Q \in E$ such that $zy^{-1} \notin (\mathcal{L}_{n-r})_Q$, the stalk of $\mathcal{L}_{n-r}$ at Q. Set $P = \sigma^{-1}(Q)$. Since $\mathcal{L}$ is generated by global sections, there exists $t \in H^0(\mathcal{L})$ such that

$$(\mathcal{L}_{n-r+1})_P \cong (\mathcal{L})_P \otimes_{\mathcal{O}_{E,P}} (\mathcal{L}^\sigma_{n-r})_P \cong t\mathcal{O}_{E,P} \otimes (\mathcal{L}^\sigma_{n-r})_P \cong t(\mathcal{L}^\sigma_{n-r})_P.$$

Since $B_1zy^{-1} \subseteq B_{n-r+1} \subseteq (\mathcal{L}_{n-r+1})_P$, this certainly implies that $t(zy^{-1})^\sigma \in t(\mathcal{L}^\sigma_{n-r})_P$. Thus $(zy^{-1})^\sigma \in (\mathcal{L}^\sigma_{n-r})_P$ and $zy^{-1} \in (\mathcal{L}_{n-r})_Q$, giving the required contradiction.

3.9. Combined with Lemma 3.3, Theorem 3.4(iii) implies that $B = \phi(S)$ is a homomorphic image of S and so a natural question is to determine the kernel.

Corollary. *Consider the elements*

$$\Omega_1 = -x_0^2 + x_1^2 + x_2^2 + x_3^2 \quad \text{and} \quad \Omega_2 = x_1^2 + \tfrac{1+\alpha}{1-\beta}x_2^2 + \tfrac{1-\alpha}{1+\gamma}x_3^2.$$

Then each Ω_i is a central element of S and $B = S/S\Omega_1 + S\Omega_2$.

Proof. That the Ω_i are central is given in [Sk1, Theorem 2]. By construction, for any i and j, $\phi(x_ix_j) = x_ix_j^\sigma$, where the right hand side is regarded as an element of $k(E)$. Now use the explicit formula for σ given by Corollary 2.8 to show that $\phi(\Omega_i) = 0$ for each i. In the notation of Theorem 3.4(iv), the set $Z = \{\Omega_1, \Omega_2, f_1, \dots, f_6\}$ is linearly independent in J_2. Since $\dim B_2 = 8$, this implies that Z spans J_2. Thus Theorem 3.4(iv) implies that $B = S/S\Omega_1 + S\Omega_2$.

3.10 Remark. When $k = \mathbb{C}$, the fact that $\Omega_i \in ker(\phi)$ is also implicit in the construction of S in terms of theta functions. To see this, consider the description of Γ (and hence the geometric ring B) given in (2.10–2.13). Thus Γ is determined by six equations corresponding to the six addition formulae for the g_{ab} given by (2.10.1). However, using the ideas of [We], one can find two further relations between the g_{ij}, these being:

$$g_{11}(z)g_{11}(z+\tau) - g_{00}(z)g_{00}(z+\tau) - g_{01}(z)g_{01}(z+\tau) - g_{10}(z)g_{10}(z+\tau) = 0,$$

$$g_{00}(z)g_{00}(z+\tau) + \tfrac{1+\alpha_{00}}{1-\alpha_{01}}g_{01}(z)g_{01}(z+\tau) + \tfrac{1-\alpha_{00}}{1+\alpha_{10}}g_{10}(z)g_{10}(z+\tau) = 0.$$

These equations simply say that the homogeneous forms $\widetilde{\Omega}_i$ are zero on Γ. Thus, by the argument of Lemma 3.3, the elements Ω_i do indeed lie in $\ker(\phi)$.

4. THE DUAL OF THE SKLYANIN ALGEBRA.

4.1. Throughout this section we assume that $\alpha, \beta, \gamma \in k$ satisfy either (1.9.1) or (1.9.4) or (in order to complete the description of the degenerate cases of Section 1) that $\alpha = 0$, $\beta \neq -1$ and $\gamma \neq 1$.

In order to prove that the Sklyanin algebra S is regular, one must at least prove that the trivial module ${}_Sk$ has a finite projective resolution. In this section, we find a complex of free S-modules $(K_\bullet, d) \to {}_Sk \to 0$ which is a potential resolution of ${}_Sk$. The exactness of this complex will be proved in Section 5 using the interplay between S and the geometric ring B.

It is easy, though tedious, to construct a potential resolution for ${}_Sk$ using the generators and relations for S. However, we will use a more conceptual, though no less tedious approach based on Manin's construction of Koszul complexes for quadratic algebras. One advantage of this approach is to show that the Gorenstein condition for S is an easy consequence of the proof that S has finite global dimension.

4.2. Write $S = T(V)/\langle W\rangle$, where V is the vector space spanned by $\{x_0, \ldots, x_3\}$ and W is the subspace of $V \otimes V$ spanned by the defining relations $\{f_i : 1 \leq i \leq 6\}$ of S. In this section, Z^* will always denote the vector space dual of the k-vector space Z. Let $W^\perp$ denote the orthogonal to W in $V^* \otimes V^*$. The *dual algebra to* S is defined to be $S^! = T(V^*)/\langle W^\perp\rangle$. Thus $S^!$ is also a graded algebra, $S^! = \bigoplus_{n\geq 0} S^!_n$. Let $\{\xi_0, \xi_1, \xi_2, \xi_3\} \subset V^*$ be the dual basis to the generators $\{x_j\}$ for S and set

$$e \;=\; \sum_{j=0}^{3} \xi_j \otimes x_j \;\in\; S^!_1 \otimes S_1.$$

Then $e^2 = 0$ [Ma2, Lemma 9.1].

Define $K^\bullet = \bigoplus_{n\geq 0} K^n$, where $K^n = S^!_n \otimes S$, and let $d^*: K^n \to K^{n+1}$ denote left multiplication by e. Since $e^2 = 0$, $(K^\bullet, d^*)$ is a complex of free right S-modules, called a *Koszul complex*. The second complex of interest is the dual of $(K^\bullet, d^*)$, namely the *Koszul complex* of free left S-modules

$$(K_\bullet,\, d) \;=\; (K_\bullet(S),\, d) \;=\; Hom_S((K^\bullet,\, d^*), S).$$

More explicitly, the left regular representation of $S^!$ on itself induces a right $S^!$-module structure on $(S^!)^* = \bigoplus_{n\geq 0}(S^!_n)^*$. Thus, if $K_n = S \otimes (S^!_n)^*$ and $K_\bullet = \bigoplus_{n\geq 0} K_n$, then right multiplication by e induces a map $d : K_n \to K_{n-1}$ and hence a differential $d : K_\bullet \to K_\bullet$.

4.3. The considerations in (4.2) apply to any graded algebra generated in degree 1 and defined by homogeneous relations of degree 2. Such an algebra is called a *quadratic algebra*, and the next (folklore) result is stated in that context.

Lemma. *Let $S = T(V)/\langle W\rangle$ be a quadratic algebra with defining relations $W \subset V \otimes V$. Then the "right hand end"*

$$S \otimes (S^!_2)^* \xrightarrow{d} S \otimes (S^!_1)^* \xrightarrow{d} S \otimes (S^!_0)^* \xrightarrow{\epsilon} k \longrightarrow 0 \tag{4.3.1}$$

of the augmented Koszul complex $(K_\bullet, d)$ is an exact sequence of left S-modules. It is isomorphic to the exact sequence

$$S^{(dim\, W)} \xrightarrow{\bullet M} S^{(dim\, S_1)} \xrightarrow{\bullet \mathbf{x}} S \xrightarrow{\epsilon} k \longrightarrow 0 \tag{4.3.2}$$

obtained from the generators and relations for S. More precisely, let $\{x_0, x_1, \dots\}$ be a basis for S_1, and suppose that $\{f_i = \sum_p m_{ip}x_p : m_{ip} \in V\}$ is a basis for W. Then $\bullet\mathbf{x}$ denotes right multiplication by $\mathbf{x} = (x_0, x_1, \dots)^T$ and $\bullet M$ denotes right multiplication by the matrix $M = (m_{ip})$.

Proof. It is well-known, and elementary, that the second sequence is exact. Hence the proof is simply a matter of carefully interpreting the Koszul complex. First, $S \otimes (S^!_0)^* \cong S \otimes S_0 \cong S$, while $S \otimes (S^!_1)^* \cong S \otimes S_1$ and $S \otimes (S^!_2)^* \cong S \otimes W$. Now check that the map "right multiplication by e" coincides with the homomorphisms in the second sequence.

4.4. The key to understanding the complex $(K_\bullet, d)$ is the structure of $S^!$ and especially its multiplication table. By construction, $S^!$ is generated by $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ with defining relations

$$\xi_0^2 = \xi_1^2 = \xi_2^2 = \xi_3^2 = 0$$

$$\begin{aligned}
2\xi_2\xi_3 + (\alpha+1)\xi_0\xi_1 - (\alpha-1)\xi_1\xi_0 &= 0, & 2\xi_3\xi_2 + (\alpha-1)\xi_0\xi_1 - (\alpha+1)\xi_1\xi_0 &= 0,\\
2\xi_3\xi_1 + (\beta+1)\xi_0\xi_2 - (\beta-1)\xi_2\xi_0 &= 0, & 2\xi_1\xi_3 + (\beta-1)\xi_0\xi_2 - (\beta+1)\xi_2\xi_0 &= 0,\\
2\xi_1\xi_2 + (\gamma+1)\xi_0\xi_3 - (\gamma-1)\xi_3\xi_0 &= 0, & 2\xi_2\xi_1 + (\gamma-1)\xi_0\xi_3 - (\gamma+1)\xi_3\xi_0 &= 0.
\end{aligned}$$

To see this, note that $\dim W^\perp = 16 - \dim W = 10$. Since the elements given above are clearly linearly independent and do lie in $W^\perp$, they must form the required basis.

There is an involutive anti-automorphism σ of S defined by $\sigma(x_0) = -x_0$, and $\sigma(x_i) = x_i$ for $i \neq 0$. By duality, σ induces an involutive anti-automorphism of $S^!$, which is also denoted by σ; that is $\sigma(\xi_0) = -\xi_0$, and $\sigma(\xi_i) = \xi_i$ for $i \neq 0$. This induces an anti-automorphism, still denoted by σ, of $S \otimes S^!$. This has the useful property that $\sigma(e) = e$, where e is defined in (4.2).

4.5. It is easy to use the relations given in (4.4) to write down a natural candidate for the basis of $S^!$.

Proposition. *If* $n \geq 5$*, then* $S^!_n = 0$*. If* $n \leq 4$*, then* $S^!_n$ *is spanned by the following elements:*

$$\begin{array}{ll} S^!_0 : & 1 \\ S^!_1 : & \xi_0,\ \xi_1,\ \xi_2,\ \xi_3 \\ S^!_2 : & \xi_0\xi_1,\ \xi_0\xi_2,\ \xi_0\xi_3,\ \xi_1\xi_0,\ \xi_2\xi_0,\ \xi_3\xi_0 \\ S^!_3 : & \xi_0\xi_1\xi_0,\ \xi_0\xi_2\xi_0,\ \xi_0\xi_3\xi_0,\ \xi_1\xi_0\xi_1 \\ S^!_4 : & \xi_0\xi_1\xi_0\xi_1 \end{array}$$

Proof. Everything is obvious for $n \leq 2$, and for $n \geq 3$, one uses the relations in (4.4). Let $F = k.\xi_1 + k.\xi_2 + k.\xi_3$. Then $S^!_2 = \xi_0 F + F\xi_0$, and

$$S^!_3 \ = \ S^!_2 S^!_1 \ = \ F\xi_0 F + \xi_0 F\xi_0 + \xi_0 F^2 \ = \ F\xi_0 F + \xi_0 F\xi_0.$$

Now check that $F\xi_0 F \ \subseteq \ \xi_1\xi_0\xi_1 \cdot k + \xi_0 F\xi_0$, thus giving the result for $S^!_3$.

To complete the proof, one uses the same idea, together with the following identities (which will also be used later on):

$$\xi_0\xi_j\xi_0\xi_j \ = \ -\xi_j\xi_0\xi_j\xi_0 \quad \text{for} \quad 1 \leq j \leq 3, \qquad \xi_0\xi_i\xi_0\xi_j \ = \ 0 \quad \text{for} \quad i \neq j,$$

$$\xi_0\xi_3\xi_0\xi_3 = \tfrac{1+\alpha}{1-\gamma}\xi_0\xi_1\xi_0\xi_1, \qquad \xi_0\xi_2\xi_0\xi_2 \ = \ \tfrac{1+\gamma}{1-\beta}\tfrac{1+\alpha}{1-\gamma}\xi_0\xi_1\xi_0\xi_1$$

Note that, if $\{\alpha, \beta, \gamma\}$ satisfy (1.9.1), then this last formula reduces to the expression $\xi_0\xi_2\xi_0\xi_2 = \frac{1-\alpha}{1+\beta}\xi_0\xi_1\xi_0\xi_1$. Of course, this expression is undefined if the parameters satisfy (1.9.4).

4.6 Theorem. *The elements given in Proposition 4.5 form a basis for* $S^!$*. In particular,* $dim S^!_n = \binom{4}{n}$.

Proof. As typically happens for rings defined by generators and relations, there seems to be no proof of this result that does not require a considerable amount of computation. One proof can be given using Bergman's Diamond Lemma [Be] (since $S^!$ is known to be finite dimensional, the algorithm for the Diamond Lemma given in [An, §2] will be a finite procedure). This can be checked easily on a computer, using Schelter's *Affine* program, and we are grateful to him for sharing this program with us.

An alternative proof, and one that is more amenable to hand computations, is as follows. Suppose for a moment that the elements given in (4.5) form a basis for $S^!$. Then $\dim_k S^! = 16$, and the right regular representation of $S^!$ gives a map $\rho : S^! \to M_{16}(k)$, from $S^!$ to the ring of 16×16 matrices. Using (4.4), one can explicitly write down the $\rho(\xi_i)$ (in fact, this is the same calculation that will be required to write down the matrices N and N_1 in (4.11), below). For example,

$$\begin{aligned}\rho(\xi_1) \;=\; & e_{1,3} + 2e_{2,6} + (1-\gamma)e_{4,8} + (1+\gamma)e_{4,11} - (\beta+1)e_{5,7} + (\beta-1)e_{5,10} \\ & + \tfrac{1}{2}(\gamma+1)e_{7,14} + \tfrac{1}{2}(\beta-1)e_{8,13} + e_{9,15} + \tfrac{(\alpha-1)(1-\gamma)}{2(\alpha+1)}e_{10,14} \\ & + \tfrac{1}{4}(\alpha+1)(\beta+1)e_{11,13} + e_{12,16},\end{aligned}$$

where $e_{i,j}$ denote the usual matrix units.

Now reverse this procedure. Consider the subalgebra R of $M_{16}(k)$ generated by the 4 matrices $r_i = \rho(\xi_i)$. Now show that the r_i satisfy the relations of the ring $S^!$ and that $\dim_k R \geq 16$. It then follows that $\dim_k S^! \geq \dim R \geq 16$, as required. The details are straightforward, although tedious, and so are omitted.

It goes without saying that a more conceptual proof of this theorem would be welcome. One possibility is that there exists an intrinsic definition of $S^!$ in terms of the elliptic curve E.

4.7 Corollary. *The augmented Koszul complex* $(K_\bullet, d)$ *is a complex of graded left* S-*modules*

$$0 \longrightarrow S \xrightarrow{d} S^{(4)} \xrightarrow{d} S^{(6)} \xrightarrow{d} S^{(4)} \xrightarrow{d} S \xrightarrow{\epsilon} k \longrightarrow 0 \qquad (4.7.1)$$

for which each differential d *is graded of degree* 1. *Moreover, the right hand end*

$$S^{(6)} \longrightarrow S^{(4)} \longrightarrow S \longrightarrow k \longrightarrow 0$$

of this sequence is exact.

4.8. A quadratic algebra S with the property that the augmented Koszul complex $(K_\bullet,\, d) \to {}_S k \to 0$ is exact is called a *Koszul algebra* **[Pr, Ma1, Ma2]**. A general property of a Koszul algebra, say A, is that the Hilbert series $H_A(t) = \sum \dim_k(A_n)t^n$ of the algebra A and that of its dual $A^!$ are related by the equation $H_A(t) \cdot H_{A^!}(-t) = 1$. Now, the polynomial ring $k[z_0,\, z_1,\, z_2,\, z_3]$ is a Koszul algebra, with Koszul dual the exterior algebra $\Lambda(k^{(4)})$. Further, by Theorem 4.6, $S^!$ has the same Hilbert series as $\Lambda(k^{(4)})$. Thus, any proof of the exactness of (4.7.1) will also show that S has the same Hilbert series as the polynomial ring $k[z_0,\, z_1,\, z_2,\, z_3]$. However, as will be seen in Section 5, the proof that (4.7.1) is exact is inextricably entwined with the proof that S has the same Hilbert series as the polynomial ring.

4.9. The next step is to show that $R = S^!$ is a *Frobenius algebra*, in the sense that $({}_R R)^* \cong R_R$ as right R-modules. The significance of this is that it allows us to prove that the dual complex $(K^\bullet, d^*) = \mathrm{Hom}_S((K_\bullet, d),\, S)$, which is a potential resolution for k_S, is exact if and only if $(K_\bullet, d)$ is exact. Thus, the Gorenstein property for S will immediately follow from the exactness of $(K_\bullet, d)$.

Proposition. *The ring $S^!$ is a Frobenius algebra. The isomorphism $\psi : S^!_{S^!} \to ({}_{S^!}S^!)^*$ is induced from the multiplication map $S^!_n \times S^!_{4-n} \to S^!_4$ which gives a non-degenerate pairing for each n. In particular ψ restricts to an isomorphism $S^!_n \to (S^!_{4-n})^*$ for each n.*

Proof. Given $a, b \in S^!$, write ab as a sum of homogeneous elements and let $\Psi(a, b)$ be the coefficient of $\eta = \xi_0\xi_1\xi_0\xi_1$ in this expansion. Clearly, Ψ is a bilinear form. Let a be a homogeneous element in $S^!$. Then the identities at the end of the proof of (4.5) can be used to find an element $b \in S^!$ for which $ab = \eta$. Thus Ψ is nondegenerate. Finally, as $\Psi(ar, b) = \Psi(a, rb)$ for all $a, b, r \in S^!$, this implies that the map $\psi : a \mapsto \Psi(a,\ )$ gives the required isomorphism from $S^!_{S^!}$ to $({}_{S^!}S^!)^*$.

4.10 Proposition. **(i)** *The complexes $(K_\bullet, d)$ and $(K^\bullet, d^*)$ are isomorphic as complexes of vector spaces.*

(ii) *If the augmented complex $(K_\bullet, d) \to {}_Sk \to 0$ is exact, then S is Gorenstein.*

Proof. **(i)** Consider the complexes in question:

$$(K_\bullet, d) \qquad \cdots \xrightarrow{d} S \otimes (S^!_n)^* \xrightarrow{d} S \otimes (S^!_{n-1})^* \xrightarrow{d} \cdots$$

$$(K^\bullet, d^*) \qquad \cdots \xrightarrow{d^*} S \otimes S^!_n \xrightarrow{d^*} S \otimes S^!_{n+1} \xrightarrow{d^*} \cdots$$

By Proposition 4.9, $(K_\bullet, d)$ is isomorphic to the complex of left S-modules:

$$(K'_\bullet, \partial) \qquad \cdots \xrightarrow{\partial} S \otimes S^!_{4-n} \xrightarrow{\partial} S \otimes S^!_{4-n+1} \xrightarrow{\partial} \cdots$$

Since the isomorphism ψ in Proposition 4.9 is a right $S^!$-module map, the differential ∂ is still given by right multiplication by the element $e \in S \otimes S^!$. Now apply the anti-automorphism σ, described in (4.4), to show that $\sigma(K'_\bullet, \partial)$ is isomorphic to the complex

$$\cdots \xrightarrow{\partial^*} S \otimes S^!_m \xrightarrow{\partial^*} S \otimes S^!_{m+1} \xrightarrow{\partial^*} \cdots$$

where the differential is left multiplication by $\sigma(e)$. Since $\sigma(e) = e$, this complex is none other than $(K^\bullet, d^*)$.

(ii) By Lemma 4.3, the complex $(K_\bullet,\ d)$ can be augmented to a complex $(K_\bullet,\ d) \to {}_Sk \to 0$. By part(i), therefore, $(K^\bullet,\ d^*)$ can be extended to a complex $(K^\bullet,\ d^*) \to k_S \to 0$ and this complex is exact if and only if the first augmented complex is exact. Since $(K^\bullet,\ d^*) = \mathrm{Hom}_S((K_\bullet,\ d),\ S)$, this proves the result.

4.11. When we prove the exactness of $(K_\bullet, d)$ in Section 5, we will require an explicit description of the "left hand end" of this complex. This is provided by the next lemma.

Lemma. *Consider the left hand end*

$$0 \longrightarrow S \xrightarrow{d_1} S^{(4)} \xrightarrow{d_2} S^{(6)}$$

of the complex $(K_\bullet, d)$. *With respect to the appropriate bases of* $S^{(n)}$, *the homomorphisms* d_1 *and* d_2 *are given by right multiplication by the matrices* $\mathbf{x} = (x_0, x_1, x_2, x_3)$, *respectively,*

$$N = \frac{1}{2}\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ 0 & (1-\beta)x_3 & -(1+\gamma)x_2 & 2x_0 & (1+\beta)x_3 & -(1-\gamma)x_2 \\ -(1+\alpha)x_3 & 0 & (1-\gamma)x_1 & -(1-\alpha)x_3 & 2x_0 & (1+\gamma)x_1 \\ (1-\alpha)x_2 & -(1+\beta)x_1 & 0 & (1+\alpha)x_2 & -(1-\beta)x_1 & 2x_0 \end{pmatrix}.$$

Proof. Just as the right hand end of $(K_\bullet, d)$ is easily described, it is the left hand end of $(K^\bullet, d^*)$ that is most useful. But, by the proof of (4.10), $(K_\bullet, d)$ is isomorphic to the complex

$$(K'_\bullet, \partial) \qquad\qquad 0 \longrightarrow S \otimes S^!_0 \xrightarrow{\partial_1} S \otimes S^!_1 \xrightarrow{\partial_2} S \otimes S^!_2 \xrightarrow{\partial_3} \cdots$$

where each ∂_i is given by right multiplication by $e = \sum x_i \otimes \xi_i$. Give each $S^!_n$ the ordered basis described in Proposition 4.5. Then $\partial_1(1 \otimes 1) = \sum x_i \otimes \xi_i = \sum x_i(1 \otimes \xi_i)$, and so ∂_1 is given by right multiplication by $\mathbf{x}$.

Similarly, using (4.4), the second row of the matrix N is obtained from the calculation

$$\begin{aligned} \partial_2(1 \otimes \xi_1) &= x_0 \otimes \xi_1\xi_0 + x_2 \otimes \xi_1\xi_2 + x_3 \otimes \xi_1\xi_3 \\ &= x_0(1 \otimes \xi_1\xi_0) + x_2\Big(1 \otimes \tfrac{1}{2}(\gamma-1)\xi_3\xi_0 - 1 \otimes \tfrac{1}{2}(\gamma+1)\xi_0\xi_3\Big) \\ &\qquad + x_3\Big(1 \otimes \tfrac{1}{2}(1-\beta)\xi_0\xi_2 + 1 \otimes \tfrac{1}{2}(1+\beta)\xi_2\xi_0\Big). \end{aligned}$$

Similar expressions for the other $\partial_2(1 \otimes \xi_j)$, give the remaining rows of N.

For completeness, we describe the other maps in $(K'_\bullet, \partial)$. Give $S^!_n$ the ordered basis in (4.5). Then $\partial_2 : S \otimes S^!_2 \to S \otimes S^!_3$ is given by right multiplication by the matrix

$$N' = \begin{pmatrix} x_0 & \frac{1+\beta}{2}x_3 & \frac{\gamma-1}{2}x_2 & 0 \\ \frac{\alpha-1}{2}x_3 & x_0 & \frac{1+\gamma}{2}x_1 & 0 \\ \frac{1+\alpha}{2}x_2 & \frac{\beta-1}{2}x_1 & x_0 & 0 \\ 0 & \frac{1-\gamma}{1+\gamma}\frac{1-\beta}{2}x_3 & \frac{1+\gamma}{2}\frac{1+\beta}{1-\beta}x_2 & x_1 \\ \frac{1+\gamma}{1-\gamma}\frac{1+\alpha}{2}x_3 & 0 & \frac{1-\alpha}{1+\alpha}\frac{\gamma-1}{2}x_1 & \frac{1+\alpha}{1-\beta}\frac{1+\gamma}{1-\gamma}x_2 \\ \frac{1+\gamma}{\gamma-1}\frac{\alpha+1}{2}x_2 & \frac{1-\gamma}{1+\gamma}\frac{1-\beta}{2}x_1 & 0 & \frac{1+\alpha}{1-\gamma}x_3 \end{pmatrix}.$$

Similarly, $\partial_3 : S \otimes S^!_3 \to S \otimes S^!_4$ is given by right multiplication by the matrix

$$\begin{pmatrix} x_1 & \frac{1+\alpha}{1-\beta}\frac{1+\gamma}{1-\gamma}x_2 & \frac{1+\alpha}{1-\gamma}x_3 & -x_0 \end{pmatrix}^T.$$

4.12. By the remarks made in (1.9), the results of this section also apply in two of the degenerate cases, namely (1.9.2) and (1.9.3). This is achieved by re-indexing the $\{x_i : i \geq 1\}$. The elements given in Proposition 4.5 will still form a basis for $S^!$, although the details of the proof must be changed a little to avoid a coefficient of the form 0/0.

4.13. We end this section by completing the discussion of the degenerate cases covered by Lemmas 1.4 and 1.6. Thus we may assume that $\alpha = 0$ and $-\gamma = \beta \neq -1$. Then, either by Lemma 1.4 or by Lemma 1.6, S is isomorphic to an iterated Ore extension. It follows that S has a PBW basis, in the sense of [Pr, §5] and hence, by [Pr, Theorem 5.3], S is a Koszul algebra. By [Ma1, Theorem 6], this implies that $(K_\bullet, d) \to {}_Sk \to 0$ is exact. Thus Corollary 1.7 and Proposition 4.10 combine to prove:

Corollary. *If α, β, γ satisfy the hypotheses of either (1.4) or (1.6), then the Sklyanin algebra $S(\alpha, \beta, \gamma)$ is a regular graded algebra.*

5. EXACTNESS OF THE KOSZUL COMPLEX.

5.1. In this section we complete the proof of Theorem 0.3 of the introduction. The idea of the proof is to pull back information from the geometric ring $B = B(\alpha, \beta, \gamma)$, defined in Section 3, to provide information about the Sklyanin algebra $S = S(\alpha, \beta, \gamma)$. The main point is to prove that the augmented Koszul complex $(K_\bullet, d) \to k \to 0$ of Corollary 4.7 is exact, and hence that S is a 4-dimensional regular algebra, in the sense of (0.1). Curiously, though, this requires one to prove simultaneously that S has the same Hilbert series as the commutative polynomial ring on 4 indeterminates, and that the central elements Ω_1, Ω_2 form a regular sequence.

5.2. Throughout this section, $(K_\bullet, d)$ will stand for the Koszul complex $(K_\bullet(S), d)$ defined in (4.2). As remarked in (4.3), the right hand end of the augmented complex $(K_\bullet, d) \to k \to 0$ is exact, and we begin this section by proving the exactness of the left hand end of the complex $B \otimes_S (K_\bullet, d)$.

Lemma. *Assume that α, β, γ satisfy (1.9.1) or (1.9.4). Then:*

1. *If $i \neq j$ then $x_iB + x_jB \supset B_2$.*
2. *The ring B is an Ore domain.*

Proof. The proof is given for the case $i = 0, j = 1$; the other cases are similar. Set $I = x_0B + x_1B$. By Corollary 3.9, B is a homomorphic image of S. Thus one may use the relations (0.2.2) to prove that

$$I \supset \{x_2x_0,\, x_2x_1,\, x_2x_3,\, x_3x_0,\, x_3x_1,\, x_3x_2\}.$$

By Corollary 3.9 the images of Ω_1 and Ω_2 in B are both zero, so $\Omega_1, \Omega_2 \in I$. It follows that $\{x_2^2, x_3^2\} \subset I$, and hence that $B_2 \subset I$.

By Theorem 3.4, B is a domain of finite Gelfand-Kirillov dimension and so, by [KL, Theorem 4.12, p.52], B is an Ore domain; that is any two non-zero left ideals of B have non-zero intersection.

5.3 Proposition. *Assume that α, β, γ satisfy (1.9.1) or (1.9.4). Then the sequence of left B-modules*

$$0 \longrightarrow B \xrightarrow{d} B^{(4)} \xrightarrow{d} B^{(6)},$$

at the left hand end of $B \otimes_S (K_\bullet, d)$ is exact.

Proof. Choose bases for $B^{(n)}$ as in Lemma 4.11. Then the map $B \xrightarrow{d} B^{(4)}$ is right multiplication by $\mathbf{x} = (x_0, x_1, x_2, x_3)$. Since B is a domain, this map is injective. The map $B^{(4)} \xrightarrow{d} B^{(6)}$ is right multiplication by the matrix N defined in Lemma 4.11. Suppose that $\mathbf{u} = (u_1, u_2, u_3, u_4) \in B^{(4)}$ satisfies $\mathbf{u}N = 0$. Since the entries in N are all linear we may assume that all the u_i are homogeneous of the same degree.

Sublemma. *There exist non-zero homogeneous* $s, t \in B$ *such that* $s\mathbf{u} = t\mathbf{x}$.

Proof of the sublemma. Since $\mathbf{x}N = 0$, we can freely replace $\mathbf{u}$ by $s\mathbf{u} - t\mathbf{x}$, for any $s,\, t \in B \setminus \{0\}$. Since B is an Ore domain, we can assume, by making such a change, that $u_3 = 0$. Then, from the last two columns of N, it follows that $u_1x_3 = \lambda u_2x_0 \in u_2B$, and $u_1x_2 = \nu u_2x_1 \in u_2B$ for some $\lambda, \nu \in k \setminus \{0\}$. Hence $u_1(x_3B + x_2B) \subset u_2B$ and therefore, by Lemma 5.2, $u_1B_2 \subset u_2B$. By Theorem 3.4(v) this implies that $u_1 = u_2b$ for some $b \in B$. Thus $u_2(bx_3 - \lambda x_0) = 0$. Since B is a domain, $u_2 = 0$. Hence $u_1 = 0$. Finally, from the first column of N, it follows that $u_0x_1 = 0$, whence $u_0 = 0$. Thus $\mathbf{u} = 0$, and the sublemma is true.

Now return to the proof of the theorem, and choose $s, t \in B$ as in the sublemma. Since $\mathbf{x} = (x_0,\, x_1,\, x_2,\, x_3)$, this implies that $tB_1 \subseteq sB$. By Theorem 3.4(v), this implies that $t = sa$ for some $a \in B$. Thus, $s(\mathbf{u} - a\mathbf{x}) = 0$. Since B is a domain, $\mathbf{u} = a\mathbf{x}$, as required.

5.4. We now prove the main theorem. As in (3.4), we adopt the convention that a k-vector space will be denoted by an upper case letter, and its dimension will be denoted by the corresponding lower case letter. For example, the polynomial ring in 4 commuting variables will be written $C = k[z_0,\, z_1,\, z_2,\, z_3]$, graded by giving each z_i degree one. Thus $c_n = \dim_k C_n$.

Theorem. *Assume that* $\{\alpha,\, \beta,\, \gamma\}$ *satisfy (1.9.1) or (1.9.4), and let* $S = S(\alpha,\, \beta,\, \gamma)$ *be the corresponding Sklyanin algebra. Then*

(i) the augmented Koszul complex $(K_\bullet(S), d) \to {}_Sk \to 0$ *is exact;*

(ii) S *has the same Hilbert series as the polynomial ring* $C = k[z_0,\, z_1,\, z_2,\, z_3]$*;*

(iii) $\{\Omega_1, \Omega_2\}$ *is a regular sequence in* S*; that is,* Ω_1 *is a non-zero divisor in* S *and* Ω_2 *is a non-zero divisor in* $S/S\Omega_1$.

Remark. By Corollary 3.9, the elements $\Omega_1,\, \Omega_2$ are central, homogeneous elements of degree 2, and $B = S/(S\Omega_1 + \Omega_2)$. For applications elsewhere, we will prove the theorem with the centrality of the Ω_i replaced by the weaker assumption that $(\Omega_1,\, \Omega_2)$ is merely a normalising sequence; that is, $S\Omega_1 = \Omega_1 S$ and $A\Omega_2 = \Omega_2 A$, where $A = S/S\Omega_1$.

Since the Ω_i are assumed to be homogeneous, this still implies that $S_n\Omega_1 = \Omega_1 S_n$ and $A_n\Omega_2 = \Omega_2 A_n$ for each $n \geq 0$. We will use this observation several times, without comment.

Proof. The theorem is proved by rephrasing each assertion in terms of the homology groups of certain complexes and then using a simultaneous induction to prove that these groups are all zero. First, by Corollary 4.7, one has the graded complex

$$0 \longrightarrow S_{n-4} \xrightarrow{d_1} S_{n-3}^{(4)} \xrightarrow{d_2} S_{n-2}^{(6)} \xrightarrow{d_3} S_{n-1}^{(4)} \xrightarrow{d_4} S_n \xrightarrow{\epsilon} k_n \longrightarrow 0 \qquad (5.4.1)$$

By Corollary 4.7, again, only the first three homology groups, $P_n = \mathrm{Ker}(d_1)$, $Q_n = \mathrm{Ker}(d_2)/\mathrm{Im}(d_1)$ and $R_n = \mathrm{Ker}(d_3)/\mathrm{Im}(d_2)$ can be nonzero. Thus,

$$s_{n-4} - 4s_{n-3} + 6s_{n-2} - 4s_{n-1} + s_n - \delta_{n,0} \;=\; p_{n-4} - q_{n-3} + r_{n-2}.$$

Since the Koszul complex for the polynomial ring C is exact,

$$c_{n-4} - 4c_{n-3} + 6c_{n-2} - 4c_{n-1} + c_n - \delta_{n,0} \;=\; 0.$$

Thus, setting $d_n = s_n - c_n$, it follows that

$$d_{n-4} - 4d_{n-3} + 6d_{n-2} - 4d_{n-1} + d_n \;=\; p_{n-4} - q_{n-3} + r_{n-2}. \qquad (5.4.2)$$

Remember, we are assuming that (Ω_1, Ω_2) is a normalising sequence of homogeneous elements of S of degree two and that $B = S/S\Omega_1 + S\Omega_2$. Thus, write

$$A \;=\; S/S\Omega_1, \qquad Z \;=\; \ell\text{-}ann_S(\Omega_1) \;=\; \{s \in S : s\Omega_1 = 0\} \quad \text{and} \quad W \;=\; \ell\text{-}ann_A(\Omega_2).$$

Then multiplication by Ω_1 on S, and by Ω_2 on A gives exact sequences

$$0 \longrightarrow Z_{n-2} \longrightarrow S_{n-2} \xrightarrow{\bullet\Omega_1} S_n \longrightarrow A_n \longrightarrow 0$$

$$0 \longrightarrow W_{n-2} \longrightarrow A_{n-2} \xrightarrow{\bullet\Omega_2} A_n \longrightarrow B_n \longrightarrow 0.$$

Thus $z_{n-2} - s_{n-2} + s_n - a_n = 0$ and $w_{n-2} - a_{n-2} + a_n - b_n = 0$. But, $b_n = c_n - 2c_{n-2} + c_{n-4}$, by Theorem 3.4(vi). These three equations imply that

$$w_{n-2} \;-\; z_{n-4} \;+\; d_{n-4} \;-\; 2d_{n-2} \;+\; z_{n-2} \;+\; d_n \;=\; 0. \qquad (5.4.3)$$

In order to determine these integers, we will induct on the subscript, and the crucial inductive steps are provided by the next result.

Sublemma. *Consider the complex*

$$0 \longrightarrow A \xrightarrow{d_1} A^{(4)} \xrightarrow{d_2} A^{(6)}$$

at the left hand end of the complex $A \otimes_S (K_\bullet, d)$. Write U and V for its (graded) first and second homology groups. Then, for any $j \in \mathbb{Z}$, the following implications hold.

$$w_{j-1} = u_{j-2} = 0 \quad \Rightarrow \quad u_j = 0 \qquad (5.4.4)$$

$$u_j = z_{j-1} = p_{j-2} = 0 \quad \Rightarrow \quad p_j = 0 \qquad (5.4.5)$$

$$w_{j-1} = v_{j-2} = 0 \quad \Rightarrow \quad v_j = 0 \qquad (5.4.6)$$

$$v_j = z_{j-1} = q_{j-2} = 0 \quad \Rightarrow \quad q_j = 0 \qquad (5.4.7)$$

$$w_{j-1} = z_{j-1} = u_{j-2} = p_{j-2} = 0 \quad \Rightarrow \quad u_j = p_j = 0 \qquad (5.4.8)$$

$$w_{j-1} = z_{j-1} = v_{j-2} = q_{j-2} = 0 \quad \Rightarrow \quad v_j = q_j = 0 \qquad (5.4.9)$$

proof. By Lemma 4.11, the map d_1 of (5.4.1) is given by multiplication by the element $\mathbf{x} = (x_0, x_1, x_2, x_3)$. Thus $P = \operatorname{Ker} d_1 = \{s \in S : sS_1 = 0\} = \operatorname{Soc} S_S$. Similarly, $U = \operatorname{Soc} A_A$.

(5.4.4) Let $u \in U_j$. Then $uA_1 = 0$. But, by Theorem 3.4, B is a domain; so $u = u'\Omega_2$ for some $u' \in A_{j-2}$. Thus $0 = u'\Omega_2 A_1 = u'A_1\Omega_2$, and so $u'A_1 \subseteq W_{j-1} = 0$. Hence $u' \in U_{j-2} = 0$, and so $u = 0$.

(5.4.5) Let $p \in P_j$. Then $[p + S\Omega_1]/S\Omega_1 \in U_j = 0$. Therefore $p = p'\Omega_1$ for some $p' \in S_{j-2}$. Thus $0 = pS_1 = p'\Omega_1 S_1 = p'S_1\Omega_1$, whence $p'S_1 \subseteq Z_{j-1} = 0$. Hence $p' \in P_{j-2} = 0$, and so $p = 0$.

(5.4.6) Let $\mathbf{v} = (\nu_1, \nu_2, \nu_3, \nu_4) \in V_j = \operatorname{Ker}\left\{d_2 : A_j^{(4)} \to A_{j+1}^{(6)}\right\}$. We must prove that $\mathbf{v} \in \operatorname{Im}(d_1)$. Since $0 \longrightarrow B \xrightarrow{d_1} B^{(4)} \xrightarrow{d_2} B^{(6)}$ is exact (see Proposition 5.3), $\overline{\mathbf{v}} = [\mathbf{v} + \Omega_2 A^{(4)}]$ is in the image of $B \xrightarrow{d_1} B^{(4)}$. Thus Lemma 4.11 implies that $\overline{\mathbf{v}} = (bx_0, bx_1, bx_2, bx_3) = b\mathbf{x}$ for some $b \in B_{j-1}$. Let $a \in A_{j-1}$ be a preimage of b. Then $\mathbf{v} = a\mathbf{x} + \Omega_2\mathbf{v}'$, for some $\mathbf{v}' \in A_{j-2}^{(4)}$. Now

$$0 = d_2(\mathbf{v}) = d_2(a\mathbf{x} + \Omega_2\mathbf{v}') = d_2(\Omega_2\mathbf{v}') = \Omega_2 d_2(\mathbf{v}').$$

By hypothesis, $W_{j-1} = \{t \in A_{j-1} : t\Omega_2 = 0\} = 0$ and so $\dim A_{j-1}\Omega_2 = \dim A_{j-1}$. But, as $A_{j-1}\Omega_2 = \Omega_2 A_{j-1}$, this implies that $\dim \Omega_2 A_{j-1} = \dim A_{j-1}$ and hence that $\{t \in A_{j-1} : \Omega_2 t = 0\} = 0$. Since $d_2(v') \in A_{j-1}$, the last displayed equation therefore implies that $d_2(v') = 0$. Thus $\mathbf{v}' \in V_{j-2} = 0$. Hence $\mathbf{v} = a\mathbf{x} \in \operatorname{Im}(d_1)$, as required.

(5.4.7) This is similar to (5.4.6), so the details are left to the reader.

(5.4.8) Just combine (5.4.4) and (5.4.5).

(5.4.9) Just combine (5.4.6) and (5.4.7).

Now return to the proof of the theorem. We must prove, for all j, that

(1) $p_j = q_j = r_j = 0$,
(2) $d_j = 0$,
(3) $w_j = z_j = 0$.

The method of proof is induction. We say that $H(m)$ holds if the following equalities are true for all $i \geq 0$:

$$H(m) \quad 0 = p_{m-i} = q_{m-i} = u_{m-i} = v_{m-i} = z_{m-i} = w_{m-i} = r_{m-i} = d_{m+2-i}$$

If $\beta = a, b, \ldots, z$, observe that $\beta_n = 0$ for $n < 0$ and, except for $\beta = d$, that $\beta_n \geq 0$. Thus $H(m)$ does hold if $m \leq -3$. Suppose that $H(m)$ is true, for some m. Then (5.4.8) and (5.4.9), with $j = m + 1$, imply that

$$p_{m+1} = q_{m+1} = u_{m+1} = v_{m+1} = 0.$$

Similarly, if $n = m + 3$, then (5.4.2) implies that $d_{m+3} = r_{m+1} \geq 0$, while (5.4.3) implies that $d_{m+3} = -(w_{m+1} + z_{m+1}) \leq 0$. It follows that $d_{m+3} = r_{m+1} = w_{m+1} = z_{m+1} = 0$. Hence $H(m+1)$ is true, and the theorem is proved.

5.5. Finally, we can combine the results of the paper to prove Theorem 0.3 of the introduction.

Theorem. *Assume that k is any field (not necessarily algebraically closed) of characteristic not equal to 2. Let $S = S(\alpha, \beta, \gamma)$ be defined as in (0.2). Assume that $\{\alpha, \beta, \gamma\}$ is not equal to $\{-1, +1, \gamma\}$, $\{\alpha, -1, +1\}$ or $\{+1, \beta, -1\}$. Then:*

(i) S is a regular graded algebra of dimension 4.

(ii) $H_S(t)$ equals the Hilbert series of a commutative polynomial ring in 4 variables.

(iii) S is a Noetherian domain.

(iv) S is a Koszul algebra.

For other values of α, β, γ, S has many zero-divisors.

Proof. Assume, first, that k is algebraically closed. Then Corollary 1.3 implies that S has zero-divisors in the excluded cases. If one of α, β, γ is zero, but Corollary 1.3 is not applicable, then the result follows from Corollary 1.7 combined with Corollary 4.13. All other possibilities are covered by the cases mentioned in (1.9). In these cases, parts (i), (ii) and (iv) of the theorem follow from Theorem 5.4.

The Noetherian property may be proved by mimicking the proof of [ATV1, §8], although since we have assumed throughout the paper that k is algebraically closed, a couple of comments are in order. The idea of the proof in [ATV1] is as follows. First, in order to prove that S is Noetherian, it suffices to show that B is Noetherian (see [ATV1, Lemma 8.2]). Secondly, by the usual kind of reduction, one may assume that this ring $B(k)$ is constructed from the geometric data $(E, \sigma, \mathcal{L})$, defined over a finite field k (see [ATV1, Lemma 8.9], and the comments before that lemma). Now, if $\bar{k}$ is the algebraic closure of k, then the corresponding geometric ring $B(\bar{k})$ defined over $\bar{k}$ is isomorphic to $B(k) \otimes \bar{k}$. Since this is a faithfully flat extension, in order to prove that $B(k)$ is Noetherian, it suffices to show that $B(\bar{k})$ is Noetherian. By [ATV1, Lemma 8.5], this will be the case if the automorphism σ has finite order. However, since E is an elliptic curve, the group of translations is a normal subgroup of $\mathrm{Aut}(E)$ of finite index (use [Ha, Corollary 4.7 and Lemma 4.9, pp.321-322]). Thus, replacing σ by σ^n, for some n, we may assume that σ is given by translation by some point $p \in E$. But, p is a k_1-rational point on E for some finite subfield k_1 of $\bar{k}$. Since there are only finitely many k_1-rational points on E, this certainly implies that p and hence σ has finite order. Thus S is Noetherian. Finally, [ATV2, Theorem 3.9] implies that S is a domain.

Now consider the case when k is not algebraically closed. Then $\bar{S} = S \otimes_k \bar{k}$ is a faithfully flat extension of S for which the assertions of the theorem hold. It follows easily that the theorem also holds for S (that S is regular follows from the standard change of rings theorem [Ro, Theorem 11.56, p.360]).

Remark. It is clear from the comments at the end of Section 2 that the condition (0.2.1) on the scalars $\{\alpha, \beta, \gamma\}$ is a necessary ingredient in Sklyanin's construction of S in terms of theta functions. Moreover, this condition has been used in most of the computations of Sections 2, 3 and 4. A natural question is, therefore, what is the structure of S if one does not assume (0.2.1)? The answer is that the ring collapses. More formally, let $S' = S'(\alpha, \beta, \gamma)$ be the k-algebra, with the same generators and relations as those of S, but such that $\{\alpha, \beta, \gamma\}$ are now algebraically independent over the prime subfield of k. Then computer calculations show that, for large n, one has $dim_k(S'_n) \leq 20 \ll dim_k(S_n) = O(n^3)$ and, moreover, S' has many zero-divisors.

5.6. One of the aims of this paper has been to show that the techniques developed in [ATV1] to deal with regular rings of dimension three have direct analogues in dimension four. For example, the results of this section show that, for rings with four generators, one can always pull back results about regularity and Hilbert series from an appropriately nice factor ring. For completeness, we note the (rather large) number of conditions required of a ring A for it to satisfy the conclusions of Theorem 5.5. Assume that:

(i) A is a graded k-algebra with 4 generators and quadratic relations.

(ii) The dual algebra $A^!$ has the same Hilbert series as the exterior algebra $\Lambda(k^{(4)})$.

(iii) There exists an elliptic curve $E \subset \mathbb{P}^3$ with an automorphism σ and corresponding geometric ring B for which there exists an isomorphism $\phi : A_1 \to B_1$ that induces a homomorphism $\phi : A \to B$. (Of course, the obvious way to find such a ϕ is through the methods of Section 2.)

(iv) By Theorem 3.4, Ker(ϕ) is generated by two elements, say $\Omega_1, \Omega_2 \in A_2$. Assume that (Ω_1, Ω_2) form a normalising sequence.

(v) Proposition 5.3 holds for B.

Then A is a Noetherian Koszul algebra of dimension 4, with the same Hilbert series as a polynomial ring in 4 variables (in this generality, one needs an unpublished result of Artin and van den Bergh in order to prove that A is Noetherian). If A is also Gorenstein (which would follow if $A^!$ were Frobenius, with an appropriate antiautomorphism, as in (4.4)), then A will be a domain.

5.7. In [OF1] and [OF2] various algebras are constructed from the data $\{E, \sigma, \mathcal{L}\}$, where σ is translation by a point on the elliptic curve E and $\mathcal{L}$ is a line bundle of degree n. These algebras are generated by n elements and, when $n = 4$, they are the Sklyanin algebras $S(\alpha, \beta, \gamma)$. When $n \geq 5$, these algebras seem to be less amenable to our methods, in particular because the kernel of the corresponding map $\phi : S \to B$ will not be generated by a regular sequence. However, we expect that they will still be Noetherian, regular domains of dimension n, with the same Hilbert series as the polynomial ring in n variables.

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