

Part A.

Short answer questions

- (1) Choose values for a and b so that $\begin{pmatrix} a & 1 & 0 \\ 1 & b & 2 \\ -1 & 1 & 1 \end{pmatrix}$ is singular.

There are infinitely many solutions to this problem: any pair (a, b) such that a is non-zero and $b = 2 + \frac{3}{a}$ will work because

$$-\frac{3}{a}(a, 1, 0) + (1, 2 + \frac{3}{a}, 2) - 2(-1, 1, 1) = (0, 0, 0).$$

But you only need to find one solution, so let me show you how to do that. A matrix is singular if its rows are linearly dependent, so you must choose a and b so that happens. Let's write the rows as \underline{v}_1 , \underline{v}_2 , and \underline{v}_3 . Looking at the third column, notice that $0 = 2 - 2 \times 1$, so let's pick a and b so that $\underline{v}_1 + \underline{v}_2 - 2\underline{v}_3 = 0$. Looking at the second column, we need $1 + b - 2 \times 1 = 0$, so $b = 1$. Looking at the first column, we need $a + 1 - 2(-1) = 0$, so $a = -3$. That is ONE solution to the problem.

At the end of these notes I show how to find all solutions.

- (2) Choose values for a and b so that the columns of the matrix in the previous question form a basis for \mathbb{R}^3 .

There are infinitely many correct answers to this, any values of a and b except those in the previous question will work. Any three linearly independent vectors in \mathbb{R}^3 form a basis for it, so we must pick a and b so that the columns (and/or the rows) of the matrix are linearly independent. For example, take $a = b = 0$.

- (3) How many equations and how many unknowns are there in the system of linear equations whose augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 3 & 6 & 4 & 2 & 1 \\ 0 & 0 & 4 & -3 & -1 \end{pmatrix}?$$

3 equations and 4 unknowns.

- (4) Compute A^{-1} when

$$(AB)^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B^T = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$$

Since $(AB)^{-1} = B^{-1}A^{-1}$, $A^{-1} = B(AB)^{-1} = (B^T)^T(AB)^{-1}$. So take the transpose of B^T , then form the product $B(AB)^{-1}$.

- (5) If the following system of equations in the unknowns x, t, w is written as a matrix equation $A\underline{x} = \underline{b}$, what are A , \underline{x} , and \underline{b} ?

$$\begin{aligned} 2x + 3w - t &= -1 \\ t + 2 - x - w &= 0 \\ 2x + 3t &= 1 + 2t \end{aligned}$$

You should rewrite the system so all the unknowns are on the left of the = signs and the constants are on the right of the = signs:

$$2x + 3w - t = -1$$

$$t - x - w = -2$$

$$2x + t = 1$$

This can be rewritten as the matrix equation

$$\begin{pmatrix} 2 & 3 & -1 \\ -1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ w \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

So

$$A = \begin{pmatrix} 2 & 3 & -1 \\ -1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x \\ w \\ t \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

- (6) Write down the system of linear equations you need to solve in order to find the parabola $y = ax^2 + bx + c$ passing through the points $(-4, 2)$, $(1, -3)$, $(2, 0)$.

Plugging in the first point gives $2 = 16a - 4b + c$. Plugging in the second point gives $-3 = a + b + c$. Plugging in the third point gives $0 = 4a + 2b + c$. Those are the three equations.

Part B.

Complete the definitions.

You do not need to write down the part I have already written.

Just complete the sentence.

- (1) A subset $\{\underline{v}_1, \dots, \underline{v}_d\}$ of \mathbb{R}^n is linearly dependent if ...
if there are numbers $a_1, \dots, a_d \in \mathbb{R}$, not all zero, such that $a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_d\underline{v}_d = \underline{0}$.
- (2) An $n \times n$ matrix A is non-singular if ...
the only solution to the equation $A\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$.
- (3) A subset $S = \{\underline{v}_1, \dots, \underline{v}_d\}$ of \mathbb{R}^n is a basis for \mathbb{R}^n if ...
it spans \mathbb{R}^n (i.e., every element in \mathbb{R}^n is a linear combination of elements of S) and the expression for each element of \mathbb{R}^n as a linear combination of elements of S is unique.
A shorter version of this: every element of \mathbb{R}^n can be written as a linear combination of the elements in S in exactly one way.
- (4) The dimension of a subspace $W \subset \mathbb{R}^n$ is ...
the number of elements in a basis for W .
- (5) A subset W of \mathbb{R}^n is a subspace if ...
it contains $\underline{0}$, and whenever $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} \in W$, and whenever $\underline{u} \in W$ and $a \in \mathbb{R}$ then $a\underline{u} \in W$.
- (6) The linear span $\langle \underline{v}_1, \dots, \underline{v}_d \rangle$ of a subset $\{\underline{v}_1, \dots, \underline{v}_d\} \subset \mathbb{R}^n$ is ...
the set of all vectors $a_1\underline{v}_1 + \dots + a_d\underline{v}_d$ as a_1, \dots, a_d take all possible values in \mathbb{R} .

Part C.

True or False—just write T or F

- (1) The matrix $\begin{pmatrix} a & 1 \\ -2 & b \end{pmatrix}$ is non-singular except when $ab = -2$.
T
- (2) Every set of five vectors in \mathbb{R}^4 is linearly dependent.
T
- (3) If a subset of \mathbb{R}^4 spans \mathbb{R}^4 it is linearly independent.
F For example, if $\underline{e}_1, \dots, \underline{e}_4$ are the “usual” basis vectors for \mathbb{R}^4 , then $\{\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4, \underline{e}_1 + \underline{e}_2\}$ spans \mathbb{R}^4 but is linearly dependent.
- (4) A homogeneous linear system of 15 equations in 16 unknowns always has a non-zero solution.
T
- (5) If S is a linearly dependent subset of \mathbb{R}^n so is every subset of \mathbb{R}^n that contains S .
T
- (6) Every subset of a linearly independent set is linearly independent.
T
- (7) A square matrix is singular if and only if its transpose is singular.
T
- (8) $2I - I^2 + I^{-1}$ is singular.
F Notice that $I^2 = I$ and $I^{-1} = I$, so this expression simplifies to $2I - I + I = 2I$ and the inverse of this is $\frac{1}{2}I$.
- (9) $2I - I^2 - I^{-1}$ is singular.
T because it is $= 0$.
- (10) The set $W = \{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4 = 0\}$ is a subspace of \mathbb{R}^4 .
T
- (11) The set $W = \{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a subspace of \mathbb{R}^4 .
F It doesn't contain $\underline{0}$.
- (12) The solutions to a system of linear equations always forms a subspace.
F The solutions to a **homogeneous** system of linear equations is a subspace.
- (13) If $\langle \underline{u}, \underline{v}, \underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle$, then \underline{u} is a linear combination of \underline{v} and \underline{w} .
T
- (14) If \underline{u} is a linear combination of \underline{v} and \underline{w} , then $\langle \underline{u}, \underline{v}, \underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle$.
T
- (15) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in \mathbb{R}^n , then $\{\underline{v}_1 - 2\underline{v}_2, 2\underline{v}_2 - 3\underline{v}_3, 3\underline{v}_3 - \underline{v}_1\}$ is linearly dependent.
T because $1(\underline{v}_1 - 2\underline{v}_2) + 1(\underline{v}_2 - 3\underline{v}_3) + 1(3\underline{v}_3 - \underline{v}_1) = \underline{0}$.
- (16) The null space of an $m \times n$ matrix is contained in \mathbb{R}^n .
T
- (17) The range of an $m \times n$ matrix is contained in \mathbb{R}^n .
F No it is contained in \mathbb{R}^m because $A\underline{x}$ is an $m \times 1$ matrix when $\underline{x} \in \mathbb{R}^n$.
- (18) If V and W are subspaces of \mathbb{R}^n such that $V \subset W$, then $V \cup W$ is a subspace of \mathbb{R}^n .
T because $V \cup W = W$ in this case.
- (19) Let A and B be $n \times n$ matrices. If A is singular so is AB .

T First recall that a matrix is singular if and only if it does not have an inverse. Second, if a matrix has an inverse so does its transpose. here we are assuming A is singular, so it has no inverse, whence A^T is singular. Hence there is a non-zero vector \underline{x} such that $A^T \underline{x} = 0$. It follows that $B^T A^T \underline{x} = 0$, so $B^T A^T$ is singular. But $B^T A^T = (AB)^T$, so we conclude that $(AB)^T$ is singular, and hence AB is singular.

- (20) Let A and B be $n \times n$ matrices. If B is singular so is AB .

T There is a non-zero \underline{x} such that $B\underline{x} = 0$, whence $(AB)\underline{x} = A(B\underline{x}) = A\underline{0} = \underline{0}$.

- (21) If A and B are non-singular $n \times n$ matrices, so is AB .

T Non-singularity is equivalent to having an inverse, so AB has an inverse $(BA)^{-1}$. Now $AB(AB)^{-1} = I$ and $B(AB)^{-1}A = I$, so $B(AB)^{-1}$ is the inverse of A , whence A is non-singular.

- (22) If A and B are non-singular $n \times n$ matrices, so is $A + B$.

F for example, take $A = I$ and $B = -I$.

- (23) Let A be an $n \times n$ matrix. If the rows of A are linearly dependent, then A is singular.

T

- (24) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

T

- (25) There is a matrix whose inverse is $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$.

- (26) If $A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

F

- (27) Any linearly independent set of five vectors in \mathbb{R}^5 is a basis for \mathbb{R}^5 .

T

- (28) A square matrix B such that $B^{13} = 0$ must be singular.

T

- (29) The row space of the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a basis for \mathbb{R}^3 .

T

Part A, Question 1. Now I will show you how to find all solutions to that problem. We need to pick a and b so that there are numbers $c, d, e \in \mathbb{R}$, not all zero, such that $c\underline{v}_1 + d\underline{v}_2 + e\underline{v}_3 = 0$. A calculation shows that

$$c\underline{v}_1 + d\underline{v}_2 + e\underline{v}_3 = (ca + d - e, c + db + e, 2d + e).$$

We therefore need to find $c, d, e \in \mathbb{R}$, not all zero, such that

$$(ca + d - e, c + db + e, 2d + e) = (0, 0, 0).$$

Notice this is impossible if $d = e = 0$ because that would then force $c = 0$ (because $c + db + e = c$). From the equation $2d + e = 0$, it follows that both d and e are non-zero. Because $2d + e = 0$, $e = -2d$. Now the equations $ca + d - e = 0$ and $c + db + e = 0$ become

$$ca + 3d = 0, \quad c + db - 2d = 0.$$

Notice that because $d \neq 0$, the first of these equations now implies that $c \neq 0$. The first equation tells us that $d = -\frac{1}{3}ac = -\frac{ac}{3}$. Substituting this into the second equation we get

$$0 = c + db - 2d = c + (b - 2)d = c - (b - 2)\frac{ac}{3} = c\left(1 - \frac{(b - 2)a}{3}\right).$$

Since $c \neq 0$, the only way this can be zero is if

$$1 - \frac{(b - 2)a}{3} = 0.$$

This gives $b = 2 + \frac{3}{a}$. So the full set of solutions is

$$(a, b) = \left(a, 2 + \frac{3}{a}\right), \quad a \neq 0.$$

Some general comments on the exam.

Use the symbols correctly. For example, write \mathbb{R}^n , not $\mathbb{R}n$ or Rn or R^n . Also the symbol “ \in ” means “is an element of” or “belongs to”. Thus, if you write “ \underline{x} is $\in W$ ” this reads as “ \underline{x} is an element of W .” It is good to use the standard notation: for example, write $16a$ rather than $a16$. I was often confused when someone wrote \underline{x} then spoke as if \underline{x} were a number; the underlining is used to denote a vector, not a number.

Read the questions carefully. For example, in Part A you were not asked to solve the systems of equations in questions 5 and 6. You were not asked in question 3 to write out the equations, or solve them. In Part C, the true/false questions, you were not asked to give reasons for your answers, simply the answers.

Part B, the definitions, was the most challenging part of the exam. I think I will give more definitions on the final exam. The definitions are very, very important.

You must distinguish between the definition of a term, and a result about that term. For example, the book defines an $n \times n$ matrix A to be non-singular if the only solution to the equation $A\underline{x} = 0$ is $\underline{x} = 0$. There is a *theorem* saying that a square matrix is non-singular if and only if its columns are linearly independent; but that is a *result* about non-singular matrices, not the definition of a non-singular matrix.

Your definitions must be precise. I suggest you write down in one place all the definitions that have arisen in this course, using the exact wording the book uses, or I used when I wrote the definitions on the blackboard. Then practice writing these out several times until you have them memorized. If you like, I would be happy to look at the definitions you write down and tell you whether they are correct. You might also find it helpful to show me your answers to Part B and receive some feedback. I will ask for perhaps as many 10 definitions on the final exam.

The reason I did not ask for proofs on the midterm is because with a one hour exam there simply wasn't sufficient time. However, asking for the definitions addressed some of the same issues, namely the necessity for clarity, precision, brevity, and sentences that are grammatically correct. I think that if you can get to the point where your definitions are clear, accurate, and succinct you will be getting close to the point where you can write down proofs that are also clear, accurate, and succinct.

For question 4 on Part B, the definition of the dimension of a subspace W , some people said it was p and $W = \{\underline{w}_1, \dots, \underline{w}_p\}$, but what that means is that W has exactly p elements in it, whereas W always contains infinitely many elements unless $W = 0$. I think what you intended to say/write was $W = \langle \underline{w}_1, \dots, \underline{w}_p \rangle$ which is the correct notation for saying that “ W is the linear span of $\underline{w}_1, \dots, \underline{w}_p$ ”. Some people said the dimension of W is the number of elements in W , but W has infinitely many elements. Also the dimension is the number of elements in *a* basis for W , not in *the* basis for W ; the latter suggests that W has only one basis, but it has infinitely many.

Few people used the “if...then...” terminology when defining a subspace. For example, one of the conditions a subset W of \mathbb{R}^n must satisfy in order to be a subspace is the following: if \underline{u} and \underline{v} are in W , then $\underline{u} + \underline{v} \in W$. You could also say this as “ $\underline{u} + \underline{v} \in W$ whenever $\underline{u}, \underline{v} \in W$ ”.

Quite a few of the definitions people wrote were long and muddled. Read your own sentences and see if they make sense.

When discussing a subspace W of \mathbb{R}^n someone spoke of the “rows of W ”. A matrix has rows but a subspace does not.

Some people in discussing whether a matrix were non-singular said “ A is a basis for \mathbb{R}^n ” when what they should have said is “the columns of A are a basis for \mathbb{R}^n ”. Remember, that a basis consists of vectors, and a matrix is not a vector (unless $m = 1$ or $n = 1$); the *columns* of A are vectors.