Math 308 **Midterm Answers** **November 4, 2011**

Instructions. There are three parts to the exam. You will get 7 points for turning up to the exam and filling in your name and student ID **correctly** and **legibly** on the scantron and on the blue book, green book, or other work you hand in, *and* for writing neatly. *I will subtract from those 7 points if I have any difficulty reading your writing.*

Part A consists of questions that require a short answer. There is no partial credit and no need to show your work. Part A is worth a total of 21 points, 1 point per question.

Part B is worth 38 points, 2 points per question.

Part C consists of true/false questions. Use the scantron/bubble sheet with the convention that A =True and B =False. Part C is worth 34 points - you will get

- +1 for each correct answer,
- -1 for each incorrect answer, and
- 0 for no answer at all.

The maximum possible score for the test is 100 points (7 of which are freebies if you provide your name, section, and ID number, and write neatly). **If you score 65 or more you have a pretty good understanding of the material.**

Part A.

Short answer questions

(1) What is the 23-entry in the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 3 & 6 & 4 & 2 & 1 \\ 0 & 0 & 5 & -3 & -1 \\ 1 & 2 & 3 & 0 & 0 \end{pmatrix}?$$

Answer: 4

Comments: The 23-entry means the entry in row 2 and column 3.

(2) The ij-entry in the 9×9 matrix A is $i - j$. What is the 46-entry in its transpose?

Answer: 2

Comments: The transpose is defined in such a way that 46-entry in A^T is the 64-entry in A which is $6 - 4 = 2$. I did not accept $6 - 4$ as an answer because that answer leaves the grader to do the work. Similarly, if the answer to a question is 2 but you write $\sqrt{9 \times 19 - 13 \times 13 + 20 \times 5 - 2 \times 7 \times 7}$, you are leaving the grader to do your work for you.

(3) How many equations and how many unknowns are there in the system of linear equations whose augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 0 & | & 2 \\ 3 & 6 & 2 & 1 & | & 4 \\ 0 & 0 & -3 & -1 & | & 6 \\ 1 & 3 & 0 & 0 & | & 2 \\ 2 & 6 & 0 & 0 & | & 3 \end{pmatrix}?$$

Answer: Five equations, 4 unknowns

Comments: Everyone got this correct. Good.

(4) Why is the system of equations in the previous question inconsistent? You do NOT need to perform any elementary row operations to answer this question, just look carefully at the rows and think of the corresponding equations.

Answer: The 4^{th} row corresponds to the equation $x_1 + 3x_2 = 2$. The 5^{th} row corresponds to the equation $2x_1 + 6x_2 = 3$. However, if $x_1 + 3x_2 = 2$, then $2x_1 + 6x_2 = 2(x_1 + 3x_2) = 4$, not 3. So there is no solution to the system.

Comments: Mine is not the only correct answer. If you subtract twice the 4^{th} row from the 5^{th} row the 5^{th} row then has a 1 to right of the vertical line and 0s in the rest of the row. That is one criterion for the system to be inconsistent. Variations of this and my answer above are correct.

(5) Write down the system of linear equations you need to solve in order to find the curve $y = ax^2 + bx + c$ passing through the points $(2, 1)$, $(1, 3)$, $(-1, 1)$.

Answer: The problem is to find a, b, and c such that

$$\begin{aligned} 4a + 2b + c &= 1 \\ a + b + c &= 3 \\ a - b + c &= 1. \end{aligned}$$

Comments: Everyone got this correct. :) I did get one answer where the first equation was given as $a \cdot 4 + b \cdot 2 + c = 1$. That should be your own private work and when revealed to the public should be presented as $4a + 2b + c = 1$.

(6) Write the system of linear equations in the previous question as a matrix equation $A\underline{x} = \underline{b}$. What are A, $\underline{x}$, and $\underline{b}$?

Answer:

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad \underline{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Comments: The only error was that some people wrote

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{instead of} \qquad \underline{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(7) I want a geometric description of the solutions: the set of solutions to a 2×4 system of linear equations is either
   (a) the _________ set
   (b) or _________ in ___
   (c) or _________ in ___
   (d) or $\underline{\mathbb{R}^?}$

Answer: The empty set, or a plane in $\mathbb{R}^4$, or a 3-plane in $\mathbb{R}^4$, or $\mathbb{R}^4$.

Comments: Several people said *"An empty set"*, but there is only one empty set so they should say *"the empty set"*; of course, I had already given you the clue in (a) by writing *the _________ set.*

Some people said *"zero set"* or *"trivial set"*. That's not right. We call the empty set the empty set. I agree that the empty set is pretty trivial, and I agree that the number of elements in it is zero. Nevertheless, it is called the empty set. Compare with an *"empty chair"*. It is not the same as the trivial chair or the zero chair (whatever they might be).

A common error was to give an answer which said *"a plane through the origin"* and/or a *"a 3-plane containing the origin"* but the question does not say the system of equations is homogeneous so the set of solutions need not be a subspace. For example, the solutions to the equations $x_1 = 1$ and $x_2 = 2$, the points on the plane $\{(1, 2, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\}$ and this is *not* a subspace because it does not contain $\underline{0}$.

One proposed answer that appeared several times really puzzled me, namely *"a plane in $\mathbb{R}^3$"*. If anyone could help me understand what lies behind such a response I would be very grateful. There are 4 unknowns so all solutions are points in $\mathbb{R}^4$. Similarly, a couple of people proposed *"a line in $\mathbb{R}^2$"* as an answer. Perhaps I just need to place a little more emphasis that a solution to a system of equations in n unknowns is a point in $\mathbb{R}^n$. This is true even outside the realm of linear equations. For example every solution to the system of equations $x_1x_2x_3x_4 = 3$ and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5$ is a point in $\mathbb{R}^4$.

A line in $\mathbb{R}^4$ can NOT be the solution to a 2×4 system of linear equations. You need 3 linear equations to specify a line in $\mathbb{R}^4$ (compare with a line in $\mathbb{R}^3$ which requires two equations to specify it). To get a line you need exactly 1 independent variable. Or, in more familiar terms, a line is given by a set of parametric equations with exactly *one* parameter, e.g., the line $(1 + 2t, 3 - t, 4 + 3t, 5t)$, $t \in \mathbb{R}$. The number of independent variables for an $m \times n$ system $A\underline{x} = \underline{b}$ is $n - \text{rank}(A)$. But $\text{rank}(A) \leq m$ and $\leq n$, so for a 2×4 system, $\text{rank}(A) \leq 2$, and the number of independent variables is $\geq 4 - 2 = 2$.

Another way to see this is to think in terms of dimension even though we haven't yet got a formal definition of that. One linear equation usually drops the dimension by 1 so, in $\mathbb{R}^4$, the solution to one linear equation is a 3-plane (except in exceptional situations), and the solution to two linear equations is a

2-plane (except in exceptional situations in which it is $\mathbb{R}^4$, or a 3-plane, or the empty set).

(8) The set of solutions to a 4×4 system of linear equations is one of the four possibilities in the previous answer
(a) or a _________ in ___
(b) or a _________ in ___

Answer: A point in $\mathbb{R}^4$, or a line in $\mathbb{R}^4$.

Comments: As explained in the comments on the previous question, the line need not contain $\underline{0}$.

(9) I want a geometric description of the possibilities for the range of a 2×4 matrix. The range of a 2×4 matrix is either
(a) _________
(b) or _________ in ___
(c) or $\underline{\mathbb{R}^?}$

Answer: The zero vector $\underline{0}$, or a line in $\mathbb{R}^2$, or $\mathbb{R}^2$ itself.

Comments: The range of A is $\mathcal{R}(A) = \{A\underline{x} \mid \underline{x} \in \mathbb{R}^4\}$.

In contrast to the previous two questions, the range of a matrix *is* a subspace–we proved it in class. Check the proof if you are not sure why it is a subspace.

The most common error was saying that the range is contained in $\mathbb{R}^4$. It is contained in $\mathbb{R}^2$ because when A is 2×4, the $\underline{x}$ in $A\underline{x}$ is a 4×1 matrix and $A\underline{x}$ is a 2×1 matrix, i.e., an element in $\mathbb{R}^2$.

Some people gave *"a 2-plane in $\mathbb{R}^2$"* as a possibility. *But* there is only one 2-plane in $\mathbb{R}^2$, namely $\mathbb{R}^2$ itself. This error makes me think a good question for the final would be to ask how many 3-planes there are in $\mathbb{R}^3$.

You don't need to say *"The zero vector $\underline{0}$"* as I do. You could simply write $\{\underline{0}\}$. But it would not be correct to write 0 because 0 denotes the number 0, not the vector $\underline{0}$. It is also wrong to write $\{\emptyset\}$ when you mean $\{\underline{0}\}$. The notation $\{\emptyset\}$ means *the set containing the empty set*. Notice, for example, that $\{\emptyset\}$ is a set having one element, that element being the empty set. Likewise, $\{\emptyset, \{\emptyset\}\}$ has two elements. The set $\{\{1, 2, 3, 4\}, \{6, 7, 8, \}\}$ has two elements.

(10) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form _________ where _________

Answer: $\underline{u} + \underline{v}$ where $\underline{v}$ is a solution to the equation $A\underline{x} = \underline{0}$.

Comments: One proposed answer said *"$(\underline{u} + \underline{v})A = \underline{b}$ where $A\underline{v} = \underline{0}$"*. If the products $A\underline{u}$ and $A\underline{v}$ make sense, then the product $(\underline{u} + \underline{v})A$ will never make sense unless A, $\underline{u}$, and $\underline{v}$, are 1×1 matrices, i.e., numbers. It is OK to say *"$\underline{u} + \underline{v}$ where $A\underline{v} = \underline{0}$"*.

Another proposed answer said *"$A(\underline{u}+\underline{v}) = \underline{b}$ where $A\underline{v} = \underline{0}$"*. But read the question carefully: it says *"every solution [...] is of the form"* so you are supposed to talk about the solution, not the equation.

(11) In the previous question let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. If $A\underline{u} = \underline{b}$, then

$$T = \{\ldots \mid \ldots\}.$$

Your answer should involve $\underline{u}$ and S and the symbol $\in$ and some more.

Answer: $T = \{\underline{u} + \underline{v} \mid \underline{v} \in S\}$.

Comments: One proposed answer was $\{x \mid u \in \mathbb{R}$ and $v \in \mathbb{R}\}$. In translation, this says *"the set of all x such that u and v are real numbers"*. When we use set notation $\{P \mid Q\}$ the P is usually a collection of things and Q a condition that places restrictions on that collection. For example, the set of women with two or more brothers would be written

$$\{\text{women } x \mid x \text{ has at least two brothers}\}.$$

In particular, what is placed where the Q is refers back to what is placed where P is. As another example, we define the null space of a matrix A as $\{\underline{x} \mid A\underline{x} = \underline{0}\}$. Some people would be more strict than I am and say that the null space of an $m \times n$ matrix A is $\{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{0}\}$, i.e., the set of all elements $\underline{x}$ in $\mathbb{R}^n$ such that $A\underline{x} = \underline{0}$. Another problem with the proposed answer is that u and v should be vectors in $\mathbb{R}^n$, and therefore underlined; they are not in $\mathbb{R}$ unless A is an $m \times 1$ matrix.

Another proposed answer was $T = \{A(\underline{u} + \underline{x}) \mid \underline{x} \in S\}$. Just delete the A to obtain a correct answer.

One proposed answer was $T = \{S + \underline{v} \mid \underline{v} \in \mathbb{R}\}$. The write should try reading that: it says T *is the set consisting of the elements $S + \underline{v}$ as $\underline{v}$ runs over all real numbers."* It is hard to know where to start in saying something helpful: $S + \underline{v}$ denotes the sum of a set and a vector. But we add vectors to vectors, not to sets; $\underline{v} \in \mathbb{R}$ says that $\underline{v}$ is a real number.

(12) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \ldots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

Answer: $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$

Comments: It is important to take care. For example, whoever wrote *"$\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ where $x \in \mathbb{R}$"* could produce a correct answer with a little more care. He/she need only put an A before the $\underline{x}$ on the left and, if they wish to say more about $\underline{x}$, could add *"where*

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and $x_1, \ldots, x_n \in \mathbb{R}$." In fact, we should probably all tell the reader that $x_1, \ldots, x_n$ are the entries of $\underline{x}$. I have been a little lazy because I always adopt the convention that when $\underline{x}$ and $x_1, \ldots, x_n$ appear in the same paragraph the x_is are assumed to be the entries of $\underline{x}$ and I don't say that explicitly.

Some people wrote $A\underline{x} = \underline{A}_1 x_1 + \cdots + \underline{A}_n x_n$. Although not really incorrect it appears clunky and goes against convention. In high school you would write an equation as $y = 3x^2 + 2x - 7$ not as $y = x^2 3 + x2 - 7$. The latter is clunky and goes against convention. It is also open to some confusion when you write things like $x^3 3$ or $x7y$. Get in the habit of writing $x_1 \underline{A}_1 + \cdots + x_n \underline{A}_n$. The numbers, constants, coefficients, are almost always written to the left of other things like matrices, vectors, unknowns. Think about your street cred.

(13) The matrix $\begin{pmatrix} x & t \\ r & s \end{pmatrix}$ is invertible if and only if ____________.

Answer: $xs - rt \neq 0$

Comments: You MUST know the answer to this and the next question. A few people said *"non-singular"* but I wanted a something with more content. The formula gives a computation that can be carried out whereas saying *non-singular* is sort of passing the buck.

(14) If $\begin{pmatrix} x & t \\ r & s \end{pmatrix}$ is invertible its inverse is ____________.

Answer:

$$\frac{1}{xs - rt} \begin{pmatrix} s & -t \\ -r & x \end{pmatrix}$$

Comments:

(15) Let $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ be the columns of a 4×4 matrix A and suppose that $2\underline{A}_1 + 2\underline{A}_2 = \underline{A}_3 - 3\underline{A}_4$. Write down a solution

$$\underline{x} = \begin{pmatrix} ? \\ 2 \\ ? \\ ? \end{pmatrix}$$

to the equation $A\underline{x} = \underline{0}$.

Answer: $\underline{x} = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 3 \end{pmatrix}$

Comments: You are told that $2\underline{A}_1 + 2\underline{A}_2 - \underline{A}_3 + 3\underline{A}_4 = \underline{0}$ Since $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$, and you are asked to find $\underline{x}$ such that $A\underline{x} = \underline{0}$ and $x_2 = 2$, the answer is obvious.

Part B.

Complete the definitions.

There is a difference between theorems and definitions. Here I am asking for the definition.

Don't write the part of the definition I have already written. Just fill in the blank.

(1) A matrix A is non-singular if the only ______________

Answer: solution to $A\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$.

Comments: One proposed answer was *"solution is $\underline{x} = \underline{0}$"* but that doesn't make sense because the writer does not mention the equation. Whenever you use the word *solution* it must be clear to the reader what equation you are talking about. Solutions belong to equations. Compare this to the phrase *"son is good"*; it must be clear which son you are talking about.

Another proposed answer was *"solution to the matrix is the θ vector."* A matrix does not have a solution. An equation involving a matrix might or might not have a solution. The equation $A\underline{x} = \underline{0}$ has a solution but A does not have a solution. To speak of a solution there must be an equation in the background, and equations *always* involve the (sacred) $=$ sign.

We are using the symbol $\underline{0}$ for the zero vector, not θ.

(2) The system of equations $A\underline{x} = \underline{b}$ is homogeneous if _________

Answer: $\underline{b} = \underline{0}$.

(3) Two systems of linear equations are equivalent if _________

Answer: they have the same solutions.

Comments: The answer *"they have the same set of solutions"* is correct but it can be shorter—the words *"set of"* add no information. It is a bit like saying *"every football player on the football team played well"*—you convey the same information by saying *"everyone on the football team played well"*.

One proposed answer was *"one is a linear combination of the others"* but the subject of the sentence is "Two systems" and we can not form a linear combination of *two systems*. We can form a linear combination of two, or more, *vectors*.

(4) The linear span of $\underline{v}_1, \dots, \underline{v}_n$ is _________

Answer: all linear combinations of $\underline{v}_1, \dots, \underline{v}_n$.

Comments: You don't need to say "all *possible* linear combinations of $\underline{v}_1, \dots, \underline{v}_n$." The word *possible* does not add information; all is all.

One proposed answer was $\langle \underline{v}_1, \dots, \underline{v}_n \rangle$. That is a *notation* for the linear span of $\underline{v}_1, \dots, \underline{v}_n$, quite a common notation even though we are not using it, but it is not the *definition* of the linear span.

One incorrect solution said, in its entirety, *"$a_1v_1 + \cdots + a_nv_n$ for $a \in \mathbb{R}$"*. That person should write $\underline{v}_1$, etc., i.e., the symbol for the vector should be underlined. Also there is no symbol a preceding the mathematical phrase "$a \in \mathbb{R}$" so the a in "$a \in \mathbb{R}$" is not referring to anything. What should have been written was "$a_1, \dots, a_n \in \mathbb{R}$". But even that is not sufficient because the proposed definition would then read

> The linear span ... is $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for $a_1, \ldots, a_n \in \mathbb{R}$

which does not quite get across the idea that the linear span is *all* linear combinations. One could give a short answer along the proposed lines by writing

> The linear span ... is $\{a_1\underline{v}_1 + \cdots + a_n\underline{v}_n \mid a_1, \ldots, a_n \in \mathbb{R}\}$

because the set symbol $\{\quad\}$ may be interpreted as *"the set of all"*.

Someone who probably knew the answer wrote *"the linear combinations of the vectors* $\underline{v}$. Presumably what was meant was *"the linear combinations of the vectors* $\underline{v}_1, \ldots, \underline{v}_n$. Alas, you are being graded on the basis of what you say, not what you intend to say. The reader is not expected to fill in the gaps after guessing what the writer intends to say.

One too brief answer was "$a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$". Is this person meaning that the linear span consists of a single vector?

(5) $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly dependent if ________

Answer: $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ for some numbers $a_1, \ldots, a_n$ that are not all equal to zero.

Comments: Alternatively, you could say: the equation $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ has a non-trivial solution.

Some said "$\underline{v}_i = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ *for some* i *and some* $a_1, \ldots, a_n \in \mathbb{R}$" but this is wrong on two counts. First, there is a *Theorem* that says

> *A set of vectors* $\{\underline{v}_1, \ldots, \underline{v}_n\}$ *is linearly dependent if and only if some* $\underline{v}_i$ *is a linear combination of the other* $\underline{v}_j$*s.*

That is a consequence of the definition, not the definition itself. Second, it is always true that $\underline{v}_i = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some i and some $a_1, \ldots, a_n \in \mathbb{R}$! For example, given any $\underline{v}_1, \ldots, \underline{v}_4$, $\underline{v}_2 = 0.\underline{v}_1 + 1.\underline{v}_2 + 0.\underline{v}_3 + 0.\underline{v}_4$. Notice the statement of the theorem has the word *other* in it. That is important. You would have to write $\underline{v}_i = a_1\underline{v}_1 + \cdots + a_{i-1}\underline{v}_{i-1} + a_{i+1}\underline{v}_{i+1} + \cdots + a_n\underline{v}_n$ for some i and some $a_1, \ldots, a_n \in \mathbb{R}$ in order to indicate that you are omitting the $\underline{v}_i$ term on the right-hand side of the $=$ sign.

One proposed solution said *"if there is non-zero* a_i *such that* $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$". Does the writer mean one a_i is non-zero, or all are non-zero, or some a_i is non-zero. (And, the zero vector is denoted by $\underline{0}$, not 0.) A similar error appeared in the answer " $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ *for non-zero* $\{a_1, \ldots, a_n\}$". It is not clear if the writer is requiring that all a_is be non-zero or just that some a_i is non-zero. Also, the set symbol adds confusion in that answer–get rid of it.

(6) A matrix E is in echelon form if
(a) ________
(b) ________
(c) ________

Answer: all rows consisting entirely of 0s are below the non-zero rows; the left-most non-zero entry in each non-zero row is a 1 (we call it a *leading 1*); each leading 1 is to the right of the leading 1 in the row above it.

Comments: It is difficult to express this definition in a short, comprehensible, and correct way. Almost everyone could benefit from trying to write it out half-a-dozen times. Few answers were perfect. For example, most people used the term *"leading 1"* without defining it or saying what it means. Even if your answer was graded as correct you would benefit from polishing, shortening, and perfecting your answer. Please try it.

One proposed answer for (b) was *"All non-zero rows have a left-most leading 1"*. This is not quite correct because it uses the term *"left-most leading 1"* before that term has been defined. When you write a definition put yourself in the shoes of someone who is seeing your definition for the first time. I do not think the proposed answer conveys the correct idea. Carry out the following experiment with a friend who knows no mathematics. Say to him/her *"I want you to write down a list of 8 numbers in a row from left to right. At least one of those numbers must be non-zero and the first non-zero entry in your list must be a 1."* Then say to him/her *"I want you to write down a list of 8 numbers in a row from left to right. At least one of those numbers must be non-zero and the row must have a left-most leading 1."* I expect the second set of instructions will cause confusion. Try it!

Another proposed answer that suffers from the same defect is *"get the sum of the two rows"*. Put yourself in the place of someone who has never seen the definition before. Does *get* mean the same as *fetch*. What do I do after I *get* the sum of the two rows. There is no room for confusion if one says *"replace* R_i *by* $R_i + R_j$*"* (provided the reader knows that R_i denotes the i^{th} row, and I assuming the reader does know that.

We try to make definitions easy to understand. Often that is impossible, but it must be our goal.

Another proposed answer for (b) was *"the first entry in each non-zero row is 1."* This is not correct because *first* means *first*: the first entry in the row $(0\ 0\ 1\ 2)$ is 0. You must say *"the first non-zero entry in each non-zero row is 1."*

(7) A matrix is in row reduced echelon form if it is ________ and ________ .

Answer: in echelon form and all other entries in the columns that contain a leading 1 are 0.

Comments: The comments on the previous question about making your definition precise so it is comprehensible to someone who is reading it for the first time applies to this question too. For example, one proposed answer included the words *"the only entry in a column is the leading one of a row"*. Which column, which row? And surely the writer means the only *non-zero* entry.

(8) In this question use R_i to denote the i^{th} row of a matrix. The three elementary row operations are

(a) _________
(b) _________
(c) _________

Answer: swap R_i with R_j; replace R_i by cR_i where c is a non-zero number; replace R_i by $R_i + R_j$.

Comments: My answers are complete and as short as is possible. For example, you do not need to include *"replace R_i by $R_i + cR_j$"* because that operation is either doing nothing when $c = 0$ or, when $c = 0$ can be performed by replacing R_j by cR_j, then replacing R_i by $R_i + R_j$.

You do not need to say *"replace R_i by $R_i + R_j$ where $j \neq i$"* because replacing R_i by $R_i + R_i$ is allowed (and is the same as replacing R_i by $2R_i$) and by saying *"replace R_i by $R_i + R_j$"* you are allowing the possibility that $i \neq j$.

Many answers would be improved by using verbs *"swap"*, *"switch"*, or *"replace"*, instead of the symbol $\leftrightarrow$. *I* know what you mean, but when I ask for a definition I want it to be "book-perfect". When you use an active verb like *"swap"*, *"switch"*, or *"replace"*, there is no doubt about what you mean. Absolutely no doubt.

(9) A matrix A is invertible if there is a matrix B such that _________

Answer: $AB = BA = I$.

(10) A vector $\underline{x}$ is a linear combination of $\underline{v}_1, \ldots, \underline{v}_n$ if _________

Answer: $\underline{x} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$.

Comments: It is important to include the words *"for some"*. It is incorrect to say *"for some $(a_1, \ldots, a_n) \in \mathbb{R}$"* because the parentheses indicate that$(a_1, \ldots, a_n)$ is a vector and as such belongs to $\mathbb{R}^n$, not $\mathbb{R}$. It would be OK to say *"for some $(a_1, \ldots, a_n) \in \mathbb{R}^n$"*.

I would also accept the answer *$\underline{x}$ is in the linear span of $\underline{v}_1, \ldots, \underline{v}_n$* provided you gave a correct answer to question (4) (which asked for the definition of the linear span).

One valiant attempt was *"$\underline{x}$ is an element of the sums $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$*. The phrase *"an element of the sums"* doesn't make sense. A *set* has elements but a *sum* doesn't. One way to correct it would be to use the phrase *"an element of the set of all sums"*. A better alternative would be to simplify the answer: *"$\underline{x}$ is equal to $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$"*.

(11) Let A be an $m \times n$ matrix. Then
(a) $\mathcal{N}(A) = \{\cdots \mid \cdots\}$ and
(b) $\mathcal{R}(A) = \{\cdots \mid \cdots\}$.

Answer: $\mathcal{N}(A) = \{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{0}\}$ and $\mathcal{R}(A) = \{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}$.

(12) A subset W of $\mathbb{R}^n$ is a subspace if

(a) ________
(b) ________
(c) ________

Answer: $\underline{0} \in W$; $\underline{u}+\underline{v} \in W$ whenever $\underline{u}$ and $\underline{v}$ are in W; $c\underline{u} \in W$ for all $c \in \mathbb{R}$ and $\underline{u} \in W$.

Comments: By using the words *"such that"* where I use the word *"wherever"* you create a grammatically incorrect sentence; furthermore, the meaning is obscure.

Using *"where"* in place of *"whenever"* makes a sentence that is difficult to understand. The definition must convey the idea that if $\underline{u}$ and $\underline{v}$ are in W so is $\underline{u}+\underline{v}$.

A couple of people said *"whenever $\underline{u}$ and $\underline{v}$ are solutions in W"* but elements of W are *not* solutions (they might be sometimes). Also, whenever you speak of solutions it must be clear what they are solutions to. In this definition no equation is mentioned so the reader will befuddled by the appearance of the word *solutions*.

Some people confused the symbols $\in$ and $\subset$. In one proposed answer, (b) was stated as "$\underline{u} \subset W$, $\underline{v} \subset W$; $\underline{u}+\underline{v} \subset W$". Apart from the use of the subset symbol $\subset$ there must be something, perhaps a word or two, in the answer that indicates a condition and conclusion, e.g., *"if $\underline{u} \in W$ and $\underline{v} \in W$, then $\underline{u}+\underline{v} \in W$."*

One proposed answer for (b) was *"There exist vectors $\underline{u}$ and $\underline{v}$ in W such that $\underline{u}+\underline{v} \in W$.* This fails because of the words *"There exist"*; (b) must state a property about *all* $\underline{u}$ and $\underline{v}$ in W. For example, the set $W = \{(0,0),(0,1)\} \subset \mathbb{R}^2$ satisfies the condition that there exist vectors $\underline{u}$ and $\underline{v}$ in W such that $\underline{u}+\underline{v} \in W$; take $\underline{u}=(0,0)$ and $\underline{v}=(0,1)$, for example; but this W is not closed under addition; for example, $(0,1)+(0,1) \notin W$.

Part C.

True or False

The three numbers after the answer are the % of students who were correct, wrong, did not answer. For example, **40, 25, 35** says that 40% gave the correct answer, 25% gave the wrong answer, and 25% did not answer the question.

(1) If a system of linear equations has five different solutions it must have infinitely many solutions.

True. [**100, 0, 0**] Yeah!

Comments:

(2) The matrix $\begin{pmatrix} a & b^2 \\ 3 & b \end{pmatrix}$ is singular if $a = 3b$ or $b = 0$ and non-singular otherwise.

True. [**69.6, 24.6, 5.8**]

Comments: The matrix is singular if and only if its determinant is zero. The determinant of this matrix is $ab - 3b^2 = (a-3b)b$. Since $(a-3b)b = 0$ if and only if $a - 3b = 0$ or $b = 0$, the statement is true.

(3) A homogeneous system of 7 linear equations in 8 unknowns always has a non-trivial solution.

True. [**81.2, 15.9, 2.9**]

Comments: There is a theorem that says a system of homogeneous equations having more unknowns than equations always has a non-trivial solution.

(4) The set of solutions to a system of 5 linear equations in 6 unknowns can be a 3-plane in $\mathbb{R}^6$.

True. [**68.1, 13.0, 18.8**]

Comments: For example, take the system $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_1 + x_2 = 0$, $x_1 + x_3 = 0$. The independent variables are x_4, x_5, and x_6; they can take on any values, but we must have $x_1 = x_2 = x_3 = 0$. The set of solutions to this system is a 3-plane in $\mathbb{R}^6$.

(5) If $B^3 - 2B^2 + 5B = 2I$, then $\frac{1}{2}(B^2 - 2B + 5I)$ is the inverse of B.

True. [**34.8, 18.8, 46.3**]

Comments: The equation $B^3 - 2B^2 + 5B = 2I$ can be rewritten as $\frac{1}{2}(B^2 - 2B + 5I)B = I$ and as $B.\frac{1}{2}(B^2 - 2B + 5I) = I$. I think most of you were scared off by what appeared like a complicated equation. However, if you simply factor the left-hand side you see it is a multiple of B on both sides. The left-hand side is of the form BC and of the form CB so the equation is telling you that $BC = CB = 2I$ so $\frac{1}{2}C$ is B^{-1}.

(6) The rank of a matrix is the number of non-zero rows in it.

False. [**53.6, 44.9, 1.4**]

Comments: The rank of a matrix is the number of non-zero rows *in its row reduced echelon form*, and this might be different. For example, the row reduced echelon form of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ so

$$\operatorname{rank} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1.$$

The rank of a matrix is also the number of linearly independent rows in it.

(7) Row equivalent matrices have the same rank.

True. [**76.8, 11.6, 11.6**]

Comments: Row equivalent matrices have the same row reduced echelon form and hence the same rank.

(8) Row equivalent matrices have the same row reduced echelon form.

True. [**78.3, 15.9, 5.8**]
Comments:

(9) Let A and B be $n \times n$ matrices. If A and B are invertible then $(AB)^{-1} = A^{-1}B^{-1}$.

False. [**66.7, 29.0, 4.3**]

Comments: The inverse of AB is $B^{-1}A^{-1}$. To see this just compute,

$$ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$$

Remember that in general AB is not the same as BA so $ABA^{-1}B^{-1}$ need not equal $BAA^{-1}B^{-1}$.

(10) There are more than 8 subspaces of $\mathbb{R}^3$.

True. [**52.2, 27.5, 20.3**]

Comments: The subspaces of $\mathbb{R}^3$ are
(a) the zero subspace $\{\underline{0}\}$ consisting only of the vector $\underline{0}$;
(b) every line through the origin;
(c) every plane through the origin;
(d) $\mathbb{R}^3$ itself.
There are infinitely many lines through the origin so although there are only four *types* of subspaces in $\mathbb{R}^3$, there are infinitely many subspaces.

(11) If A and B are non-singular $n \times n$ matrices, so is AB.

True. [**94.2, 1.4, 4.3**]

Comments: A matrix is non-singular if and only if it has an inverse so the statement is equivalent to the statement *If A and B are invertible $n \times n$ matrices, so is AB.* The latter statement is true because $B^{-1}A^{-1}$ is the inverse of AB.

(12) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.

True. [**87.0, 5.8, 7.2**]

Comments: Let $\underline{A}_1, \dots, \underline{A}_n$ be the columns of A. Since $A\underline{x} = x_1\underline{A}_1 + \dots + x_n\underline{A}_n$ the equation $A\underline{x} = \underline{b}$ is equivalent to the equation

$$x_1\underline{A}_1 + \dots + x_n\underline{A}_n = \underline{b}$$

and this has a solution if and only if $\underline{b}$ is a linear combination of $\underline{A}_1, \dots, \underline{A}_n$.

(13) Let A be an $n \times n$ matrix. If the rows of A are linearly dependent, then A^T is singular.

True. [**81.2,7.2,11.6**]

Comments: We proved the following 3 conditions on a square matrix A are equivalent:
(a) A is non-singular;
(b) A has an inverse;
(c) the columns of A are linearly independent.
Replace A by A^T in this to see that the following 3 conditions are equivalent:
(a) A^T is non-singular;
(b) A^T has an inverse;
(c) the columns of A^T are linearly independent.
But the columns of A^T are the rows of A so the following 3 conditions are equivalent:
(a) A^T is non-singular;
(b) A^T has an inverse;
(c) the rows of A are linearly independent.
It follows that A^T is singular if and only if the rows of A are linearly dependent.

(14) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

True. [**95.7, 2.9, 1.4**]

Comments: Good. It is important to know that if B is obtained from A by a single elementary row operation, then A can be obtained from A by a single elementary row operation (that undoes the first operation).

(15) There is a matrix whose inverse is $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix}$.

False. [**68.1, 23.2, 8.7**]

Comments: The columns of the matrix are linearly dependent because

$$\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

This question might have looked tricky because you did not recognize that *a square matrix A has an inverse if and only if A is the inverse of a matrix.*

This is simply because the symmetry in the definition: *A has an inverse if and only if there is a matrix B such that $AB = BA = I$.* The equation $AB = BA = I$ says that B is the inverse of A *and* that A is the inverse of B.

(16) If $A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

False. [**65.2, 24.6, 10.1**]

Comments: The product BA only makes sense if B is an $m \times 2$ for some m but in that case BA is an $m \times 2$ matrix. Since E is a 2×3 matrix, E can not equal BA.

(17) If $B^{13} - B = 0$, then B singular.

False. [**36.2, 29.0, 34.8**]

Comments: The identity matrix has an inverse but $I^{13} = I$ so $I^{13} - I = 0$.

(18) A matrix can have more than one inverse.

False. [**94.2, 2.9, 2.9**]

Comments: If B and C are inverses of A, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

(19) If $\underline{u}$, $\underline{v}$, and $\underline{w}$, are any vectors in $\mathbb{R}^4$, then $\{3\underline{u} - 2\underline{v}, 2\underline{v} - 4\underline{w}, 4\underline{w} - 3\underline{u}\}$ is linearly dependent.

True. [**68.1, 7.2, 24.6**]

Comments: $1.(3\underline{u} - 2\underline{v}) + 1.(2\underline{v} - 4\underline{w}) + 1.(4\underline{w} - 3\underline{u}) = \underline{0}$.
You were given very little data. To answer it you must ask whether there were numbers a, b, c such that $a(3\underline{u} - 2\underline{v}) + b(2\underline{v} - 4\underline{w}) + c(4\underline{w} - 3\underline{u}) = \underline{0}$. This equation can be rearranged as $3(a - c)\underline{u} + 2(b - a)\underline{v} + 4(c - b)\underline{w} = \underline{0}$. Once the equation is in this form you can see that $a = b = c = 1$ is a solution.

(20) If

$$A^T = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad (AB)^{-1} = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$$

then

$$B^{-1} = \begin{pmatrix} 0 & -3 \\ -1 & 1 \end{pmatrix}.$$

True. [**44.9, 26.1, 29.0**]

Comments: Since $(AB)^{-1} = B^{-1}A^{-1}$,

$$B^{-1} = (AB)^{-1}A = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -1 & 1 \end{pmatrix}.$$

(21) Let A be a 4×5 matrix and $\underline{b} \in \mathbb{R}^5$. Suppose the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & 3 & 0 & | & 2 \\ 0 & 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The independent variables are x_2 and x_4; they can take any values and all solutions are given by $x_1 = 2 - 2x_2 - x_4$, $x_3 = 2 - 3x_4$, and $x_5 = 0$.

True. **[92.8, 2.9, 4.3]**

Comments: Good to see you can all do this.

(22) If $A = A^{\mathsf{T}}$, then A is a square matrix.

True. **[87.0, 10.1, 2.9]**

Comments: If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix but the only way an $m \times n$ matrix can equal an $n \times m$ matrix is if $m = n$.

This question tests whether you know the definition of the transpose and its effect on the size of a matrix.

(23) If $A = BC$, then every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.

True. **[69.6, 10.1, 20.3]**

Comments: If $C\underline{x} = \underline{0}$, then $A\underline{x} = (BC)\underline{x} = B(C\underline{x}) = B\underline{0} = \underline{0}$.

A lot of you chose not to answer this. It would have been good to simply ask yourself, as the question does, *is* $A\underline{x} = \underline{0}$, and the substitute BC for A.

(24) If $A = BC$, then

$$\{\underline{w} \mid C\underline{w} = \underline{0}\} \subset \{\underline{x} \mid A\underline{x} = \underline{0}\}.$$

True. **[60.9, 8.7, 30.4]**

Comments: This True/False statement is just a restatement of the statement in question (23). It is difficult to see that though if you can't read set notation. First, $\{\underline{w} \mid C\underline{w} = \underline{0}\} \subset \{\underline{x} \mid A\underline{x} = \underline{0}\}$ reads *"the set of all $\underline{w}$ such that $C\underline{w} = \underline{0}$ is a subset of the set of all $\underline{x}$ such that $A\underline{x} = \underline{0}$"*. A shorter way of saying this is *"if $C\underline{w} = \underline{0}$, then $A\underline{w} = \underline{0}$"*, and this is true by the argument in (23).

(25) The linear span of the vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

True. [**56.5, 4.3, 39.1**]

Comments: First, the set of solutions to the equation $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ is a 3-plane in $\mathbb{R}^4$; if you like, x_2, x_3, and x_4, are independent variables. The vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ lie on the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ because they are solutions to the equation $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$. To see that just plug in.

(26) A matrix having a column of zeroes never has an inverse.

True. [**81.2, 13.0, 5.8**]

Comments: Every set of vectors that contains the zero vector is linearly dependent. But the columns of an invertible matrix are linearly independent.

(27) A square matrix is singular if and only if it does not have an inverse.

True. [**76.8, 13.0, 10.1**]

Comments: It is important to remember the 6 equivalent conditions we gave for a matrix to have an inverse.

(28) $\text{span}\{\underline{u}, \underline{v}, \underline{w}\} = \text{span}\{\underline{v}, \underline{w}\}$ if and only if $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$.

True. [**84.1, 2.9, 13.0**]

Comments: If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $\underline{u} = a\underline{v} + b\underline{w}$ for some $a, b \in \mathbb{R}$. Hence

$$\lambda\underline{u} + \mu\underline{v} + \nu\underline{w} = \lambda(a\underline{v} + b\underline{w}) + \mu\underline{v} + \nu\underline{w} = (\lambda a + \mu)\underline{v} + (\lambda b + \nu)\underline{w}.$$

This calculation shows that every linear combination of $\underline{u}$, $\underline{v}$, and $\underline{w}$, is a linear combination of $\underline{v}$, and $\underline{w}$, i.e., $\text{span}\{\underline{u}, \underline{v}, \underline{w}\} \subset \text{span}\{\underline{v}, \underline{w}\}$.

The opposite inclusion $\text{span}\{\underline{v}, \underline{w}\} \subset \text{span}\{\underline{u}, \underline{v}, \underline{w}\}$ is always true because $c\underline{v} + d\underline{w} = 0.\underline{u} + c\underline{v} + d\underline{w}$.

(29) Every set of 8 vectors in $\mathbb{R}^7$ is linearly dependent.

True. [**71.0, 13.0, 15.9**]

Comments: We proved a general result saying that any set of $\geq n+1$ vectors in $\mathbb{R}^n$ is linearly dependent. The proof of that result used the result that a homogeneous system of linear equations always has a non-zero solution if there are more unknowns than equations/

(30) Every system of 3 equations in 8 unknowns has a solution.

False. [**71.0, 17.4, 11.6**]

Comments: The system might be inconsistent. For example, the system of equations $x_1 = 1$, $x_1 = 2$, $x_1 + x_2 + x_8 = 0$, has no solution. If I had said "Every system of 3 *homogenoeus* equations in 8 unknowns has a solution", the answer would be true.

(31) The phrase *"if a matrix is linearly independent"* makes sense.

False. [**78.3, 11.6, 10.1**]

Comments: It only makes sense to speak of a set of vectors being linearly independent.

(32) The linear span of $(1, 2, 3)$ and $(4, 5, 6)$ is a subspace of $\mathbb{R}^3$.

True. **78.3, 11.6, 10.1**
Comments:

(33) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 = x_2 - x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

True. **69.6, 1.4, 29.0**
Comments:

(34) The set $\{x_1, x_2, x_3, x_4) \mid x_1x_3 = x_2x_4\}$ is a subspace of $\mathbb{R}^4$.

False. **43.5, 17.4, 39.1**
Comments:

A final word. If you do any of the following things you are simply throwing away points:

(1) write R instead of $\mathbb{R}$;
(2) do not underline vectors, i.e., write v instead of $\underline{v}$;
(3) write 0 when you mean the vector $\underline{0}$;
(4) call the empty set the zero set;
(5) spell words incorrectly;
(6) write grammatically incorrect sentences;
(7) use a singular noun with a plural verb, or vice versa;
(8)