Math 308 **Midterm Answers and Comments** **July 18, 2011**

Part A.

Short answer questions

(1) Compute the determinant of the matrix $\begin{pmatrix} a & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -a & 3 \end{pmatrix}$.

The determinant is $2a^2 - 12$.

Comments: Everyone seemed to know that the determinant is $a(3+2a) - 3(3+2) + 3(-a+1)$ and that they needed to simplify that expression. But quite a lot of people made errors in that calculation. Please take care. Simplifying the expression is high school algebra (10th grade?). Long before the end of high school you should be able to perform such tasks with the same ease and agility as you ride a bike.

(2) Find the two values for a that make the matrix in question 1 singular.

$\pm\sqrt{6}$

Comments: A square matrix is singular if and only if its determinant is zero. So the two values of a that make the matrix singular are the two values of a that make $2a^2 - 12 = 0$.

(3) How many equations and how many unknowns are there in the system of linear equations whose augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 0 & 2 \\ 3 & 6 & 4 & 2 & 1 & 4 \\ 0 & 0 & 4 & -3 & -1 & 6 \\ 1 & 2 & 3 & 0 & 0 & 0 \end{array}\right)?$$

4 equations in 5 unknowns.

Comments: I think everyone got this correct.

(4) Compute A^{-1} when

$$B^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad (AB)^{-1} = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$$

$\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$

Comments: You must perform the calculation

$$A^{-1} = B(AB)^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix} = \cdots .$$

Notice the statement of the question tells you what B^{T} and you need to take the transpose to get B. Some people made arithmetic errors. Take care.

(5) Write down the system of linear equations you need to solve in order to find the cubic curve $y = ax^3 + bx^2 + cx + d$ passing through the points $(-1, 5)$, $(0, 2)$, $(1, 3)$, $(2, 1)$.

$$\begin{aligned} -a + b - c + d &= 5 \\ d &= 2 \\ a + b + c + d &= 3 \\ 8a + 4b + 2c + d &= 1 \end{aligned}$$

Comments: Almost everyone got this correct. Good. You know how to turn the question into a system of linear equations. You plug in $x = -1$ and $y = 5$ to the equation $y = ax^3 + bx^2 + cx + d$ to get the equation $5 = -a + b - c + d$, and so on with the other three points.

(6) Write the system of linear equations in the previous question as a matrix equation $A\underline{x} = \underline{b}$. What are A, $\underline{x}$, and $\underline{b}$?

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

Comments: Almost everyone got this correct. Good. You know how to turn the system of linear equations in the previous equation into a single matrix equation.

(7) I want a geometric description of the solutions: the set of solutions to a 2×3 system of linear equations is either
   (a) the __________ set
   (b) or __________ in ____
   (c) or __________ in ____
   (d) or $\underline{\mathbb{R}^?}$

   (a) the empty set,
   (b) or a line in $\mathbb{R}^3$,
   (c) or a plane in $\mathbb{R}^3$,
   (d) or $\mathbb{R}^3$.

Comments: Difficult question for almost everyone. That's a problem. It is important to understand that the set of solutions can be thought of as a geometric object. This relation between solutions to an equation and a geometric object is similar to what you learned at high school when considering the graph of equations like $y = 2x^2 - 5x + 3$ or $x^2 + y^2 = 9$ or $2x^2 + xy - y^2 + x = 7$. The points on the graph are those whose coordinates (x, y) provide a solution to the given equation. You also encounter this relationship in calculus, especially Math 126, where you think of the solutions to an equation like $x^2 + 3xy - y^3 + z^2 - 1 = 0$ as forming a surface in $\mathbb{R}^3$.

A 2×3 system of linear equations is one of the form

$$\begin{aligned} ax + by + cz =& d \\ a'x + b'y + c'z =& d' \end{aligned}$$

where $a, b, c, d, a', b', c', d'$ are some numbers and x, y, z are the unknowns. So the solutions are coordinates (x, y, z) of points in 3-space, $\mathbb{R}^3$. The solutions to a single equation of the form $ax + by + cz = d$ form the points on a plane in $\mathbb{R}^3$ (except in 2 cases: when $a = b = c = 0$ and $d \neq 0$ there are no solutions and so the set of solutions is the empty set; when $a = b = c = d = 0$, every $(x, y, z) \in \mathbb{R}^3$ is a solution.

In the question you have two such equations, so the solution set is the intersection of the solutions to $ax + by + cz = d$ and the solutions to $a'x + b'y + c'z = d'$. Typically this is an intersection of two planes which might be empty (when the planes are parallel), or a plane (when the two planes are the same), or a line.

(8) The set of solutions to a 3×3 system of linear equations is one of the four possibilities in the previous answer or a _________ in ___.

a point in $\mathbb{R}^3$.

Comments: Very few people got this correct. The analysis in the previous question applies but now, with 3 equations instead of 2, it is possible that the set of solutions is the intersection of three planes in $\mathbb{R}^3$. An intersection of 3 planes in $\mathbb{R}^3$ can be a single point. That is the typical case. Every point in $\mathbb{R}^3$ is the intersection of 3 planes, hence the answer to this question.

(9) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form _________ where _________

$\underline{u} + \underline{v}$ where $A\underline{v} = \underline{0}$.

Comments: Very few people got this correct. The answer lies in understanding the relation between solutions to $A\underline{x} = \underline{0}$ and solutions to $A\underline{x} = \underline{b}$. We proved this result in class and discussed it detail the case of 1 equation in 2 unknowns: the solutions to $ax + by = 0$ form a line in $\mathbb{R}^2$ passing through the origin and the solutions to a non-homogeneous equation $ax + by = c$ form a line in $\mathbb{R}^2$ that is parallel to $ax + by = 0$. See the discussion on *Homogeneous systems* in section 4.6 of my online notes.

This question and the next are really the same question.

(10) In the previous question let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. If $A\underline{u} = \underline{b}$, then

$$T = \{\dots \mid \dots\}.$$

Your answer should involve $\underline{u}$ and S and the symbol $\in$ and some more.

$T = \{\underline{u} + \underline{v} \mid \underline{v} \in S\}$.

Comments: Less than 5 people got this correct. This question is the same as the previous question but asks for the answer in a different format.

The answer says that T is the set of all elements in $\mathbb{R}^n$ that are obtained by adding $\underline{u}$ to an element of S.

Set-theoretic notation is a wonderful, brief, precise, language for mathematics. You only need to understand a few symbols in this course. Go to the homepage for Math 308 and print off the notes I link to under the heading *Set notation and language*. You should read and absorb those 3 pages. If you have not done that already you have handicapped yourself over the past several weeks. When I say *absorb* I don't mean get a superficial familiarity so that you can just say "oh, yes, I've seen that before". Understand this notation, completely. Test your understanding.

(11) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

$A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$.

Comments: In the answer $x_1, \dots, x_n$ denote the entries (numbers) in $\underline{x}$ with x_1 being the first entry (i.e., the one at the top) and so on.

This is one of the most important formulas in this course.

If you don't have this at your fingertips *and* understand it you will probably fail this course. Here are some incorrect answers:

- $A\underline{x} = a_1x_1 + \cdots + a_nx_n$. That is no good because the answer does not say what $a_1, \dots, a_n$ are. We have never used the notation $a_1, \dots, a_n$ for the columns of the matrix. In fact, my guess is that the notation a_1 and a_n has been used *exclusively for integers*.
- $A\underline{x} = \underline{A}_1\underline{x}_1 + \cdots + \underline{A}_n\underline{x}_n$. That is no good because you do not say what $\underline{x}_1, \dots, \underline{x}_n$ are. In fact, I have always reserved the underline notation $\underline{x}$ for vectors.
- $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ where $\underline{x} \in \mathbb{R}$. But $\underline{x}$ is not in $\mathbb{R}$, it is in $\mathbb{R}^n$.

All these errors are a matter of precision. Someone giving one of these answers has some idea what the answer is, perhaps even knows exactly what the answer is, but must write down the correct answer to get the points. Unfortunately, they do not get points for knowing the correct answer but writing down an incorrect answer.

Look at the comments for question 17 below to gain a better understanding of this question.

(12) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a ________

$\underline{b}$ is a linear combination of the columns of A.

Comments: Most people got this correct. The answers of some people suggested or said that $\underline{b}$ is a number in $\mathbb{R}$. That isn't right: $\underline{b}$ is a vector, an $m \times 1$ matrix. Whenever I underline a symbol $\underline{u}$, $\underline{v}$, $\underline{x}$, $\underline{b}$, the underlined thing is vector. Also, think about how $\underline{b}$ is obtained from the system of equations; $\underline{b}$ is formed from the terms to the right of the equal signs in the m equations

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \qquad 1 \le i \le m,$$

namely

$$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

(13) The matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible if and only if ___________.

$wz - xy \neq 0$.

Comments: I gave no points for the answer: if there is a matrix B such that

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} B = B \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This question is about a particular 2×2 matrix and there is a simple way to decide if it is invertible: compute $wz - xy$. Likewise, I gave no points for the answer "*if the columns are linearly independent*".

(14) If $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible its inverse is ___________.

$$\frac{1}{wz - xy} \begin{pmatrix} z & -x \\ -y & w \end{pmatrix}$$

Comments:

(15) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \ldots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

(16) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix with columns $\underline{B}_1, \ldots, \underline{B}_p$. Express the columns of AB in terms of the columns for B.

$AB = [A\underline{B}_1, \ldots, A\underline{B}_p]$ where the notation on the right denotes the matrix with columns $A\underline{B}_1, \ldots, A\underline{B}_p$.

Comments: This equation is fundamental (compare with question 11). Most people were unable to answer this question. Not good. Some answers involved expressions like $\underline{A}_1\underline{B}_1$; presumably that means the first column of A times the first column of B. But you can't multiply two column matrices (unless one is a 1×1 column).

(17) Let A be an invertible 3×3 matrix. Let M_{ij} be the matrix obtained by deleting row i and column j from A. If $A_{ij} =$ ___________, then $A^{-1} =$ ___________.

(18) Let $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ be the columns of a 4×4 matrix A and suppose that $\underline{A}_1 + 2\underline{A}_2 = \underline{A}_3 - 3\underline{A}_4$. Write down a solution

$$\underline{x} = \begin{pmatrix} 2 \\ ? \\ ? \\ ? \end{pmatrix}$$

to the equation $A\underline{x} = \underline{0}$.

$$\underline{x} = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 6 \end{pmatrix}.$$

Comments: Most people got this wrong. One of the most effective things you can do to get the kind of grade you want in this course is to understand what I have to say below.

This question tests whether you *understand* the meaning of the correct answer to Question 11. I was startled to learn that most people do *not* understand the meaning of the answer to Question 11. If you got question 11 correct but question 17 wrong you have something important to learn.

Question 11 asks you to express $A\underline{x}$ as a linear combination of its columns. The answer is that $A\underline{x} = x_1\underline{A}_1 + \ldots + x_n\underline{A}_n$ where $\underline{A}_1, \ldots, \underline{A}_n$ are the columns of A and $x_1, \ldots, x_n$ are the entries of $\underline{x}$. Therefore the equation $A\underline{x} = 0$ is "the same" as the equation $x_1\underline{A}_1 + \ldots + x_n\underline{A}_n = 0$.

The statement of question 17 tells you that $\underline{A}_1 + 2\underline{A}_2 = \underline{A}_3 - 3\underline{A}_4$ or, equivalently, $\underline{A}_1 + 2\underline{A}_2 - \underline{A}_3 + 3\underline{A}_4 = 0$. Rewrinting this in the form $A\underline{x}$, it is saying

$$A\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or, equivalently, that one solution to the equation $A\underline{x} = \underline{0}$ is

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}.$$

However, if $\underline{u}$ is a solution to $A\underline{x} = \underline{0}$ so is $l\underline{u}$ for all $l \in \mathbb{R}$. Taking $l = 2$ in the present problem another solution to $A\underline{x} = \underline{0}$ is

$$\underline{x} = 2\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 6 \end{pmatrix}.$$

Perhaps the above comments will help you understand why it is so important to get the notation *completely* correct in answering question 11. For example, if someone gives the answer $A\underline{x} = A_1\underline{x}_1 + \cdots + A_n\underline{x}_n$ to question 11 it is unlikely he/she will answer question 17 correctly. For example, many people treated $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ as numbers when answering this question. For example, an answer like $\underline{A}_1 = 2, \underline{A}_2 = -1, \underline{A}_3 = 3, \underline{A}_4 = 1$ shows a fundamental misunderstanding of the notation, and a disregard of the information presented in the question itself which says that $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ are the columns of a 4×4 matrix.

As another illustration of how notation can be helpful consider this. If we multiply a matrix A by the number 2 we usually write $2A$, not $A2$. This is consistent with notation like $2x$ or $2x^3$. This is a standard convention in mathematics. You first encountered it in high school algebra. When we write $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ we are consistent with that convention: x_i is a number and $\underline{A}_i$ is a matrix, a column of A. Someone who write $\underline{A}_1 x_1 + \cdots + \underline{A}_n x_n$ in answer to question 11 is breaking that convention and is therefore unlikely to see the connection between the equation $\underline{A}_1 + 2\underline{A}_2 - \underline{A}_3 + 3\underline{A}_4 = 0$ and the equation $x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$. Some people wrote $x_1\underline{A}_1 + 2x_2\underline{A}_2 - x_3\underline{A}_3 + 3x_4\underline{A}_4 = 0$ in trying to answer question 17; they are probably not paying enough attention to what they write.

Likewise, someone who writes $A\underline{x} = A_1x_1 + \cdots + A_nx_n$ is not paying enough attention to pick up on the cues/clues I am giving when I use the same notation $\underline{A}_i$ in questions 11 and 17.

Part B.

Complete the definitions.

There is a difference between theorems and definitions. Here I am asking for the definition.

Don't write the part of the definition I have already written. Just fill in the blank.

(1) (My definition, not the one in the book!) A matrix A is <u>non-singular</u> if ________________

solution to $A\underline{x} = 0$ is $\underline{x} = 0$.

Comments:

(2) A system of linear equations is <u>consistent</u> if ________________

(3) A system of linear equations is <u>inconsistent</u> if ________________

(4) The system of equations $A\underline{x} = \underline{b}$ is <u>homogeneous</u> if __________

(5) Two systems of linear equations are <u>equivalent</u> if __________

they have the same solutions.

Comments: The answer "*they have the same row reduced echelon form*" is not correct. There is a Theorem that says Two systems of linear equations are equivalent if and only if they have the same row reduced echelon form, but that is a consequence of the definition. The importance of that theorem is that it provides a method to decide if two systems of equations have the same set of solutions.

It is obviously important to know when two systems of linear equations have the same set of solutions so we introduce a single word for two systems having that property, namely "equivalent". It is easier to say two systems are equivalent than two systems have the same solutions.

(6) A matrix E is in <u>echelon form</u> if
 (a) __________
 (b) __________
 (c) __________

(7) A matrix is in <u>row reduced echelon form</u> if it is __________ and __________ .

(8) In this question use R_i to denote the i^{th} row of a matrix. The three <u>elementary row operations</u> are
 (a) __________
 (b) __________

(c) _________

(9) A matrix A is invertible if there is a matrix B such that _________

Part C.

True or False

All assertions are TRUE with the exception of numbers 5, 13, 17, 27, 30, 31, 34.

I made an error in the statement of 18. It should read *A square matrix B such that $B^{12} \neq I$ and $B^{13} - B = 0$ must be singular.* The new statement is TRUE. If B is invertible and $B^{13} - B = 0$, then

$$0 = B^{-1}(B^{13} - B) = B^{-1}B(B^{12} - I) = B^{12} - I;$$

i.e., $B^{12} = I$. But $B^{12} \neq I$ so we conclude B is not invertible, and therefore singular.

(1) If a system of linear equations has five different solutions it must have infinitely many solutions.
(2) The matrix $\begin{pmatrix} a & 2 \\ -3 & b \end{pmatrix}$ is non-singular except when $ab = -6$.
(3) A homogeneous system of 10 linear equations in 12 unknowns always has a non-zero solution.
(4) A homogeneous system of 12 linear equations in 10 unknowns always has a solution.
(5) A system of 10 linear equations in 12 unknowns always has a solution.
(6) The solutions of a system of 12 linear equations in 12 unknowns can be a line in $\mathbb{R}^{12}$.
(7) $2I^2 - 2I + I^{-1}$ is non-singular.
(8) $3I - I^2 - I^{-1} - I^4$ is singular.
(9) The determinant of a lower triangular matrix is the product of the entries on the diagonal.
(10) Let A and B be $n \times n$ matrices. If A and B are invertible so is AB.
(11) Let A and B be $n \times n$ matrices. If A or B is singular so is AB.
(12) If A and B are non-singular $n \times n$ matrices, so is AB.
(13) If A and B are non-singular $n \times n$ matrices, so is $A + B$.
(14) Let A be an $n \times n$ matrix. If the rows of A are linearly dependent, then A is singular.
(15) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.
(16) There is a matrix whose inverse is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$.
(17) If $A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.
(18) A square matrix B such that $B^{12} \neq I$ and $B^{13} + B = 0$ must be singular.
(19) A matrix can have at most one inverse.

(20) One can't add matrices of different sizes.

(21) Let A be a 4×5 matrix and $\underline{b} \in \mathbb{R}^5$. Suppose that the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & \mid & 2 \\ 0 & 0 & 1 & 3 & 0 & \mid & 2 \\ 0 & 0 & 0 & 0 & 1 & \mid & 0 \\ 0 & 0 & 0 & 0 & 0 & \mid & 0 \end{pmatrix}$$

The independent variables are x_2 and x_4; they can take any values and all solutions are given by $x_1 = 2 - 2x_2 - x_4$, $x_3 = 2 - 3x_4$, and $x_5 = 0$.

(22) If $A = A^{\mathsf{T}}$, then A is a square matrix.

(23) If $A = BC$, then every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.

(24) If $A = BC$, then

$$\{\underline{w} \mid C\underline{w} = \underline{0}\} \subset \{\underline{x} \mid A\underline{x} = \underline{0}\}.$$

(25) The points $(1, 1, 1)$ and $(1, 2, 3)$ lie on the plane $-x + 2y - z = 0$ in $\mathbb{R}^3$.

(26) A matrix having a column of zeroes never has an inverse.

(27) A matrix is invertible if and only if its determinant is 1.

(28) A square matrix is singular if and only if it does not have an inverse,

(29) The cube of the transpose of the inverse of the transpose of the inverse of a matrix is invertible,

(30) $(AB)^{-1} = A^{-1}B^{-1}$

(31) $(AB)^{\mathsf{T}} = A^{\mathsf{T}}B^{\mathsf{T}}$.

(32) The determinant of a 5×5 matrix is a sum of 120 terms.

(33) Each term in the determinant of a 5×5 matrix is a product of 5 entries in the matrix.

(34) If A is a 6×6 matrix the term $a_{24}a_{13}a_{31}a_{15}a_{56}a_{62}$ appears in the expression for the determinant.

(35) If A is a 6×6 matrix the term $a_{45}a_{56}a_{22}a_{13}a_{31}a_{64}$ appears in the expression for the determinant.