Math 308 **Final** **December 10, 2012**

Instructions. No need to show your work. In Parts A and B you get 1 point per question or 1 point for each part of a multi-part question. Part C: use the scantron/bubble sheet with A =True and B =False: for Part C you will get • +1 for each correct answer, • -1 for each incorrect answer, and • 0 for no answer at all.

Part A.

(1) If you know a single solution, $\underline{v}$ say, to the equation $A\underline{x} = \underline{b}$ and all solutions to the equation $A\underline{x} = \underline{0}$, then the set of all solutions to $A\underline{x} = \underline{b}$ is $\{...|....\}$.

Answer: $\{\underline{u} + \underline{v} \mid A\underline{v} = \underline{0}\}$

Comments: One student wrote $\{\underline{u} + \underline{v} \mid \underline{v} \in \mathcal{N}(A)\}$. That's correct.
Some wrong answers:

- $\{\underline{u} + \underline{v} \mid \underline{v} \in A\underline{x} = 0 \mid$. The symbol $\in$ means "belongs to" or "is in". To use it correctly $\in$ must be followed by a set and preceded by an element of that set. For example, Bill Clinton $\in$ {all past and present US presidents}. It would be wrong to write Obama $\in$ the White House because the White House is not a set. It is also wrong to write $6/3 \in 2$ because 2 is a number. It would be correct to write $6/3 \in \{2\}$. In this question $A\underline{x}$ is a vector, $A\underline{x} = \underline{0}$ is an equation, and $\underline{v}$ is a vector; neither $A\underline{x}$ nor $A\underline{x} = \underline{0}$ is a set.
  Perhaps the author meant to say that $\underline{v}$ is in the set $\{\underline{x} \mid A\underline{x} = \underline{0}\}$. The best way to say that is to write $A\underline{v} = \underline{0}$.
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(2) Express $A\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ as a linear combination of the columns $\underline{A}_1, \dots, \underline{A}_n$ of A.

Answer: $c_1\underline{A}_1 + \cdots + c_n\underline{A}_n$

Comments: Too many students wrote $\underline{A}_1 c_1 + \cdots + \underline{A}_n c_n$. It is common practice to write the coefficient before the vector, i.e., $2\underline{v}$ rather than $\underline{v}2$. We use the same convention for vectors as the one you used in high school for unknowns or variables. There you wrote $2x - 3y = 4$, not $x2 - y3 = 4$. Best to follow the conventions. We adopt these conventions for a reason. One reason is that it is awkward to read an expression like $x^2 2$; that's why we prefer $2x^2$. It gets even worse if one has an expression like $x_2^2 2$. Would you prefer to read $x2y3$ or $6xy$? This convention is like the one used for adjectives in English. We say *The tall tree* not *The tree tall.*

A mathematical sentence that does not follow the usual conventions looks odd and suggests the author is somewhat unfamiliar with the material. Especially in this case. I wrote $x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ on the board almost every day of the quarter; on many days I wrote it more than once.

(3) What matrix represents the linear transformation $T \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w-x \\ x-y \\ y-z \\ z-w \end{pmatrix}$.

Answer: $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Comments:

(4) Give a basis for the kernel of the linear transformation in Question 3.

Answer: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Comments: Some students wrote down all or part of

$$\ker(T) = \left\{ \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix} \right.$$

That is a correct description of the kernel but the question asked for a basis for the kernel. Perhaps I should have had two questions, one asking for the kernel of T, the other asking for a basis for the kernel of T.

(5) Give a basis for the range of the linear transformation in Question 3.

Answer: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

Comments: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\operatorname{rank}(T) + \operatorname{nullity}(T) = n$ where $\operatorname{rank}(T) = \dim \mathcal{R}(T)$ and $\operatorname{nullity}(T) = \dim(\ker(T))$. In this question $\operatorname{nullity}(T) = 1$ so $\operatorname{rank}(T) = 3$. Therefore a basis for $\mathcal{R}(T)$ must consist of 3 vectors. Some students gave only *one* vector as the answer. The remarks I've just made show that can't be right.

One student answered $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. These vectors are in the range but they are not linearly independent: they add up to zero.

(6) If $A\begin{pmatrix}4\\3\\2\\-1\end{pmatrix} = \underline{0}$, the columns $\underline{A}_1, \ldots, \underline{A}_4$ of A are linearly dependent because $\underline{A}_4 =$ _________

Answer: $\underline{A}_4 = 4\underline{A}_1 + 3\underline{A}_2 + 2\underline{A}_3$

Comments: Most students got this right but some just said *"is a linear combination of the other columns"* which was not specific enough given the prompt $\underline{A}_4 =$.

(7) The vectors $(1,1,3,3)^T$ and $(3,3,1,1)^T$ are a basis for the set of solutions to the homogeneous equations _________ and _________

Answer: $x_1 - x_2 = x_3 - x_4 = 0$

Comments: There are many correct answers to this question but the most obvious property of the vectors $(1,1,3,3)^T$ and $(3,3,1,1)^T$ is that their first and second coordinates are equal, i.e., 1=1 for the $(1,1,3,3)^T$ and 3=3 for $(3,3,1,1)^T$, and their third and fourth coordinates are equal, i.e., 3=3 for the $(1,1,3,3)^T$ and 1=1 for $(3,3,1,1)^T$. Therefore both vectors lie on the 3-plane $x_1 = x_2$ and on the 3-plane $x_3 = x_4$. They are therefore solutions to the system of equations $x_1 - x_2 = x_3 - x_4 = 0$. It is clear that the solutions to this system form a 2-plane so, since $(1,1,3,3)^T$ and $(3,3,1,1)^T$ are linearly independent solutions they must be a basis for that 2-plane, i.e., the 2-dimensional subspace consisting of solutions to the homogeneous system of equations $x_1 - x_2 = x_3 - x_4 = 0$.

(8) The vectors $(1,1,3,3)^T$ and $(3,3,1,1)^T$ are a basis for the range of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}?\\?\\?\\?\end{pmatrix}$.

Answer: $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\x\\y\\y\end{pmatrix}$.

Comments: There are infinitely many correct solutions but mine is perhaps the simplest. Several students gave the equally correct answer

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+3y\\x+3y\\3x+y\\3x+y\end{pmatrix}.$$

It took me a while to figure out why they did that: those students had figured out that if

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+3y\\x+3y\\3x+y\\3x+y\end{pmatrix},$$

then

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\3\\3\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\3\\1\\1\end{pmatrix}.$$

I'm not sure what those students did to get that answer but the process they went through to get their answer must have differed from my approach. Let me tell you what went through my mind when I gave my answer.

My first reaction was to notice that $(1,1,0,0)^T$ and $(0,0,1,1)^T$ span the same subspace as $(1,1,3,3)^T$ and $(3,3,1,1)^T$; i.e., the range of T is spanned by $(1,1,0,0)^T$ and $(0,0,1,1)^T$. Then I noticed that $(1,1,0,0)^T$ and $(0,0,1,1)^T$ are in the range of

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\x\\y\\y\end{pmatrix},$$

and obviously a basis for that range.

(9) The vectors $(1,1,3,3)^T$ and $(3,3,1,1)^T$ are a basis for the kernel (null space) of the linear transformation $T:\mathbb{R}^4\to\mathbb{R}^3$ given by $T\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\begin{pmatrix}?\\?\\?\end{pmatrix}$.

Answer: There are infinitely many correct answers but $T\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\begin{pmatrix}w-x\\y-z\\0\end{pmatrix}$ is probably the simplest.

Comments: Many students gave answers like $T\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=\begin{pmatrix}w-x\\x-w\\y-z\end{pmatrix}$. Such an answer is correct but not as simple as it could be. You look smarter by giving a simple answer.

(10) Find a basis for the line $x_1+x_2-x_3=x_2+x_3-x_4=x_1-x_2=0$ in $\mathbb{R}^4$.

Answer: $(1,1,2,3)$

Comments:

(11) Give the equation of a 3-plane in $\mathbb{R}^4$ that contains the line in question 10.

Answer: $x_1 = x_2$.

Comments: There are infinitely many correct answers but my solution is perhaps the simplest. The question is much simpler than you might have realized because its solution is already part of question 10! Question 10 can be interpreted in the following way: The set of solutions to each of the equations $x_1+x_2-x_3 = 0$, $x_2+x_3-x_4 = 0$, and $x_1-x_2 = 0$ is a 3-dimensional subspace of $\mathbb{R}^4$ and question 10 asks for a basis for the 1-dimensional subspace that is the intersection of those three 3-dimensional subspaces. That 1-dimensional subspace is, *by definition*, contained in each of those three 3-dimensional subspaces: the intersection of three sets, say $X \cap Y \cap Z$, is contained in each of them *because that is part of the meaning of the word intersection.*

Therefore another "obvious" answer to this question is $x_1 + x_2 - x_3 = 0$; another obvious answer is $x_2 + x_3 - x_4 = 0$.

Less obvious answers can be obtained by taking linear combinations of these equations, e.g., $2(x_1 + x_2 - x_3) + 3(x_2 + x_3 - x_4) = 0$ or, more succinctly, $2x_1 + 5x_2 + x_3 - 3x_4 = 0$, is a 3-plane in $\mathbb{R}^4$ that contains the line in question 10.

Several people who gave the correct answer, $(1, 1, 2, 3)$, to question 10 gave the same incorrrect answer $x_1 + x_2 + 2x_3 + 3x_4 = 0$ to this question. I do not know why they used the coordinates of the answer to question 10 as the coefficients of $x_1 + x_2 + 2x_3 + 3x_4$. One can see that $(1, 1, 2, 3)$ *does not lie on the 3-plane* $x_1 + x_2 + 2x_3 + 3x_4 = 0$ *because plugging in* 1 *for* x_1, 1 *for* x_2, 2 *for* x_3, *and* 3 *for* x_4, *gives* $x_1 + x_2 + 2x_3 + 3x_4 = 15$, *not 0.*

A puzzling answer was

$-x_1 + x_2 + x_3 - x_4 = 0$ and $x_1 - x_2 - x_3 + x_4 = 0$.

Although these two equations are different they have exactly the same solutions, i.e., they determine the same 3-plane in $\mathbb{R}^4$. Two equations are necessary to define a 2-plane in $\mathbb{R}^4$, but only one is needed to determine a 3-plane. Did the student realize the two equations defined the same 3-plane and, if so, why did he/she write down both equations? If the student thought the two equations defined different 3-planes was he/she thinking that one needs two equations to determine a 3-plane in $\mathbb{R}^4$?

(12) The matrix representing the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a vector by θ radians in the counter-clockwise direction is _________.

Answer: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Comments:

(13) If A is an $n \times n$ matrix and the equation $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^n$, then ___________

Answer:

Comments:

(14) Find an eigenvector for $\begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}$ having eigenvalue 3.

Answer: $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Comments: A lot of students did not attempt this question. The λ-eigenspace of a matrix A is $\mathcal{N}(A-\lambda I)$. The 3-eigenspace for this matrix is therefore the null space of

$$\begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix} - 3I = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

This is a rank-one matrix. Because $\begin{pmatrix} 2 & 4 \end{pmatrix} = 2\begin{pmatrix} 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\underline{v} = \underline{0}$ if and only if $\begin{pmatrix} 1 & 2 \end{pmatrix}\underline{v} = 0$. It is obvious that $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0$ so the null space of $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is spanned by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(15) What is the other eigenvalue for the matrix in the previous question?

Answer: 8

Comments: Most students knew that the eigenvalues are the roots of the characteristic polynomial and that the characteristic polynomial of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $x^2-(a+d)x+ad-bc$. In this question the characteristic polynomial is $x^2-11x+24=(x-3)(x-8)$. Some students did not know how to factor $x^2-11x+24$ even when told (implicitly) that one of the roots is $x=3$. High school math is a prerequisite for all math courses at UW.

Part B.

Complete the definitions and Theorems.
There is a difference between theorems and definitions.
Don't write the part of the question I have already written. Just fill in the blank.

(1) **Definition:** (Do not use the phrase *"linear combinations"* in your answer.) The linear span of $\underline{v}_1,\dots,\underline{v}_n$ is the set of all vectors of the form __________

Answer: $a_1\underline{v}_1+\cdots+a_n\underline{v}_n$ where $a_1,\dots,a_n\in\mathbb{R}$
OR
$a_1\underline{v}_1+\cdots+a_n\underline{v}_n$ for some $a_1,\dots,a_n\in\mathbb{R}$.

Comments: Some incorrect answers:

- $a_1\underline{v}_1+\cdots+a_n\underline{v}_n$ where $a_1=\ldots=a_n\in\mathbb{R}$
- $a_1\underline{v}_1+\cdots+a_n\underline{v}_n$ for some numbers $a_1,\dots,a_n$ are not all zero. First, the sentence is not grammatically correct; to correct the grammar put a comma after a_n and delete "are". Second, the linear span of $\underline{v}_1,\dots,\underline{v}_n$

is the set of *all* vectors $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ as $a_1, \ldots, a_n$ range over *all* real numbers. In particular, $\underline{0}$ is in the linear span of $\underline{v}_1, \ldots, \underline{v}_n$ because it equals $0\underline{v}_1 + \cdots + 0\underline{v}_n$.

- $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for all $a_1, \ldots, a_n \in \mathbb{R}$ and $\underline{v}_1, \ldots, \underline{v}_n \in \mathbb{R}^n$. This is wrong because the last part of the answer suggests that the $\underline{v}_i$s are allowed to vary. But they are fixed. Also, the fact that there are n vectors in the question is not telling you that they belong to $\mathbb{R}^n$. In fact, the question does not tell you what the ambient vector space is. It is, however, implicit in the question that the $\underline{v}_i$s belong to a common vector space.
- $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for all $a \in \mathbb{R}$. This must be wrong because the symbol a in the last part of the answer has no meaning—it doesn't appear anywhere else in the sentence. The author of this answer doesn't understand what he or she has written.
- $\{a\underline{v}_1, \ldots, a\underline{v}_n \; a \in \mathbb{R}\}$. The author of this answer doesn't understand what he or she has written.

(2) To show that $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly independent I must show that the only solution to the equation _________

Answer: $a\underline{u} + b\underline{v} + c\underline{w} = \underline{0}$ is $a = b = c = 0$.

Comments: A popular answer was
$a_1\underline{u} + a_2\underline{v} + a_3\underline{w} = \underline{0}$ is $a_1 = a_2 = a_3 = 0$.
This answer is correct, but it is quicker and cleaner to avoid the subscripts. The reason I asked about 3 vectors, $\underline{u}, \underline{v}, \underline{w}$, rather than k vectors $\underline{v}_1, \ldots, \underline{v}_k$. was so you could avoid using subscripts.

Some incorrect answers:

- $a_1\underline{u} + a_2\underline{v} + a_3\underline{w}$ is 0 where $a_1 = a_2 = a_3 = 0$. First 0, which denotes the number zero, should replaced by $\underline{0}$, which denotes the vector zero. Second, the answer is grammatically incorrect so the reader is required to fix the grammar in order to obtain a correct answer. It is the responsibility of the person answering the question to provide the correct answer. It is not good enough to write something that the reader can respond to by saying *"I think the student means this and if he/she does mean this, then that answer would be correct"*.
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(3) To show that $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly dependent I must show that _________ $= \underline{0}$ for some _________

Answer: $a\underline{u} + b\underline{v} + c\underline{w} = \underline{0}$ for some $a, b, c \in \mathbb{R}$, not all of which equal zero.

Comments: There are other ways of saying *"not all of which equal zero"*. For example, one could say *"at least one of which is not zero"* or *"that are not all zero"*.

Some incorrect answers:

- $a_1\underline{u} + a_2\underline{v} + a_3\underline{w} = \underline{0}$ for some non-zero $a_1, \ldots, a_3 \in \mathbb{R}$. The sentence suggests that all three coefficients must be non-zero. At best the sentence is ambiguous. A statement in mathematics is no good if it is ambiguous. Always make your meaning absolutely clear.

- $a_1\underline{u} + a_2\underline{v} + a_3\underline{w} = \underline{0}$ for some $a_1 \dots a_3 \neq 0$. By omitting the commas after a_1 and before a_3 the epression $a_1 \dots a_3$ can only be interpreted as $a_1a_2a_3$, the product of the three numbers. So the written answer says the product $a_1a_2a_3$ is not zero which happens only if none of the a_is is zero. Saying all the a_is are non-zero is not the same as saying at least one of them is non-zero. Of course, I can mentally insert those commas but that is not my responsibility. It is the student's responsibility to do that. Even if commas had been inserted the answer would be wrong because I don't know what is meant when someone writes $a_1, \dots, a_3 \neq 0$. Does he/she mean all three are non-zero? that a_3 is non-zero? etc.
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(4) Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$. To show that a function $T : V \to W$ is a linear transformation, I must show that ________.

Answer: $T(a\underline{u} + b\underline{v}) = aT(\underline{u}) + bT(\underline{v})$ for all $\underline{u}, \underline{v} \in V$ and all $a, b \in \mathbb{R}$
OR
$T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ and $T(c\underline{u}) = cT(\underline{u})$ for all $\underline{u}, \underline{v} \in V$ and all $c \in \mathbb{R}$.

Comments: It is essential to say that these conditions must be satisfied for *all* $\underline{u}, \underline{v} \in V$ and *all* $a, b \in \mathbb{R}$. *All* is the best word to use when you mean *all*.

Several people gave the correct answer

$T(a_1\underline{x}_1 + \cdots + a_k\underline{x}_k) = a_1T(\underline{x}_1) + \cdots + a_kT(\underline{x}_k)$ for all $\underline{x}_1, \dots, \underline{x}_k \in V$ and all $a_1, \dots, a_k \in \mathbb{R}$.

This answer is longer than necessary. A more complicated answer than necessary suggests that the author doesn't *really* understand what is going on.

Some incorrect answers:

- $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ and $T(c\underline{u}) = cT(\underline{u})$ where $\underline{u}, \underline{v} \in V$ and $c \in \mathbb{R}$. The word *where* does not mean *for all*.
- $T(a\underline{u} + b\underline{v}) = aT(\underline{u}) + bT(\underline{v})$ for all $\underline{u}, \underline{v} \in V$. This is wrong because the words "and all $a, b \in \mathbb{R}$" are missing.
- $T(a_1\underline{x}_1 + \cdots + a_k\underline{x}_k) = a_1T_1(\underline{x}_1) + \cdots + a_kT_k(\underline{x}_k)$ for all $\underline{x}_1, \dots, \underline{x}_k \in \mathbb{R}$ and all $a_1, \dots, a_k \in \mathbb{R}$. Notice the subscripts on the Ts and the condition that the $\underline{x}_i$s belong to $\mathbb{R}$ rather than V.

(5) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. To show that $\underline{x}$ is in the kernel of T I must show that ________.

Answer: $T(\underline{x}) = \underline{0}$.

Comments: A surprisingly large number of people got this wrong. I guess that if I had asked for the definition of the kernel of T many more people would have given the correct answer, $\ker(T) = \{\underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0}\}$. This suggests that people are learning to write $\ker(T) = \{\underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0}\}$ without understanding what it means. The definition $\ker(T) = \{\underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0}\}$ *means* that $\underline{x}$ is in the kernel if and only if $T(\underline{x}) = \underline{0}$.

Be self-critical when writing things like $\ker(T) = \{\underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0}\}$. Ask yourself, *do I understand what I have just written?* Remember, our capacity for self-delusion is enormous...just think of the millions of people who say global warming is a mere fiction despite the overwhelming scientific evidence.

Some wrong answers:

- $\underline{x} \in \mathbb{R}^n$. This "answer" *must* be wrong because it doesn't mention T. Logic suggests that to decide whether $\underline{x}$ is in the kernel of T one must carry out some "test" that involves both $\underline{x}$ and T.
- $T(\underline{x}) = \underline{0}$ for some $\underline{x} \in \mathbb{R}^n$. In the question, the sentence beginning *"To show that $\underline{x}$ is ..."* indicates that I am discussing a single element $\underline{x}$, a single *specific* element.
- $T(\underline{x}) = \underline{0}$ for all $\underline{x} \in \mathbb{R}^n$. This "answer" also gives the impression that question is not about a single $\underline{x}$. In fact, if $T(\underline{x}) = \underline{0}$ for all $\underline{x} \in \mathbb{R}^n$, then T is the zero linear transformation and its kernel is $\mathbb{R}^n$.
- $T(\underline{x}) = 0$ for $x \in T$. The $x \in T$ part of the answer shows a serious misunderstanding: it suggests the author thinks that T is a set. It isn't; T is a linear transformation, a function.
- $\ker(T) = \{\underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0}\}$. This is wrong because the question does not ask for the definition of the kernel. It asks what must be proved to conclude that a vector $\underline{x}$ is in the kernel.

Some strange correct answers:

- $\underline{x} \in \{\underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0}\}$. That is a very long-winded way of saying $T(\underline{x}) = \underline{0}$.

(6) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. To show that $\underline{v}$ is in the range of T I must show that _________.

Answer: $\underline{v} = T(\underline{x})$ for some $\underline{x} \in \mathbb{R}^n$.

Comments: Similar comments as for the previous question. I guess most students would have been able to tell me that the range of T is $\{T(\underline{x}) \mid \underline{x} \in \mathbb{R}^n\}$. But what does it *mean* to say the range of T is $\{T(\underline{x}) \mid \underline{x} \in \mathbb{R}^n\}$? It means that $\underline{v}$ is in the range if $\underline{v} \in \{T(\underline{x}) \mid \underline{x} \in \mathbb{R}^n\}$, i.e., that $\underline{v} = T(\underline{x})$ for some $\underline{x} \in \mathbb{R}^n$.

Some wrong answers:

- $\underline{v} \in \mathbb{R}^n$. This "answer" *must* be wrong because it doesn't mention T. Simple logic suggests that to decide whether $\underline{v}$ is in the range of T one must carry out some "test" that involves both $\underline{v}$ and T.
- $T(\underline{x}) = \underline{v}$ for $x \in T$, $\underline{v} \in \mathbb{R}^m$. The $x \in T$ part of the answer shows a serious misunderstanding: it suggests the author thinks that T is a set. It isn't; T is a linear transformation, a function.
- $\mathcal{R}(T) = \{T(\underline{x}) \mid \underline{x} \in \mathbb{R}^n\}$. This is wrong because the question does not ask for the definition of the range. It asks what must be proved to conclude that a vector is in the range. Perhaps it helps to give an analogy. If I ask someone how to get to the University of Washington and she replies *"The University of Washington is the largest university in Washington State"* she has given a correct description of the University of Washington but has not answered my question.
- $T(\underline{v})$ for all $\underline{v} \in \mathbb{R}^n$.

- $\underline{v} \in T(\underline{x})$ for $\underline{x} \in \mathbb{R}$. The symbol $\in$ should be replaced by $=$, and the word "some" should be inserted after "for".
- $T(\underline{v}) \in \mathbb{R}^m$. This is wrong because although the range of T is contained in $\mathbb{R}^m$ it might be smaller than $\mathbb{R}^m$.

(7) **Definition:** The dimension of a subspace V of $\mathbb{R}^n$ is ______________.

Answer: the number of elements in a basis for it.

Comments:
Some incorrect answers:

- *"the number of elements in its basis."* The use of *its* suggests that V has only one basis. You should know that every non-zero subspace of $\mathbb{R}^n$ has *infinitely many bases*. You must therefore speak of *a* basis unless you are discussing a particular basis.
- *the number of elements in col*(V). Since V is a vector space it does not have columns. A matrix has columns.
-

(8) **Theorem:** A linear transformation $T : V \to W$ has an inverse if and only if _____________.

Answer: it is one-to-one and onto.

Comments: Some wrong answers:

- it is linearly independent. The student who wrote this probably knows that a square matrix has an inverse if and only if its columns are linearly independent and after seeing the word *inverse* in the question that student thought...well, I'm not sure what. The answer is completely wrong though because the term *linearly independent* is a property that a set of vector has (or does not have). A linear transformation is not a set of vectors so a phrase like *This linear transformation is linearly independent* makes no more sense than a phrase like *This bird is linearly independent* or *This dream has dimension six.*
  The author of this "answer" was probably grasping at the following fact: If A is the $n \times n$ matrix representing a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, then T is invertible if and only if the columns of A are linearly independent.
- there exists a linear transformation $S : W \to V$. The answer must be wrong because it has nothing to do with T. The definition of T having an inverse is this: A linear transformation $T : V \to W$ has an inverse if there is a linear transformation $S : W \to V$ such that $T \circ S = \mathrm{id_W}$ and $S \circ T = \mathrm{id_V}$. Obviously, the student remembered part of the definition and wrote that down. But even if the student had written down the definition correctly the answer would have been wrong because I was not asking for the definition of what it means for a linear transformation to be invertible. I was asking for a *Theorem* that gives a criterion for a linear transformation to be invertible.

(9) **Theorem:** The following conditions on an $n \times n$ matrix A are equivalent:
(a) A is invertible

(b) the rows of A are ____________
(c) $\mathcal{R}(A) =$____________
(d) $\mathcal{N}(A) =$____________
(e) $\operatorname{rank}(A) =$ ____________
(f) $\det(A)$ is ____________

Answer:
- the rows of A are linearly independent.
- $\mathcal{R}(A) = \mathbb{R}^n$.
- $\mathcal{N}(A) = \{\underline{0}\}$.
- $\operatorname{rank}(A) = n$ ____________
- $\det(A)$ is non-zero.

Comments: One person answered n for (c) but the question asked for the range of A, not for the dimension of the range. Similarly, 0 is not a correct answer to (d).

(10) **Definition:** Let λ be an eigenvalue for the $n \times n$ matrix A. The <u>λ-eigenspace</u> for A is

$$E_\lambda := \{\_\_ \mid \_\_\_\_\_\_\_\_\}$$

Answer: $\{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = \lambda \underline{x}\}$.

Comments:

(11) **Theorem:** The λ-eigenspace of A is non-zero if and only if $\det(A - \lambda I) = 0$ because $E_\lambda =$ _______________.

Answer: $\mathcal{N}(A - \lambda I)$.

Comments:

(12) **Theorem:** The roots of the _______________ of a square matrix A are its_______________.

Answer:

Comments:

(13) **Theorem:** Let $T : \mathbb{R}^p \to \mathbb{R}^q$ be a linear transformation. Then there is a unique ________ matrix A such that ________ for all ________. We call A the matrix that <u>represents</u> T.

Answer:

Comments:

(14) **Theorem:** The j^{th} column of the matrix representing T is _________.

Answer:

Comments:

(15) **Definition:** A set of non-zero vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is orthogonal if ________.

Answer:

Comments:

(16) **Theorem:** Let $\{\underline{u}_1, \dots, \underline{u}_k\}$ be an orthonormal basis for a subspace V of $\mathbb{R}^n$. The orthogonal projection of $\mathbb{R}^n$ onto V is the linear transformation $P : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula $P(\underline{x}) =$ ________. Furthermore,
(a) $P(\underline{v}) = \underline{v}$ for all ________
(b) $\mathcal{R}(P) =$ ________
(c) $\ker(P) =$________
(d) $\ker(P) \cap \mathcal{R}(P) =$________
(e) $\ker(P) + \mathcal{R}(P) =$________

Answer: $P(\underline{v}) = (\underline{v} \cdot \underline{u}_1)\underline{u}_1 + \cdots + (\underline{v} \cdot \underline{u}_k)\underline{u}_k$
- $P(\underline{v}) = \underline{v}$ for all $\underline{v} \in V$.
- $\mathcal{R}(P) = V$.
- $\ker(P) = V^{\perp}$
- $\ker(P) \cap \mathcal{R}(P) = \{\underline{0}\}$
- $\ker(P) + \mathcal{R}(P) = \mathbb{R}^n$

Comments: A lot of people did not remember the formula for $P(\underline{v})$ and among those who did many failed to use the fact that the formula when $\{\underline{u}_1, \dots, \underline{u}_k\}$ is an *orthonormal* basis is simpler than the formula when $\{\underline{u}_1, \dots, \underline{u}_k\}$ is just an *orthogonal* basis. When $\{\underline{u}_1, \dots, \underline{u}_k\}$ is an orthogonal basis for V the formula is

$$P(\underline{v}) = \left(\frac{\underline{v} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1}\right)\underline{u}_1 + \cdots + \left(\frac{\underline{v} \cdot \underline{u}_k}{\underline{u}_k \cdot \underline{u}_k}\right)\underline{u}_k$$

However, when $\{\underline{u}_1, \dots, \underline{u}_k\}$ is an *orthonormal* basis $\underline{u}_i \cdot \underline{u}_i = 1$ for all i so the formula simplifies to

$$P(\underline{v}) = (\underline{v} \cdot \underline{u}_1)\underline{u}_1 + \cdots + (\underline{v} \cdot \underline{u}_k)\underline{u}_k.$$

Some students gave the answer 0 to part (d). However, in this course we have been using the convention that 0 denotes the number zero and we use the symbol $\underline{0}$ for the zero vector.

The number n turned up as an "answer" to (c), (d), or (e), but the left-hand side of the $=$ sign in (c), (d), and (e) is a *subspace* of $\mathbb{R}^n$ so the right-hand side of that $=$ sign *must also be a subspace*. A subspace can never equal a number—subspaces and numbers are different animals. The dimension of a subspace is a number but the subspace itself is not a number.

(17) **Definition:** Let A be an $m \times n$ matrix. A vector $\underline{x}^*$ in $\mathbb{R}^n$ is a least-squares solution to $A\underline{x} = \underline{b}$ if ________ is as close as possible to ________ .

Answer: if $A\underline{x}^*$ is as close as possible to $\underline{b}$
OR
$||A\underline{x}^* - \underline{b}|| \leq ||A\underline{x} - \underline{b}||$ for all $\underline{x} \in \mathbb{R}^n$.

Comments:
Some incorrect answers

- $||A\underline{x}^* - \underline{b}||$ is as close as possible to $||A\underline{x} - \underline{b}||$. This is a mixture of the two alternative answers I give above.

(18) **Theorem:** If $A^TA\underline{x}^* = A^T\underline{b}$, then $\underline{x}^*$ is ________.

Answer: the least-squares solution to the equation $A\underline{x} = \underline{b}$.

Comments: Some people gave the answer *the least-squares solution*. But the least-squares solution to *what?*. Whenever you say something like *this is the solution* you must tell the reader *what* it is the solution to. Sometimes it is clear from the context.

(19) **Definition:** An $n \times n$ matrix Q is orthogonal if ________.

Answer: $Q^TQ = I_n$ or $Q^TQ = I$.

Comments: Some people said *"the columns of Q are orthogonal"*. That is not correct: the columns of $A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$ are orthogonal to each other but

$$A^TA = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}.$$

Some people said *"the columns of Q are orthonormal"*. That's not quite correct: one needs to say *"the columns of Q are an orthonormal basis for $\mathbb{R}^n$"* (see part (4) of the Theorem in the next question).

(20) **Theorem:** The following 4 conditions on an $n \times n$ matrix Q are equivalent:
(a) Q is orthogonal;
(b) ________ for all $\underline{x} \in \mathbb{R}^n$;
(c) ________ for all $\underline{u}, \underline{v} \in \mathbb{R}^n$.
(d) the ________ an orthonormal basis ________

Answer:

Comments:

Part C.

True or False

Remember +1, 0, or -1.

Comments: The data are interesting. There are 85 questions in this section. Scores can range from $+85$ to -85. The average was 51 points and 43% of the class got less than 50 points.

(1) Let A and B be matrices having the same row-reduced echelon form, namely

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The equations $A\underline{x} = \underline{b}$ and $B\underline{x} = \underline{b}$ have the same solutions.

(2) Let A and B be as in question 1. Every solution to the equation $A\underline{x} = \underline{b}$ is a solution to the equation $(A+B)\underline{x} = 2\underline{b}$.

(3) Let A be as in question 1. The set of solutions to the equation $A\underline{x} = \underline{0}$ is a subspace of $\mathbb{R}^4$.

(4) Let A be as in question 1. The set of solutions to the equation $A\underline{x} = \underline{0}$ is a subspace having dimension 2.

(5) Let A be as in question 1. The function $T(\underline{x}) = A\underline{x}$ is a linear transformation having rank 3 and nullity 2.

(6) The kernel of a non-zero linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ is a subspace of $\mathbb{R}^2$ whose dimension is either 0 or 1.

(7) Every non-zero linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^1$ is onto.

(8) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is onto.

(9) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is one-to-one.

(10) Every non-zero linear transformation $T : \mathbb{R}^1 \to \mathbb{R}^6$ has rank one.

(11) Every non-zero linear transformation $T : \mathbb{R}^1 \to \mathbb{R}^6$ is one-to-one.

(12) If $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y-z \\ z-x \\ x-y \end{pmatrix}$, then $\ker(T) = \mathbb{R}\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

(13) The vectors $\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ are a basis for the range of the linear transformation $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ x_2 \end{pmatrix}$.

(14) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^6$ such that $\mathcal{R}(T) = \mathcal{N}(T)$.

(15) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^6$ such that $\dim \mathcal{R}(T) = \dim \mathcal{N}(T)$.

(16) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$. The null space of T is $\{(x_4, 0, 0, 0) \mid x_4 \text{ is a real number}\}$.

(17) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x_1, x_2) = (x_2, 0)$. The null space of T is $\{(x_3, 0) \mid x_3 \in \mathbb{R}\}$.

(18) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x_1, x_2) = (x_2, 0)$. Then $\{(5, 0)\}$ is a basis for the null space of T.

(19) The set $\{(-2, 0, 0, 0), (0, 0, 1, 1)\}$ is a basis for the range of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_2)$.

(20) The vector $(0, 0, 1)$ spans the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_2)$.

(21) The formula $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ ab \end{pmatrix}$ defines a linear transformation.

(22) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range is the set of solutions to the equation $x^2 + y^2 = z^2$.

(23) The linear transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$ is one-to-one.

(24) The linear transformation in question 23 is onto.

(25) The linear transformation in question 23 is its own inverse.

(26) -1 is an eigenvalue of the linear transformation in question 23.

(27) $\dim(E_1) = 2$ for the linear transformation in question 23.

(28) The linear transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x \end{pmatrix}$ is one-to-one.

(29) The linear transformation in question 28 is onto.

(30) The inverse of the linear transformation in question 28 is $\frac{1}{2}T$.

(31) If $T(\underline{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $T(\underline{e}_2) = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$, then $T\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

(32) Let S and T be the linear transformation $T(x, y) = (x + 2y, 2x + y)$ and $S(x, y) = (-x + 2y, 2x - y)$. Then $S \circ T = T \circ S$.

(33) Let A be any square matrix. Then A^2 commutes with $I - A$.

(34) If an $n \times n$ matrix has n distinct eigenvalues, then its columns form a basis for $\mathbb{R}^n$.

(35) 2 is an eigenvalue of the matrix $\begin{pmatrix} 3 & 2 & 3 \\ 4 & 7 & 6 \\ 5 & 7 & 11 \end{pmatrix}$.

(36) Let V be a 4-dimensional subspace of $\mathbb{R}^5$. Every set of five vectors in V is linearly dependent.

(37) Let V be a 4-dimensional subspace of $\mathbb{R}^5$. Every set of four vectors in V spans V.

(38) A 3-plane and a 2-plane in $\mathbb{R}^5$ can meet at a point.

(39) Let V be a 4-dimensional subspace of $\mathbb{R}^5$. There is a basis for $\mathbb{R}^5$ consisting of 4 vectors in V and one vector not in V.

(40) If V is a subspace of $\mathbb{R}^n$, then $(V^\perp)^\perp = V$.

(41) $\{3\underline{u} - 2\underline{v}, 2\underline{v} - 4\underline{w}, 4\underline{w} - 3\underline{u}\}$ is linearly dependent for all choices of $\underline{u}$, $\underline{v}$, and $\underline{w}$.

(42) Every set of orthogonal vectors in $\mathbb{R}^n$ is linearly independent.

(43) If $A = BC$, then every solution to $A\underline{x} = \underline{0}$ is a solution to $C\underline{x} = \underline{0}$.

(44) Let A be an $m \times n$ matrix and B an $n \times p$ matrix. If $C = AB$, then $\{C\underline{x} \mid \underline{x} \in \mathbb{R}^p\} \subset \{A\underline{w} \mid \underline{w} \in \mathbb{R}^n\}$.

(45) The linear span of the vectors $(4, 0, 0, 1)$, $(0, 3, 2, 0)$ and $(4, 3, 2, 1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

(46) The linear span of the vectors $(4, 0, 0, 1)$, $(0, 2, 0, -1)$ and $(4, 3, 2, 1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

(47) Let $\underline{u}$, $\underline{v}$, and $\underline{w}$ be vectors in $\mathbb{R}^n$. Then $\mathsf{span}\{\underline{u}, \underline{v}, \underline{w}\} = \mathsf{span}\{\underline{u} + 2\underline{v}, \underline{u} + 3\underline{v}, \underline{u} + \underline{v} + \underline{w}\}$.

(48) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 = x_2 - x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

(49) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 - x_2 + x_4 = 1\}$ is a subspace of $\mathbb{R}^4$.

(50) Every subspace of $\mathbb{R}^n$ has an orthogonal basis.

(51) Every subspace of $\mathbb{R}^n$ has an orthonormal basis.

(52) $\operatorname{span}\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}5\\6\\7\\8\end{pmatrix}, \begin{pmatrix}9\\10\\11\\12\end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\end{pmatrix}\right\}$.

(53) Every subset of a linearly independent set is linearly independent.

(54) The dimension of a subspace is the number of elements in it.

(55) Let X and Y be subsets of $\mathbb{R}^n$. If $X \subset Y$ and X is linearly dependent, then Y is linearly dependent.

(56) The row reduced echelon form of a square matrix is the identity if and only if the matrix is invertible.

(57) Every 4×4 matrix has an eigenvector.

(58) The vectors $(2, 2, -4, 3, 0)$ and $(0, 0, 0, 0, 1)$ are a basis for the subspace $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$ of $\mathbb{R}^5$.

(59) The vectors $(1, 1)$, $(1, -2)$, $(2, -3)$ are a basis for the subspace of $\mathbb{R}^4$ that is the set of solutions to the equations $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$.

(60) $\{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = x_1 + 2x_4 = 1\}$ is a subspace of $\mathbb{R}^5$.

The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cup \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.

(61) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cap \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.

(62) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is linear combination of the columns of A.

(63) The dimension of the column space of a matrix is equal to the dimension of its row space.

(64) If V and W are subspaces of $\mathbb{R}^n$ so is $\{\underline{v} + \underline{w} \mid \underline{v} \in V \text{ and } \underline{w} \in W\}$.

(65) If V and W are subspaces of $\mathbb{R}^n$ so is $\{2\underline{v} - 3\underline{w} \mid \underline{v} \in V \text{ and } \underline{w} \in W\}$.

(66) Let $\underline{u}$ and $\underline{v}$ be linearly independent vectors that belong to the subspace of $\mathbb{R}^4$ consisting of solutions to the homogeneous system of equations

$$\begin{aligned} 4x_1 - 3x_2 + 2x_3 - x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \end{aligned}$$

The vector $(1, 2, 3, 4)$ is a linear combination of $\underline{u}$ and $\underline{v}$.

(67) Let $\underline{u}$ and $\underline{v}$ be linearly independent vectors that belong to the subspace of $\mathbb{R}^4$ consisting of solutions to the homogeneous system of equations

$$\begin{aligned} 4x_1 - 3x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 3x_2 - 4x_3 + x_4 &= 0 \end{aligned}$$

The vector $(1, 2, 3, 4)$ is a linear combination of $\underline{u}$ and $\underline{v}$.

(68) If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

(69) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

(70) There are vectors $\underline{u}, \underline{v} \in \mathbb{R}^4$ such that $\underline{v} \cdot \underline{u} = 2012$.

(71) It is possible to choose values for the $*$s so that the matrix

$$\begin{pmatrix} 1 & 2 & * & * \\ 0 & 2 & * & 1 \\ 1 & 0 & 0 & -3 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

is invertible.

(72) The determinant of $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$ is zero.

(73) The determinant of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$ is zero.

(74) If $\underline{x}$ is an eigenvector for A with eigenvalue 5, then $4\underline{x}$ is an eigenvector for A with eigenvalue 20.

(75) If $\underline{x}$ is an eigenvector for A with eigenvalue 5, and an eigenvector for B with eigenvalue 4, then its is an eigenvector for AB with eigenvalue 20.

(76) The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ are a basis for the subspace of $\mathbb{R}^3$ that they span.

(77) The set $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

(78) The characteristic polynomial of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is $(1+t)^2(1-t)^2$.

(79) The matrix in the previous question has exactly two eigenspaces, each of dimension 2.

(80) $\{(1,1,1),\ (1,0,1),\ (3,2,3)\}$ is a basis for $\mathbb{R}^3$.

(81) If $\{\underline{u}, \underline{v}, \underline{w}\}$ is an orthonormal set of vectors, then $\underline{u}+\underline{v}+\underline{w}$ and $\underline{u}+2\underline{v}-3\underline{w}$ are orthogonal.

(82) The length of the vector $(1,2,2,4)$ is 9.

(83) There is a 2×3 matrix A and a 3×2 matrix B such that AB is the identity matrix.

(84) If A is a 2×3 matrix and B is a 3×2 matrix, then BA can not be the identity matrix because $\mathcal{N}(A) \neq \{\underline{0}\}$.

(85) If A is a 2×3 matrix and B is a 3×2 matrix, then BA can not be the identity matrix because $\mathcal{R}(B) \neq \mathbb{R}^3$.