Math 308 **Midterm Answers** **October 29, 2012**

The maximum possible score for the test was 24+28+67= 119 points. The highest score was 108, the mean 66, and the median 65. If you scored 15 on Part A, 20 on Part B, and 45 on Part C, you have a pretty reasonable understanding of the material.

Part A.

Short answer questions

(1) An $m \times n$ system of linear equations consists of $\underline{\quad m \quad}$ equations in $\underline{\quad n \quad}$ unknowns.

(2) Write down the system of linear equations you need to solve in order to find the curve $y = ax^3 + bx^2 + cx + d$ passing through the points $(2,2)$, $(1,3)$, $(-1,1)$, $(0,1)$.

$$\begin{aligned} 8a + 4b + 2c + d &= 2 \\ a + b + c + d &= 3 \\ -a + b - c + d &= 1 \\ d &= 1 \end{aligned}$$

(3) Write the system of linear equations in the previous question as a matrix equation $A\underline{x} = \underline{b}$. What are A, $\underline{x}$, and $\underline{b}$?

$$A = \begin{pmatrix} 8 & 4 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \underline{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

(4) I want a geometric description of the solutions: the set of solutions to a 2×4 system of linear equations is either
   (a) the <u>*the empty*</u> set
   (b) or <u>*a 3-plane* in $\mathbb{R}^4$</u>
   (c) or <u>*a plane* in $\mathbb{R}^4$</u>
   (d) or <u>$\mathbb{R}^4$</u>

(5) The set of solutions to a 4×4 system of linear equations is one of the four possibilities in the previous answer
   (a) or <u>*a line* in $\mathbb{R}^4$</u>
   (b) or a <u>*a point* in $\mathbb{R}^4$</u>

(6) I want a geometric description of the possibilities for the range of a 2×3 matrix. The range of a 2×3 matrix is either
   (a) <u>*the zero vector*</u>
   (b) or <u>*a line through the origin* in $\mathbb{R}^2$</u>
   (c) or <u>$\mathbb{R}^2$</u>

(7) Let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. I want you to describe the relation between S and T: If $A\underline{u} = \underline{b}$, then

$$T = \{\ldots \mid \ldots\}.$$

Your answer should involve $\underline{u}$ and S and the symbol $\in$ and some more.

$$T = \underline{u} + S \qquad \text{or} \qquad T = \{\underline{u} + \underline{v} \mid \underline{v} \in S\}.$$

(8) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form $\underline{u} + \underline{v}$ where $\underline{A\underline{v} = 0}$

(9) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \ldots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

$$A\underline{x} = x_1\underline{A}_1 + x_2\underline{A}_2 + \cdots + x_n\underline{A}_n$$

(10) Write down a matrix A such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$.

$$A = \begin{pmatrix} \quad \end{pmatrix}$$

(11) Write down a matrix A such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ 2y + z \\ 2x + y + z \\ 0 \end{pmatrix}$.

$$A = \begin{pmatrix} \quad \end{pmatrix}$$

(12) Let $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ be the columns of a matrix A and suppose that $2\underline{A}_1 + 2\underline{A}_4 - 2\underline{A}_3 = \underline{A}_2$. Write down a solution to the equation $A\underline{x} = \underline{0}$ of the form

$$\underline{x} = \begin{pmatrix} 6 \\ ? \\ ? \\ ? \end{pmatrix}$$

(13) $(4, 4, 3, 3)$ and $(2, 2, 2, 2)$ are a basis for the subspace of $\mathbb{R}^4$ that is the set of all solutions to the equations _________ and _________.

(14) Find a basis for the line $x_1 - x_2 = x_2 + 2x_3 = 3x_1 - 2x_4 = 0$ in $\mathbb{R}^4$.

(15) Find two linearly independent vectors that lie on the plane in $\mathbb{R}^4$ given by the equations

$$x_1 - x_2 + 2x_3 - x_4 = 0$$
$$x_1 - x_2 + 3x_3 - x_4 = 0$$

(16) The vector $(2, 1, 0, 1)$ a linear combination of the vectors in your answer to the previous question because _________.

(17) Let A be a 4×5 matrix and $\underline{b} \in \mathbb{R}^5$. Suppose the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The independent or free variables are _________.

(18) Write down all solutions to the equation $A\underline{x} = \underline{b}$ in the previous equation in the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = ?+?+?$$

where the free variables appear on the right-hand side of the $=$ sign

Part B.

Complete the definitions and Theorems.
There is a difference between theorems and definitions.
Don't write the part of the question I have already written. Just fill in the blank.

(1) **Definition:** The system of equations $A\underline{x} = \underline{b}$ is homogeneous if ________
(2) **Definition:** Two systems of linear equations are equivalent if ________
(3) **Definition:** A vector $\underline{x}$ is a linear combination of $\underline{v}_1, \ldots, \underline{v}_n$ if ________
(4) **Definition:** The linear span of $\underline{v}_1, \ldots, \underline{v}_n$ is the set of all ________
(5) **Definition:** A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly independent if the only solution to the equation ________
(6) **Definition:** $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly dependent if ________ $= \underline{0}$ for some ________
(7) **Definition:** A matrix E is in row echelon form if
  (a) ________
  (b) ________
(8) **Definition:** A matrix is in row reduced echelon form if
  (a) it is ________ and
  (b) ________ and
  (c) ________.
(9) **Definition:** In this question use R_i to denote the i^{th} row of a matrix.
  The three elementary row operations are
  (a) ________
  (b) ________
  (c) ________
(10) **Definition:** The range of an $m \times n$ matrix A is $\mathcal{R}(A) = \{\cdots \mid \cdots\}$.
(11) **Definition:** A subset W of $\mathbb{R}^n$ is a subspace if
  (a) ________
  (b) ________
  (c) ________
(12) **Definition:** A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_d\}$ is a basis for a subspace V of $\mathbb{R}^n$ if ____________
(13) **Definition:** The dimension of a subspace V of $\mathbb{R}^n$ is ____________.
(14) **Definition:** The nullity of a matrix is ____________.
(15) **Theorem:** The rank of a matrix is equal to the dimension of ____________.
(16) **Theorem:** The nullity and rank of a $p \times q$ matrix are related by the formula ____________.
(17) **Theorem:** Two systems of linear equations are equivalent if the row reduced echelon forms of their augmented coefficient matrices are ____________.

(18) **Theorem:** Let A be an $n \times n$ matrix and $\underline{b} \in \mathbb{R}^n$. The equation $A\underline{x} = \underline{b}$ has a unique solution if and only if A is ____________.

(19) **Theorem:** Let A be an $m \times n$ matrix and let E be the row-reduced echelon matrix that is row equivalent to it. Then the non-zero rows of E are a basis for _________.

(20) **Theorem:** A set of vectors is linearly dependent if and only if one of the vectors is __________ of the others.

(21) **Theorem:** An $n \times n$ matrix A is invertible if and only if its columns ________________.

(22) **Theorem:** An $n \times n$ matrix A is invertible if and only if its range ________________.

Part C.

True or False

The three numbers after the answer are the % of students who were correct, wrong, did not answer. For example, **[40, 25, 35]** says that 40% gave the correct answer, 25% gave the wrong answer, and 35% did not answer the question.

On an average True/False question, 40% of the class either gave the wrong answer or chose to not answer the question. That that tells me many of you have a lot of learning to do before the final.

Advice: You should go through every question that you got wrong or did not answer and ask yourself *What am I misunderstanding, or what piece of knowledge am I missing, that explains why I did not give the correct answer?*

I will give (brief) comments on questions which appear to have been the most difficult. My answers are not complete in the sense that I only consider *some* of the many reasons someone might get a question wrong. If you are not satisfied by my responces, i.e., of the

(1) Every set of five vectors in $\mathbb{R}^2$ spans $\mathbb{R}^2$.

False. **[66,20,15]**
Comments: The vectors $\{(0,1),(0,2),(0,3),(0,4),(0,5)\}$ do not span $\mathbb{R}^2$. They span a line, the line through $(0,0)$ and $(0,1)$. The line is a 1-dimensional subspace of $\mathbb{R}^2$. You might observe that $(1,0)$ is not in the linear span of $\{(0,1),(0,2),(0,3),(0,4),(0,5)\}$ so they do not span $\mathbb{R}^2$. If you missed this question you should try computing a few linear combinations of $\{(0,1),(0,2),(0,3),(0,4),(0,5)\}$.

(2) Every set of three vectors in $\mathbb{R}^3$ is linearly independent.

False. **[98,0,2]**

(3) Every set of four vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

False. **[81,12,7]**

(4) Every set of six vectors in $\mathbb{R}^5$ is linearly dependent.

True. **[63,32,5]**
Comments: In class, we proved a theorem that says if $n > m$, then every set of n vectors in $\mathbb{R}^m$ is linearly dependent (go and find it in your notes, in the book, and in my online notes, and try to understand the proof). The key step in the proof was that a set of m homogeneous equations in n unknowns always has a non-trivial solution if $n > m$; that solution produced the coefficients for a nontrivial linear relation among the n vectors.

(5) The row space of a $p \times q$ matrix is a subspace of $\mathbb{R}^q$.

True. **[76,17,7]**

(6) The column space of a $p \times q$ matrix is a subspace of $\mathbb{R}^p$.

True. [**76,15,10**]

(7) The matrix $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ has an inverse if and only if $a \neq b$ and $a \neq -b$.

True. [**78,20,2**]

(8) A homogeneous system of 7 linear equations in 8 unknowns always has a non-trivial solution.

True. [**93,5,2**]

(9) The set of solutions to a system of 5 linear equations in 6 unknowns can be a 3-plane in $\mathbb{R}^6$.

True. [**73,0,27**]
Comments: For example, the set of solutions to the homogeneous system of the five equations $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_1 + x_2 = 0$, $x_1 + x_3 = 0$ is the 3-dimensional subspace of $\mathbb{R}^6$ spanned by e_4, e_5, and e_6.

(10) If $B^3 + B^2 = 3I - 3B$, then $\frac{1}{3}(B^2 + B + 3I)$ is the inverse of B.

True. [**68,5,27**]
Comments: In order to determine if a a statement of the form "A is the inverse of B" is true you can calculate AB and BA. If $AB = BA = I$, the claim is true; if not the claim is false. In this question, $\frac{1}{3}(B^2 + B + 3I)B = \frac{1}{3}(B^3+B^2+3B) = \frac{1}{3}(3I-3B+3B) = I$. The product of B and $\frac{1}{3}(B^2+B+3I)$ is also I so the claim is TRUE.

(11) If the dimension of $\text{span}(\underline{v}_1, \dots, \underline{v}_k)$ is k, then $\{\underline{v}_1, \dots, \underline{v}_k\}$ is linearly independent.

True. [**44,20,34**]
Comments: Serious gaps in people's understanding of this question and the next. You should know that the dimension of $\text{span}(\underline{v}_1, \dots, \underline{v}_k)$ is k if and only if $\{\underline{v}_1, \dots, \underline{v}_k\}$ is linearly independent. We may have proved this in class. If not that result is probably implicit, if not explicit, in the section about dimension and bases.

By definition, the dimension of a vector space is the number of elements in a basis for it. If $\{\underline{v}_1, \dots, \underline{v}_k\}$ is linearly dependent at least one of $\underline{v}_1, \dots, \underline{v}_k$ is a linear combination of the others. However, if $\underline{v}_k$, for example, is a linear combination of $\underline{v}_1, \dots, \underline{v}_{k-1}$, then $\text{span}(\underline{v}_1, \dots, \underline{v}_k)$ is equal to $\text{span}(\underline{v}_1, \dots, \underline{v}_{k-1})$.

(12) If $\{\underline{v}_1, \dots, \underline{v}_k\}$ is linearly independent, then the dimension of $\text{span}(\underline{v}_1, \dots, \underline{v}_k)$ is k.

True. [**51,10,39**]
Comments: See the comments to the previous question.

(13) $\mathbb{R}^1$ has infinitely many subspaces

False. **[20,41,39]**

Comments: Serious gaps in people's understanding of this question. First, you need to know that $\mathbb{R}^1$ is just a fancy notation for $\mathbb{R}$, the set of real numbers: just as $\mathbb{R}^2$ denotes all ordered pairs of real numbers, i.e., $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$, $\mathbb{R}^1 = \{a \in \mathbb{R} \mid a \in \mathbb{R}\} = \mathbb{R}$. The only subspaces of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$. If V is any non-zero subspace of $\mathbb{R}$ it contains a non-zero number, v say, and then, because V is a subspace of $\mathbb{R}$, it contains all scalar multiples of v, i.e., V contains cv for all $c \in \mathbb{R}$; however, every real number is a multiple of v; for example, $a \in \mathbb{R}$ is equal to $(av^{-1})v$.

(14) $\mathbb{R}^2$ has exactly three subspaces

False. **[86,5,10]**

(15) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.

True. **[90,7,2]**

(16) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

True. **[100,0,0]**

(17) There is a matrix whose inverse is $\begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 3 \\ 0 & 3 & 6 \end{pmatrix}$.

False. **[76,17,7]**

(18) If $B^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix A such that $AB = E$.

False. **[76,20,5]**

(19) If $B^7 = 2I$, then $B^{-3} = \frac{1}{2}B^4$.

True. **[24,10,66]**

Comments: See the comments about Question 10. To see that the claim is TRUE simply multiply B^3 by $\frac{1}{2}B^4$ in both orders and observe that the product is $\frac{1}{2}B^7 = I$. So $\frac{1}{2}B^4$ is the inverse of B^3.

Maybe this was difficult because you forgot the discussion we had about defining powers of square matrices. If B is a square matrix and n a positive number we write B^n for the product of B with itself n times. For example, B^4 is a short notation for $BBBB$. When B is an invertible matrix B^{-3} means $B^{-1}B^{-1}B^{-1}$. We use the same conventions here as we do for numbers, i.e., $x^4 = xxxx$ and $x^{-3} = x^{-1}x^{-1}x^{-1}$.

(20) If $\underline{u}$, $\underline{v}$, and $\underline{w}$, are any vectors in $\mathbb{R}^4$, then $\{3\underline{u}-2\underline{v}, 2\underline{v}-4\underline{w}, 4\underline{w}-3\underline{u}\}$ is linearly dependent.

True. [**76,7,17**]
They are linearly dependent because $1.(3\underline{u}-2\underline{v})+1.(2\underline{v}-4\underline{w})+1.(4\underline{w}-3\underline{u})=\underline{0}$.

(21) If $A = BC$, then every solution to $A\underline{x} = \underline{0}$ is a solution to $C\underline{x} = \underline{0}$.

False. [**63,15,22**]
Comments: First, look at my comment on the next question.
As you know, a product of non-zero matrices can be zero. For example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

i.e, it is possible to have non-matrices B and C such that $BC = 0$. Every $\underline{x}$ is a solution is a solution to the equation $0\underline{x} = \underline{0}$. However, if $C \neq 0$, there are vectors non-zero $\underline{x}$ such that $C\underline{x} \neq \underline{0}$.

(22) If $A = BC$, then every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.

True. [**73,12,15**]
Comments: This question can be answered by a simple calculation: if $C\underline{x} = 0$, then $A\underline{x} = BC\underline{x} = B\underline{0} = \underline{0}$. This calculation shows that if $C\underline{x} = \underline{0}$, then $A\underline{x} = \underline{0}$; i.e., every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.

(23) Let A be an $m \times n$ matrix and B an $n \times p$ matrix. If $C = AB$, then $\{C\underline{x} \mid \underline{x} \in \mathbb{R}^p\} \subset \{A\underline{w} \mid \underline{w} \in \mathbb{R}^n\}$.

True. [**39,27,34**]
Comments: This is a *very* simple question designed to determine whether you understand notation. You are told that $C = AB$, so $\{C\underline{x} \mid \underline{x} \in \mathbb{R}^p\} = \{AB\underline{x} \mid \underline{x} \in \mathbb{R}^p\}$. A vector $AB\underline{x}$ is certainly a (right) multiple of A.

(24) The linear span of the vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ is the 3-plane $2x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

False. [**66,7,27**]
Comments: The first thing to consider in answering this question is whether the vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ lie on the 3-plane $2x_1 - 2x_2 + 3x_3 - 4x_4 = 0$. They do if and only if they satisfy the equation $2x_1 - 2x_2 + 3x_3 - 4x_4 = 0$. The point $(4,0,0,1)$ does not satisfy the equation so does not lie on the 3-plane $2x_1 - 2x_2 + 3x_3 - 4x_4 = 0$.

(25) The linear span of the vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

True. [**71,2,27**]
Comments: This is identical to the previous question *except* the equation of the 3-plane is different. All three vectors satisfy the equation $x_1 - 2x_2 +$

$3x_3 - 4x_4 = 0$ so lie of the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$. Next, you need to ask if they are a basis, i.e., are they linearly independent? They are: to see that just use the definition of linear independence.

(26) If $\underline{a}$, $\underline{b}$, and $\underline{c}$ are linear combinations of $\underline{u}$, $\underline{v}$, and $\underline{w}$, and $\underline{u}, \underline{v}, \underline{w} \in \text{span}\{\underline{x}, \underline{y}, \underline{z}\}$ then $\underline{a}$, $\underline{b}$, and $\underline{c}$ are linear combinations of of $\underline{x}$, $\underline{y}$, and $\underline{z}$.

True. [**76,0,24**]

(27) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 = x_2 - x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

True. [**73,10,17**]
Comments: The set in the question is the set of solutions to the system of homogeneous linear equations

$$x_1 + x_3 = 0$$
$$x_2 - x_4 = 0.$$

The set of solutions to every system of homogeneous linear equations is a subspace. To see this first think of the system as a single equations $A\underline{x} = \underline{0}$. It is obvious that $\underline{0}$ is a solution. If $\underline{u}$ and $\underline{v}$ are solutions so is $\underline{u} + \underline{v}$ because $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} = \underline{0}$. If $l \in \mathbb{R}$ and $\underline{v}$ is a solution so is $l\underline{v}$ because $A(l\underline{v}) = lA\underline{v} = l\underline{0} = \underline{0}$. Hence the three conditions for a subset of $\mathbb{R}^4$ to be a subspace are all satisfied.

(28) $\{\underline{x} \in \mathbb{R}^4 \mid x_1 + x_3 = x_2 - x_4\}$ is a subspace of $\mathbb{R}^4$.

True. [**54,12,34**]
Comments: The set in the question is the set of solution to the single homogeneous linear equation

$$x_1 + x_3 - x_2 + x_4 = 0$$

so is a subspace for the reasons given in the comments to the previous question.

(29) $\{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a subspace of $\mathbb{R}^5$.

False. [**63,10,27**]
Comments: The vector $\underline{0}$ is not a solution to the equation $x_3 + x_4 = 1$ so the set is not a subspace: every subspace contains $\underline{0}$.

(30) For any vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, $\{\underline{u}, \underline{v}, \underline{w}\}$ and $\{\underline{u} - \underline{v}, \underline{v}, \underline{w} - \underline{u}\}$ have the same linear span.

True. [**73,2,24**]

(31) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ have the same the linear span as $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$.

True. [**37,31,31**]

(32) The dimension of a subspace is the number of elements in it.

False. **[76,10,14]**
Comments: This statement is not just false but absolutely nuts. If V is any non-zero subspace of $\mathbb{R}^n$, there are infinitely may different elements in V.

If V is a non-zero subspace it contains a non-zero element, $\underline{v}$ say, and therefore contains the infinitely many elements $\{c\underline{v} \mid c \in \mathbb{R}\}$, i.e., V contains all scalar multiples of $\underline{v}$ and there are infinitely many such multiples.

Furthermore, if V is any subspace of $\mathbb{R}^9$, for example, then $\dim(V)$ is either $0, 1, 2, \ldots$, or 9. And, if k is an integer between 0 and n, $\mathbb{R}^n$ has a subspace of dimension k.

(33) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5, \underline{v}_6\}$ is linearly independent so is $\{\underline{v}_2, \underline{v}_4, \underline{v}_6\}$.

True. **[76,7,17]**

(34) If $\{\underline{v}_2, \underline{v}_4, \underline{v}_6\}$ is linearly dependent so is $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5, \underline{v}_6\}$.

True. **[41,37,22]**
Comments: This result follows immediately from (an understanding of) the definition. Look at the definition again. If $a\underline{v}_2 + b\underline{v}_4 + c\underline{v}_6 = \underline{0}$, then $0\underline{v}_1 + a\underline{v}_2 + 0\underline{v}_3 + b\underline{v}_4 + 0\underline{v}_5 + c\underline{v}_6 = \underline{0}$.

(35) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, then $\{\underline{v}_1 + 3\underline{v}_2, 3\underline{v}_2 + \underline{v}_3, \underline{v}_3 - \underline{v}_1\}$ is linearly dependent.

True. **[46,20,34]**
Comments: This result follows immediately from (an understanding of) the definition. Look at the definition again. Now compute $(\underline{v}_1 + 3\underline{v}_2) - (3\underline{v}_2 + \underline{v}_3) + (\underline{v}_3 - \underline{v}_1) = \underline{0}$.

(36) If A is an invertible matrix, then $A^{-1}\underline{b}$ is the unique solution to the equation $A\underline{x} = \underline{b}$.

True. **[85,0,15]**

(37) $\text{span}\{\underline{u}_1, \ldots, \underline{u}_r\} = \text{span}\{\underline{v}_1, \ldots, \underline{v}_s\}$ if and only if every $\underline{u}_i$ is a linear combination of the $\underline{v}_j$s and every $\underline{v}_j$ is a linear combination of the $\underline{u}_i$s.

True. **[71,0,29]**

(38) The row reduced echelon form of a square matrix is the identity if and only if the matrix is invertible.

True. **[83,5,12]**

(39) A square matrix has an inverse if and only if its columns are linearly dependent.

False. **[73,15,12]**

(40) The row and column spaces of a matrix always have the same dimension.

True. [**61,24,15**]

(41) The rank of a matrix is at most the number of columns it has.

True. [**61,20,20**]

(42) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

True. [**85,2,12**]

(43) There is a matrix whose inverse is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

True. [**81,2,17**]

(44) The column space of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

True. [**59,12,29**]

(45) For all matrices A and B, $\mathcal{R}(AB) \subset \mathcal{R}(A)$

True. [**37,17,46**]
Comments: This result follows immediately from (an understanding of) the definition. Look at the definition again: $\mathcal{R}(A)$ consists of all right multiples of A, the vectors $A\underline{x}$, and $\mathcal{R}(AB)$ consists of all right multiples of AB. But a right multiple of AB, i.e., an element $AB\underline{x}$ is automatically a right multiple of A; after all, $AB\underline{x}$ is $A(B\underline{x})$.

(46) If $A^3 = B^3 = C^3 = I$, then $(A^4B^5C^6)^{-1} = BA^2$.

True. [**20,12,68**]
Comments: See the comments about Questions 10 and 19. To see that the claim is TRUE simply multiply $A^4B^5C^6$ on left by BA^2. to get $(BA^2)(A^4B^5C^6) = BA^6B^5C^6 = BI^2B^5I^2 = B^6 = II = I$. We also proved a theorem saying that if P and Q are square matrices, then $PQ = I$ if and only if $QP = I$.

(47) If W is a subspace of $\mathbb{R}^n$ that contains $\underline{u} + \underline{v}$, then W contains $\underline{u}$ and $\underline{v}$.

False. [**54,22,24**]

(48) If $\underline{u}$ and $\underline{v}$ are linearly independent vectors on the plane in $\mathbb{R}^4$ given by the equations $x_1 - x_2 + x_3 - 4x_4 = 0$ and $x_1 - x_2 + x_3 - 2x_4 = 0$, then $(1, 2, 1, 0)$ a linear combination of $\underline{u}$ and $\underline{v}$.

True. **[44,10,46]**

(49) The range of a matrix is its columns.

False. **[76,2,22]**

(50) The vectors $(2, 2, -4, 3, 0)$ and $(0, 0, 0, 0, 1)$ are a basis for the subspace $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$ of $\mathbb{R}^5$.

True. **[27,27,46]**

(51) The vectors $(1, 1)$, $(1, -2)$, $(2, -3)$ are a basis for the subspace of $\mathbb{R}^4$ give by the solutions to the equations $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$.

False. **[37,10,54]**

(52) The set of solutions to an $m \times n$ system of linear equations is a subspace of $\mathbb{R}^n$.

False. **[29,34,37]**

(53) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cap \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.

True. **[51,2,46]**

(54) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cap \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is equal to the set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 - x_4 = 0\}$.

True. **[32,5,63]**

(55) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is linear combination of the rows of A.

False. **[32,5,63]**

(56) The equation $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^n$ if A is an $n \times n$ matrix with rank n.

True. **[66,7,27]**

(57) There is a 3×3 matrix A such that $\mathcal{R}(A) = \mathcal{N}(A)$.

False. **[58,5,37]**

(58) There is a 2×2 matrix A such that $\mathcal{R}(A) = \mathcal{N}(A)$.

True. **[56,0,44]**

(59) There is a 2×2 matrix A such that $\mathcal{R}(A)$ is the union of the subspaces $x = y$ and $x = -y$.

False. **[5,12,83]**

(60) Let A be the 2×2 matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

The null space of A is $\left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \;\middle|\; t \text{ is a real number} \right\}$.

True. **[49,7,44]**

(61) Let A be a 4×3 matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ x_2 \end{pmatrix}.$$

The range of A has many bases; one of them consists of the vectors

$$\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

True. **[42,2,56]**

In the next 6 questions, A is a 4×4 matrix whose columns $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ have the property that $\underline{A}_1 + \underline{A}_2 = \underline{A}_3 + \underline{A}_4$.

(62) The columns of A span $\mathbb{R}^4$.

False. **[20,44,36]**

(63) A is not invertible.

True. **[63,7,29]**

(64) The rows of A are linearly dependent.

True. **[49,10,41]**

(65) The equation $A\underline{x} = 0$ has a non-trivial solution.

True. **[59,2,39]**

(66) $A \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix} = \underline{0}$.

False. $[\mathbf{41,15,44}]$

(67) $A \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \underline{0}$.

True. $[\mathbf{44,17,39}]$