Math 308 **Midterm Solutions** **July 16, 2012**

I haven't had time to write solutions for every question yet. Will try to do so but it would be helpful if you emailed me and asked for solutions to particular questions. Maybe there are some questions where I don't need to say anything.

Part A.

Short answer questions

(1) How many equations and how many unknowns are there in the system of linear equations whose augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 2 & 3| & 2 \\ 3 & 6 & 2 & 1 & 1 & 4| & 4 \\ 0 & 0 & -3 & -1 & 0 & 0| & 6 \\ 1 & 3 & 0 & 0 & 2 & 4| & 2 \\ 2 & 6 & 0 & 0 & 4 & 8| & 4 \end{pmatrix}?$$

(2) Write down the system of linear equations you need to solve in order to find the curve $y = ax^3 + bx^2 + cx + d$ passing through the points $(2, 1)$, $(1, 3)$, $(-1, 1)$, $(1, 1)$.

Almost everyone got this correct.

Comments: The funny thing is no one observed that the system of equations is inconsistent. The inconsistency arises because there is no curve of the form $y = f(x)$ that will pass through both $(1, 3)$ and $(1, 1)$! Passing through $(1, 3)$ says $f(1) = 3$ and passing through $(1, 1)$ says $f(1) = 1$. But then f would not be a function! I didn't pay enough attention when I wrote the question.

(3) Write the system of linear equations in the previous question as a matrix equation $A\underline{x} = \underline{b}$. What are A, $\underline{x}$, and $\underline{b}$?

Comments: One problem was that a couple of people wrote

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

instead of

$$\underline{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

The unknowns in the previous question are a, b, c, d not x_1, x_2, x_3, x_4.

A couple of people wrote down the augmented matrix for the problem but that wasn't what I asked for.

(4) I want a geometric description of the solutions: the set of solutions to a 2×4 system of linear equations is either

(a) the _________ set
(b) or _________ in ___
(c) or _________ in ___
(d) or $\underline{\mathbb{R}^?}$

Answer: The empty set, or a plane in $\mathbb{R}^4$, or a 3-plane in $\mathbb{R}^4$, or $\mathbb{R}^4$.

Comments: Several people said *"An empty set"*, but there is only one empty set so they should say *"the empty set"*; of course, I had already given you the clue in (a) by writing *the* _________ *set*.

Some people said *"zero set"* or *"trivial set"*. That's not right. We call the empty set the empty set. I agree that the empty set is pretty trivial, and I agree that the number of elements in it is zero. Nevertheless, it is called the empty set. Compare with an *"empty box"*. It is not the same as the trivial box or the zero box (whatever they might be).

A common error was to give an answer which said *"a plane through the origin"* and/or a *"a 3-plane containing the origin"* but the question does not say the system of equations is homogeneous so the set of solutions need not be a subspace. For example, the solutions to the equations $x_1 = 1$ and $x_2 = 2$ are the points on the plane $\{(1, 2, x_3, x_4)\ |x_3, x_4 \in \mathbb{R}\}$ and this is *not* a subspace because it does not contain $\underline{0}$.

One proposed answer that appeared several times really puzzled me, namely *"a plane in* $\mathbb{R}^3$*"*. If anyone could help me understand what lies behind such a response I would be very grateful. There are 4 unknowns so all solutions are points in $\mathbb{R}^4$. Similarly, a couple of people proposed *"a line in* $\mathbb{R}^2$*"* as an answer. Perhaps I just need to place a little more emphasis that a solution to a system of equations in n unknowns is a point in $\mathbb{R}^n$. This is true even outside the realm of linear equations. For example every solution to the system of equations $x_1x_2x_3x_4 = 3$ and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5$ is a point in $\mathbb{R}^4$.

A line in $\mathbb{R}^4$ can NOT be the solution to a 2×4 system of linear equations. You need 3 linear equations to specify a line in $\mathbb{R}^4$ (compare with a line in $\mathbb{R}^3$ which requires two equations to specify it). To get a line you need exactly 1 independent or free variable. Or, in more familiar terms, a line is given by a set of parametric equations with exactly *one* parameter, e.g., the line $(1 + 2t, 3 - t, 4 + 3t, 5t)$, $t \in \mathbb{R}$. The number of independent or free variables for an $m \times n$ system $A\underline{x} = \underline{b}$ is $n - \operatorname{rank}(A)$. But $\operatorname{rank}(A) \le m$ and $\le n$, so for a 2×4 system, $\operatorname{rank}(A) \le 2$, and the number of independent or free variables is $\ge 4 - 2 = 2$.

Another way to see this is to think in terms of dimension even though we haven't yet got a formal definition of that. One linear equation usually drops the dimension by 1 so, in $\mathbb{R}^4$, the solution to one linear equation is a 3-plane (except in exceptional situations), and the solution to two linear equations is a 2-plane (except in exceptional situations in which it is $\mathbb{R}^4$, or a 3-plane, or the empty set).

(5) The set of solutions to a 4×4 system of linear equations is one of the four possibilities in the previous answer
(a) or a _________ in ___
(b) or a _________ in ___

Answer: A point in $\mathbb{R}^4$, or a line in $\mathbb{R}^4$.

Comments: As explained in the comments on the previous question, the line need not contain $\underline{0}$.

In answering this question and the last one a few people called the range *" a 3d shape in $\mathbb{R}^4$"* and/or *" a 4D shape in $\mathbb{R}^4$"*. *"Shape"* is very vague and definitely out of place in linear algebra. The *only* geometric shapes that occur in linear algebra are points, lines, planes, 3-planes, and their higher dimensional analogues. Blobs, spheres, parabolas, ellipsoids, circles; *none* of these things turn up in linear algebra. It might sound circular, but the only "shapes" that arise in linear algebra are linear ones. By that I mean, a "shape" that has the property that if p and q belong to it so does the line though p and q and by *line* I mean the infinite line extending in both directions.

I can't overemphasize the importance of having geometric view in linear algebra and in that geometric view having no visual images other than linear ones. Eliminate all other "shapes". Oh, and apart from points every geometric object in linear algebra has infinitely many points on it and is a union of lines.

One person said *"a 4-plane in $\mathbb{R}^4$"* in (a) and then said *"a 4-dimensional subspace of $\mathbb{R}^4$"* in (b). But the only 4-plane in $\mathbb{R}^4$ is $\mathbb{R}^4$ and the only 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$ so both these answers are the same as the answer to (d) in the previous question.

(6) I want a geometric description of the possibilities for the range of a 2×3 matrix. The range of a 2×3 matrix is either
  (a) _________
  (b) or _________ in ____
  (c) or $\underline{\mathbb{R}^?}$

Answer: The zero vector $\underline{0}$, or a line in $\mathbb{R}^2$, or $\mathbb{R}^2$ itself.

Comments: The range of A is $\mathcal{R}(A) = \{A\underline{x} \mid \underline{x} \in \mathbb{R}^4\}$.

In contrast to the previous two questions, the range of a matrix *is* a subspace–we proved it in class. Check the proof if you are not sure why it is a subspace.

The most common error was saying that the range is contained in $\mathbb{R}^4$. It is contained in $\mathbb{R}^2$ because when A is 2×4, the $\underline{x}$ in $A\underline{x}$ is a 4×1 matrix and $A\underline{x}$ is a 2×1 matrix, i.e., an element in $\mathbb{R}^2$.

Some people gave *"a 2-plane in $\mathbb{R}^2$"* as a possibility. *But* there is only one 2-plane in $\mathbb{R}^2$, namely $\mathbb{R}^2$ itself. This error makes me think a good question for the final would be to ask how many 3-planes there are in $\mathbb{R}^3$.

You don't need to say *"The zero vector $\underline{0}$"* as I do. You could simply write $\{\underline{0}\}$. But it would not be correct to write 0 because 0 denotes the number 0, not the vector $\underline{0}$. It is also wrong to write $\{\emptyset\}$ when you mean $\{\underline{0}\}$. The notation $\{\emptyset\}$ means *the set containing the empty set.* Notice, for example, that $\{\emptyset\}$ is a set having one element, that element being the empty set. Likewise, $\{\emptyset, \{\emptyset\}\}$ has two elements. The set $\{\{1, 2, 3, 4\}, \{6, 7, 8, \}\}$ has two elements.

(7) Let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. I want you to describe the relation

between S and T: If $A\underline{u} = \underline{b}$, then

$$T = \{\dots | \dots\}.$$

Your answer should involve $\underline{u}$ and S and the symbol $\in$ and some more.

Comments:

(8) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form _________ where _________

Answer: $\underline{x} = \underline{u} + \underline{v}$ (OR just $\underline{u} + \underline{v}$) where $\underline{v}$ is a solution to the equation $A\underline{x} = \underline{0}$.

Comments: Notice the word "a" in my answer. If you had "the" instead of "a" your answer would be incorrect. In general the equation $A\underline{x} = \underline{0}$ will have infinitely many solutions so your answer must allow for that possibility. When you say "the solution" you imply there is only one solution.

One "answer" was $A(\underline{u} + \underline{v}) = \underline{b}$. This is certainly true if $A\underline{v} = \underline{0}$ but even if you added that it would not be correct. Look at the exact wording of the question. You must make a statement about the solution, i.e., about $\underline{x}$ and saying $A(\underline{u} + \underline{v}) = \underline{b}$ is not saying what $\underline{x}$ is.

(9) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

Answer: $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$

Comments: It is important to take care. For example, whoever wrote "$\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ *where* $x \in \mathbb{R}$" could produce a correct answer with a little more care. He/she need only put an A before the $\underline{x}$ on the left and, if they wish to say more about $\underline{x}$, could add "*where*

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and $x_1, \dots, x_n \in \mathbb{R}$." In fact, we should probably all tell the reader that $x_1, \dots, x_n$ are the entries of $\underline{x}$. I have been a little lazy because I always adopt the convention that when $\underline{x}$ and $x_1, \dots, x_n$ appear in the same paragraph the x_is are assumed to be the entries of $\underline{x}$ and I don't say that explicitly.

One person said "where $\underline{A}_1, \dots, \underline{A}_n \in \mathbb{R}$. Very wrong because $\underline{A}_i$ denotes the i^{th} column of A and will only be in $\mathbb{R}$, i.e., will only be a number, when A is a $1 \times n$ matrix.

Some people wrote $A\underline{x} = \underline{A}_1 x_1 + \cdots + \underline{A}_n x_n$. Although not really incorrect it appears clunky and goes against convention. In high school you would write an equation as $y = 3x^2 + 2x - 7$ not as $y = x^2 3 + x2 - 7$. The latter is clunky and goes against convention. It is also open to some confusion when you write things like $x^3 3$ or $x7y$. Get in the habit of writing $x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$. The numbers, constants, coefficients, are almost always written to the left of other things like matrices, vectors, unknowns. Think about your street cred.

(10) Write down a 4×3 matrix A such that $A\begin{pmatrix}x\\y\\z\end{pmatrix} = z\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + x\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$.

$$\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\\0&0&0\end{pmatrix}$$

Comments: This question and the two after it are designed to see if you really understand what your answer to Question 9 means. About 30% of the class got this question and the next two wrong even though they got Question 9 correct. This tells me those people are content to memorize something without understanding what it means. Not good - intellectually lazy.

A few people just made careless errors but it was apparent they did understand the meaning of the Question 9. It is difficult to read carefully when time is short.

(11) Write down a 4×3 matrix A such that $A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\y+z\\x+y+z\\0\end{pmatrix}$.

This question is "identical" to the previous one so the answer is the same, namely

$$\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\\0&0&0\end{pmatrix}.$$

Comments: Understand why this question is the same as the previous one.

(12) Let $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ be the columns of a 4×4 matrix A and suppose that $2\underline{A}_1 + 2\underline{A}_4 = \underline{A}_3 - 3\underline{A}_2$. Write down a solution to the equation $A\underline{x} = \underline{0}$ of the form

$$\underline{x} = \begin{pmatrix}6\\?\\?\\?\end{pmatrix}$$

Answer: $\underline{x} = \begin{pmatrix}6\\9\\-3\\6\end{pmatrix}$

Comments: You are told that $2\underline{A}_1 + 3\underline{A}_2 - \underline{A}_3 + 2\underline{A}_4 = \underline{0}$. Since $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ the equation $2\underline{A}_1 + 3\underline{A}_2 - \underline{A}_3 + 2\underline{A}_4 = \underline{0}$ is telling you that

$$A \begin{pmatrix} 2 \\ 3 \\ -1 \\ 2 \end{pmatrix} = \underline{0}.$$

But you are asked for a solution to $A\underline{x}$ in which $x_1 = 6$. If $A\underline{x} = \underline{0}$, then $Ac\underline{x} = \underline{0}$ for all $c \in \mathbb{R}$. Hence

$$3 \begin{pmatrix} 2 \\ 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -3 \\ 6 \end{pmatrix}$$

is a solution to $A\underline{x} = \underline{0}$.

(13) $(3,1,1,3)^T$ and $(1,2,2,1)^T$ are solutions to the two (different!) homogeneous equations _________ and _________

Answer: There are infinitely many answers but one of the simplest is $x_1 - x_4 = 0$ and $x_2 - x_3 = 0$.

Comments: The question asks for two equations of the form $ax_1 + bx_2 + cx_3 + dx_4 = 0$ having the property that $(x_1, x_2, x_3, x_4) = (3,1,1,3)$ and $(x_1, x_2, x_3, x_4) = (1,2,2,1)$ are solutions to those two equations; i.e., $3a + b + c + 3d = a + 2b + 2c + d = 0$. You can certainly solve the system consisting of these two equations by forming the augmented matrix, doing EROs, etc., but that is a little tedious.

Since $(3,1,1,3)^T$ and $(1,2,2,1)^T$ are solutions to $x_1 - x_4 = x_2 - x_3 = 0$ they are also solutions to every equation of the form $\lambda(x_1 - x_4) + \mu(x_2 - x_3) = 0$.

This question is equivalent to the following geometric question: *find two different 3-planes in $\mathbb{R}^4$ (I could be more precise and say that these 3-planes are actually 3-dimensional subspaces of $\mathbb{R}^4$.) each of which contains the points $(3,1,1,3)$ and $(1,2,2,1)$?* The pattern I notice is that each of these points has its first and fourth digits equal so lies on the 3-plane $x_1 = x_4$. Another pattern is that each of these points has its second and third digits equal so lies on the 3-plane $x_2 = x_3$. This gives me the homogeneous equations $x_1 - x_4 = 0$ and $x_2 - x_3 = 0$.

(14) $(3,1,1,3)^T$ and $(1,2,2,1)^T$ belong to the 2-dimensional subspace of $\mathbb{R}^4$ that is the set of all solutions to the equations _________

Answer: This is exactly the same question as the previous one! So it has the same answer $x_1 - x_4 = 0$ and $x_2 - x_3 = 0$.

Comments:

(15) Find a basis for the line $x_1 + x_2 = x_2 + x_3 = 3x_1 - 2x_4 = 0$ in $\mathbb{R}^4$.

Answer: $(2, -2, 2, 3)$ or any non-zero multiple of that vector.

Comments: Regardless of the exact equations, I am asking for a basis for a line, a 1-dimensional subspace because it is a line that contains $\underline{0}$, in $\mathbb{R}^4$ so that basis must consist of *one* vector having *four* entries. So an answer having two or more vectors will be wrong. Likewise, an answer consisting of a single vector with 3 entries, for example, $(4, 3, 1)$, will be wrong.

When I write $x_1 + x_2 = x_2 + x_3 = 3x_1 - 2x_4 = 0$ I mean the system of equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \\ 3x_1 - 2x_4 &= 0. \end{aligned}$$

A non-trivial solution to the equation $3x_1 - 2x_4 = 0$ is given by taking $x_1 = 2$ and $x_4 = 3$. From the equation $x_1 + x_2 = 0$ one sees that $x_2 = -2$. From the equation $x_2 + x_3 = 0$ one sees that $x_3 = 2$. That's where $(2, -2, 2, 3)$ came from.

You might compare this with question 19 in part A of the 2009 midterm.

(16) Find two linearly independent vectors that lie on the plane in $\mathbb{R}^4$ given by the equations

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 0 \\ 3x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

Answer: There are infinitely many answers to this question because a plane has infinitely many different bases. A simple answer is the pair of vectors $(0, 1, 1, 0)$ and $(0, 0, 1, 1)$.

Comments: Many people got this question wrong in a way that suggested they had a fundamental misunderstanding of the question from the beginning.

In $\mathbb{R}^4$, the solutions to a single homogeneous linear equation form a 3-dimensional subspace of $\mathbb{R}^4$. Conversely, every 3-dimensional subspace of $\mathbb{R}^4$ is the set of solutions to a single homogeneous linear equation. A plane in $\mathbb{R}^4$ is the intersection of two 3-dimensional subspaces so is the set of solutions to a system of two homogeneous linear equations. In other words, every 2-dimensional subspace of $\mathbb{R}^4$ is the set of solutions to a system of two homogeneous linear equation. Conversely, the set of solutions to a a system of two homogeneous linear equations is a 2-dimensional subspace of $\mathbb{R}^4$ (unless one the equations is a multiple of the other). Similar considerations apply in higher dimensions, and to larger systems of linear equations.

A vector, or point, in $\mathbb{R}^4$ has four coordinates (a, b, c, d). It will lie on the plane in question if and only if $2a - b + c - d = 0$ and $3a - b + c + d = 0$.

In other words, you are being asked to solve a system of two homogeneous equations in 4 unknowns. You could either write down the augmented matrix and do EROs to find the free (or independent variables) etc. Or you could simply eyeball the system and see from the first equation that $2x_1 = x_2 - x_3 + x_4$ and from the second equation that $3x_1 = x_2 - x_3 + x_4$. The only way this can happen is if $x_1 = 0$. So x_1 *must* be 0. Then you just have to find two linearly independent solutions to $x_2 - x_3 + x_4 = 0$. It should be pretty obvious that $(x_2, x_3, x_4) = (0, 1, 1)$ and $(x_2, x_3, x_4) = (1, 1, 0)$ are two such solutions.

With a question like this you should always check your answer–for example, many people gave the answer $(2, -1, 1, -1)$ and $(3, -1, 1, -1)$. Neither of those points is a solution to either equation.

Some people gave two vectors, one satisfying the equation $2x_1 - x_2 + x_3 - x_4 = 0$ and the other satisfying the equation $3x_1 - x_2 + x_3 - x_4 = 0$, but neither point satisfying both equations. You needed to find two points each of which satisfied both equations.

If you had read the 2009 Fall midterm (Part A, question 5) OR the Fall 2010 midterm (Part A, question 5) both of which have been on the course webpage since the beginning of the quarter, you would have seen exactly the same question and a discussion along the lines I have just given.

The message, again: read the old midterms and the answers and comments that I have posted for your benefit.

(17) Why is $(0, 1, 2, 1)$ a linear combination of the vectors in your answer to the previous equation?

Comments:

(18) Let A be a 4×5 matrix and $\underline{b} \in \mathbb{R}^5$. Suppose the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & 3 & 0 & | & 2 \\ 0 & 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The independent or free variables are _________; they can take any values and all solutions are given by _________.

Comments:

Part B.

Complete the definitions and Theorems.
There is a difference between theorems and definitions.
Don't write the part of the question I have already written. Just fill in the blank.

(1) **Definition:** The system of equations $A\underline{x} = \underline{b}$ is <u>homogeneous</u> if _________
(2) **Definition:** Two systems of linear equations are <u>equivalent</u> if _________
(3) **Definition:** A vector $\underline{x}$ is a <u>linear combination</u> of $\underline{v}_1, \ldots, \underline{v}_n$ if _________
(4) **Definition:** The <u>linear span</u> of $\underline{v}_1, \ldots, \underline{v}_n$ is the set of all _________
(5) **Definition:** A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is <u>linearly independent</u> if the only solution to the equation _________

Answer: $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ is $a_1 = a_2 = \cdots = a_n = 0$.
Comments: If you replace "is" by "where" in the above sentence the answer becomes wrong. The words "is" and "where" have completely different meanings. Compare "my dog is fierce" and "my dog where fierce". OR, "the only answer is $x = 3$" versus "the only answer where $x = 3$".

(6) **Definition:** $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is <u>linearly dependent</u> if _________ $= \underline{0}$ for some _________
(7) **Definition:** A matrix E is in <u>row echelon form</u> if
(a) _________
(b) _________

Answer: (a) all rows consisting entirely of 0s are below the non-zero rows and (b) the left-most non-zero entry in each non-zero row, which we call the *leading entry* in that row, is to the right of the leading entry in the row above it.

Comments: It is difficult to express this definition in a short, comprehensible, and correct way. Almost everyone could benefit from trying to write it out half-a-dozen times. Few answers were perfect. Several people used the term *"leading entry"* without defining it or saying what it means. For example, "the left-most non-zero entry in each non-zero row is to the right of the leading entry in the row above it" does not make sense until you have defined "leading entry".

One proposed answer for (b) was *"All non-zero rows have a left-most leading entry"*. This is not quite correct because it uses the term *"left-most leading entry"* before "leading entry" has been defined. When you write a definition put yourself in the shoes of someone who is seeing your definition for the first time. I do not think the proposed answer conveys the correct idea. Carry out the following experiment with a friend who knows no mathematics. Say to him/her *"I want you to write down a list of 8 numbers in a row from left to right. At least one of those numbers must be non-zero and the first non-zero entry in your list must be a 5."* Then say to him/her *"I want you to write down a list of 8 numbers in a row from left to right. At least one of those numbers must be non-zero and the row must have a left-most leading 5."* I expect the second set of instructions will cause confusion. Try it!

Even if your answer was graded as correct you would benefit from polishing, shortening, and perfecting your answer. Please try it.

We try to make definitions easy to understand. Often that is impossible, but it must be our goal.

(8) **Definition:** A matrix is in row reduced echelon form if
   (a) it is _________ and
   (b) _________ and
   (c) _________.

(9) **Definition:** Let A be an $m \times n$ matrix and let E be the row-reduced echelon matrix that is row equivalent to it. If $x_1, \dots, x_n$ are the unknowns in the system of equations $A\underline{x} = \underline{b}$, then x_j is a free variable if ________.

(10) **Definition:** In this question use R_i to denote the i^{th} row of a matrix.
   The three elementary row operations are
   (a) _________
   (b) _________
   (c) _________

(11) **Definition:** The null space and range of an $m \times n$ matrix A are
   (a) $\mathcal{N}(A) = \{\cdots \mid \cdots\}$ and
   (b) $\mathcal{R}(A) = \{\cdots \mid \cdots\}$.

(12) **Definition:** A subset W of $\mathbb{R}^n$ is a subspace if
   (a) _________
   (b) _________
   (c) _________

(13) **Definition:** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_d\}$ is a basis for a subspace V of $\mathbb{R}^n$ if ______________

(14) **Definition:** The dimension of a subspace V of $\mathbb{R}^n$ is ______________.

(15) **Definition:** The nullity of a matrix is ______________.

   Answer: The dimension of its null space.

   Comments:

(16) **Definition:** The rank of a matrix A is ______________.

   Answer: The number of non-zero rows in a row echelon form of A OR the number of non-zero rows in its row reduced echelon form.

   Comments: It is incorrect to say "the number of non-zero rows in row echelon form" because you must say which matrix is being put in echelon form. It would be correct to say "the number of non-zero rows in the row echelon form of the given matrix".

(17) **Theorem:** The rank of a matrix is equal to the dimension of ______________.

(18) **Theorem:** The nullity and rank of a $p \times q$ matrix are related by the formula ______________.

(19) **Theorem:** Two systems of linear equations are equivalent if the row reduced echelon forms of their augmented coefficient matrices are ______________.

(20) **Theorem:** The matrix $\begin{pmatrix} p & q \\ s & x \end{pmatrix}$ is invertible if and only if ___________.

Answer: $px - qs \neq 0$

Comments: You MUST know the answer to this and the next question. A few people said *"it has an inverse"* but that is a tautology—I wanted a something with more content. The formula gives a computation that can be carried out whereas saying *it has an inverse* is sort of passing the buck.

(21) **Theorem:** If $\begin{pmatrix} p & q \\ s & x \end{pmatrix}$ is invertible its inverse is ___________.

Answer:

$$\frac{1}{px - qs}\begin{pmatrix} x & -q \\ -s & p \end{pmatrix}$$

Comments: Almost everyone got this right.

(22) **Theorem:** A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if $\text{span}(\underline{w}, \underline{v}_1, \dots, \underline{v}_n) =$ _________

(23) **Theorem:** If the dimension of $\text{span}(\underline{v}_1, \dots, \underline{v}_k)$ is $< k$, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is _______________

(24) **Theorem:** A homogeneous system of p linear equations in q unknowns always has a non-zero solution if _______________.

(25) **Theorem:** Let A be an $n \times n$ matrix and $\underline{b} \in \mathbb{R}^n$. The equation $A\underline{x} = \underline{b}$ has a unique solution if and only if A is ___________.

(26) **Theorem:** Let A be an $m \times n$ matrix and let E be the row-reduced echelon matrix that is row equivalent to it. Then the non-zero rows of E are a basis for ________.

(27) **Theorem:** A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.

(28) **Proposition:** Let $\underline{B}_1, \dots, \underline{B}_q$ be the columns of B. The columns of the product AB are _______________.

(29) **Theorem:** An $n \times n$ matrix A is invertible if and only if its columns _______________.

(30) **Theorem:** An $n \times n$ matrix A is invertible if and only if its range _______________.

Answer: is $\mathbb{R}^n$.
OR
has dimension n.

Comments: It is not correct to say "is n". The range is not a number. It is a subspace of $\mathbb{R}^n$. Its *dimension* is a number but the range itself is not.

(31) **Theorem:** An $n \times n$ matrix A is invertible if and only if the equation $A\underline{x} = \underline{b}$ _______________.

Answer: has a unique solution.
OR
has a unique solution for all $\underline{b} \in \mathbb{R}^n$.

Comments: It is not correct to say "has the trivial solution where $\underline{b} = \underline{0}$". First, "where" should probably be "when" for the sentence to make sense. However "has the trivial solution when $\underline{b} = \underline{0}$" is wrong because *every* equation $A\underline{x} = \underline{0}$ has the solution $\underline{x} = \underline{0}$. You must say the equation has a *unique* solution—that is what invertibility does for you. If your answer does not say something about the uniqueness of the solution, whether you take the equation to be $A\underline{x} = \underline{0}$ or $A\underline{x} = \underline{b}$, your answer will not be correct.

Part C.

True or False

Remember +1, 0, or -1.

The three numbers after the answer are the % of students who were correct, wrong, did not answer. For example, [**40, 25, 35**] says that 40% gave the correct answer, 25% gave the wrong answer, and 25% did not answer the question.

(1) Every set of four vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

False. [**51,23,26**]

Comments: The vectors $(1,0,0,0)$, $(2,0,0,0)$, $(3,0,0,0)$, $(4,0,0,0)$, do not span $\mathbb{R}^4$. They span a 1-dimensional subspace, the line $\mathbb{R}(1,0,0,0)$ that passes through the origin and $(1,0,0,0)$. The vectors $(1,0,0,0)$, $(2,0,0,0)$, $(3,0,0,0)$, $(4,0,0,0)$ are multiples of each other, so they are linearly dependent. Any fact any pair of them is a linearly dependent set.

(2) Every set of five vectors in $\mathbb{R}^4$ is linearly dependent.

True. [**49,20,31**]

Comments: We proved a theorem saying if $r > n$, then any set of r vectors in $\mathbb{R}^n$ is linearly dependent. Look at the proof: it relies on the fact that a homogeneous system of n equations in r unknowns has a non-trivial solution if $r > n$.

(3) Every set of five vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

False. [**46,13,41**]

Comments: The vectors $(1,0,0,0)$, $(2,0,0,0)$, $(3,0,0,0)$, $(4,0,0,0)$, $(5,0,0,0)$, do not span $\mathbb{R}^4$.

(4) Every set of four vectors in $\mathbb{R}^4$ is linearly independent.

False. [**51,10,39**]

Comments: The set $\{(1,0,0,0),(2,0,0,0),(3,0,0,0),(4,0,0,0)\}$ is linearly dependent.

(5) The row space of an $p \times q$ matrix is a subspace of $\mathbb{R}^p$.

False. [**44,33,23**]

Comments: A $p \times q$ matrix has p rows and q columns so each row has q entries and therefore belongs to $\mathbb{R}^q$. The row space is the linear span of the rows so is a subspace of $\mathbb{R}^q$.

(6) The column space of an $p \times q$ matrix is a subspace of $\mathbb{R}^p$.

True. [**51,31,18**]

Comments: See comments to previous question and make appropriate changes.

(7) A system of linear equations can have exactly two different solutions.

False. [**87,8,5**]

Comments: A system of linear equations has either no solutions, one solution, or infinitely many solutions. If p and q are different solutions so is every point on the line through p and q. That line consists of the points $tp + (1-t)q$, $t \in \mathbb{R}$.

(8) The matrix $\begin{pmatrix} a & b \\ 3 & b \end{pmatrix}$ has an inverse if and only if $a \neq 3$ and $b \neq 0$.

True. [**90,3,7**] Good.

Comments: Compute the determinant. It is $ab - 3b = (a-3)b$. The matrix is invertible if and only if $(a-3)b \neq 0$, i.e., if and only if $a \neq 3$ and $b \neq 0$.

(9) A homogeneous system of 7 linear equations in 8 unknowns always has a non-trivial solution.

True. [**80,10,10**]

Comments: A homogeneous system of n equations in r unknowns has a non-trivial solution if $r > n$.

(10) The set of solutions to a system of 5 linear equations in 6 unknowns can be a 3-plane in $\mathbb{R}^6$.

True. [**72,3,26**]

Comments: For example, take the system $x_1 = x_2 = x_3 = x_1 + x_2 = x_1 + x_3 = 0$. The solution is the 3-plane in $\mathbb{R}^6$ consisting of all points $(0,0,0,x_4,x_5,x_6)$.

(11) If $B^3 - B^2 = 2I + 2B$, then $\frac{1}{2}(B^2 - B - 2I)$ is the inverse of B.

True. [**80,10,10**] Good

Comments:

(12) The rank of a matrix is the number of non-zero rows in it.

False. [**80,15,5**]

Comments:

(13) Row equivalent matrices have the same rank.

True. [**77,10,13**]

Comments:

(14) Row equivalent matrices have the same row reduced echelon form.

True. [**92,3,5**]

Comments:

(15) If A and B are invertible then $(AB)^{-1} = A^{-1}B^{-1}$.

False. [**92,5,3**] Good

Comments:

(16) The system of linear equations with the following augmented matrix is inconsistent.

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 2 & 3| & 2 \\ 3 & 6 & 2 & 1 & 1 & 4| & 4 \\ 0 & 0 & -3 & -1 & 0 & 0| & 6 \\ 1 & 3 & 0 & 0 & 2 & 4| & 2 \\ 2 & 6 & 0 & 0 & 4 & 8| & 3 \end{pmatrix}$$

True. [**77,3,20**] Good.

Comments:

(17) $\mathbb{R}^3$ has infinitely many subspaces

True. [**80,2,18**] Good.

Comments:

(18) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.

True. **[82,5,13]**

Comments: It is good to remember how to prove this. Let $\underline{A}_1, \dots, \underline{A}_n$ be the columns of A. If you rewrite $A\underline{x} = \underline{b}$ as $x_1\underline{A}_1 + \cdots + x_n\underline{A}_n = \underline{b}$ you see that $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of $\underline{A}_1, \dots, \underline{A}_n$.

(19) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

True. **[87,0,13]**

Comments:

(20) There is a matrix whose inverse is $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix}$.

False. **[20,51,28]**

Comments: There are many ways to see this matrix is not invertible. For example, the third row is a multiple of the second row so the rows are linearly dependent. Equivalently, a row echelon form of the matrix has a row of zeroes so the row reduced echelon form also has a row of zeroes and is not the identity matrix. Alternatively, the columns are linearly dependent–good practice to find a linear dependence relation between the columns.

(21) If $A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

False. **[62,23,15]**

Comments: A is a 2×2 matrix so BA is an $m \times 2$ matrix for some m but E is a 2×3 matrix.

(22) If $B^{12} = 0$, then B never has an inverse.

True. **[20,23,56]**

Comments: If B had an inverse, then $(B^{-1})^{12}$, which we usually write as B^{-12}, would be the inverse of B^{12} but the zero matrix does not have an inverse.

(23) If $\underline{u}$, $\underline{v}$, and $\underline{w}$, are any vectors in $\mathbb{R}^4$, then $\{3\underline{u} - 2\underline{v}, 2\underline{v} - 4\underline{w}, 4\underline{w} - 3\underline{u}\}$ is linearly dependent.

True. **[69,0,31]**

Comments:

(24) If

$$A^T = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad (AB)^{-1} = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$$

then

$$B^{-1} = \begin{pmatrix} 0 & -3 \\ -1 & 1 \end{pmatrix}.$$

True. **[33,18,49]**

Comments:

(25) If $A = BC$, then every solution to $A\underline{x} = \underline{0}$ is a solution to $C\underline{x} = \underline{0}$.

False. **[39,20,41]**

Comments:

(26) If $A = BC$, then every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.

True. **[51,10,39]**

Comments:

(27) Let A be an $m \times n$ matrix and B an $n \times p$ matrix. If $C = AB$, then $\{C\underline{x} \mid \underline{x} \in \mathbb{R}^p\} \subset \{A\underline{w} \mid \underline{w} \in \mathbb{R}^n\}$.

True. **[26,13,61]**

Comments:

(28) The linear span of the vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ is the 3-plane $2x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

False. **[56, 0, 44]**

Comments:

(29) The linear span of the vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

True. **[56,0,44]**

Comments:

(30) If $\mathsf{span}\{\underline{u}, \underline{v}\} \subset \mathsf{span}\{\underline{x}, \underline{w}, \underline{z}\}$ then every linear combination of $\underline{u}$ and $\underline{v}$ is a linear combination of $\underline{x}$, $\underline{w}$, and $\underline{z}$.

True. **[54,3,44]**

Comments:

(31) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 = x_2 - x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

True. [**59,3,39**]

Comments:

(32) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 - x_2x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

False. [**51,5,44**]

Comments:

(33) A square matrix is invertible if all its entries are non-zero.

False. [**59,3,38**]

Comments:

(34) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1/2 \\ 1/3 & 1/4 \end{pmatrix}$

False. [**67,3,30**]

Comments:

(35) An invertible matrix can have a row of zeroes.

False. [**54,5,41**]

Comments:

(36) For any vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, $\{\underline{u}, \underline{v}, \underline{w}\}$ and $\{\underline{u} - \underline{v}, \underline{v}, \underline{w} - \underline{u}\}$ have the same linear span.

True. [**44,3,54**]

Comments:

(37) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$ have the same the linear span as $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

True. [**41,3,56**]

Comments:

(38) Every subset of a linearly independent set is linearly independent.

True. [**57,3,41**]

Comments:

(39) The dimension of a subspace is the number of elements in it.

False. **[28,21,51]**

Comments:

(40) Every subset of a linearly dependent set is linearly dependent.

False. **[26,26,48]**

Comments:

(41) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, then $\{\underline{v}_1 + 3\underline{v}_2, 3\underline{v}_2 + \underline{v}_3, \underline{v}_3 - \underline{v}_1\}$ is linearly dependent.

True. **[31,3,66]**

Comments:

(42) If A is an invertible matrix, then $A^{-1}\underline{b}$ is the unique solution to the equation $A\underline{x} = \underline{b}$.

True. **[41,5,54]**

Comments:

(43) $\mathrm{span}\{\underline{u}_1, \ldots, \underline{u}_r\} = \mathrm{span}\{\underline{v}_1, \ldots, \underline{v}_s\}$ if and only if every $\underline{u}_i$ is a linear combination of the $\underline{v}_j$s and every $\underline{v}_j$ is a linear combination of the $\underline{u}_i$s.

True. **[33,0,67]**

Comments:

(44) The row reduced echelon form of a square matrix is the identity if and only if the matrix is invertible.

True. **[39,10,51]**

Comments:

(45) A square matrix has an inverse if and only if its columns are linearly dependent.

False. **[41,5,54]**

Comments:

(46) The row and column spaces of a matrix always have the same dimension.

True. **[10,31,59]**

Comments:

(47) The rank of a matrix is at most the number of columns it has.

True. [**8,33,59**]

Comments:

(48) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

True. [**46,0,54**]

Comments:

(49) There is a matrix whose inverse is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

True. [**28,10,61**]

Comments:

(50) The column space of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

True. [**26,5,69**]

Comments: OOPS! I meant to say "the columns of the matrix are a basis for $\mathbb{R}^3$" in which case it would be true. As written it is actually false but because 26% answered True and only 5% answered False my guess is that most people read what I *wanted* to write rather than what I wrote.

(51) For all matrices A and B, $\mathcal{R}(AB) \subset \mathcal{R}(A)$

True. [**10,8,82**]

Comments:

(52) If $A^2 = B^3 = C^4 = I$, then $(A^4B^2C^2)^{-1} = C^2B$.

True. [**15,8,77**]

Comments:

(53) If W is a subspace of $\mathbb{R}^n$ that contains $\underline{u} + \underline{v}$, then W contains $\underline{u}$ and $\underline{v}$.

False. [**10,26,64**]

Comments:

(54) If $\underline{u}$ and $\underline{v}$ are linearly independent vectors on the plane in $\mathbb{R}^4$ given by the equations $x_1 - x_2 + x_3 - 4x_4 = 0$ and $x_1 - x_2 + x_3 - 2x_4 = 0$, then $(1, 2, 1, 0)$ a linear combination of $\underline{u}$ and $\underline{v}$.

True. **[23,3,74]**

Comments:

(55) The range of a matrix is its columns.

False. **[36,0,64]**

Comments:

(56) The vectors $(2, 2, -4, 3, 0)$ and $(0, 0, 0, 0, 1)$ are a basis for the subspace $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$ of $\mathbb{R}^5$.

True. **[18,3,79]**

Comments:

(57) The vectors $(1, 1)$, $(1, -2)$, $(2, -3)$ are a basis for the subspace of $\mathbb{R}^4$ give by the solutions to the equations $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$.

False. **[13,3,84]**

Comments:

(58) $\{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4\}$ is a subspace of $\mathbb{R}^4$.

True. **[28,3,69]**

Comments:

(59) $\{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a subspace of $\mathbb{R}^5$.

False. **[28,0,72]**

Comments:

(60) The solutions to a system of homogeneous linear equations form a subspace.

True. **[33,3,64]**

Comments:

(61) The solutions to a system of linear equations form a subspace.

False. **[21,5,74]**

Comments:

(62) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cup \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.

False. **[8,13,79]**

Comments:

(63) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cap \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.

True. **[13,10,77]**

Comments:

(64) The linear span of a matrix is its set of columns.

False. **[21,10,69]**

Comments:

(65) U^{-1} is a subspace if U is.

False. **[8,8,84]**

Comments:

(66) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is linear combination of the rows of A.

False. **[21,10,69]**

Comments:

(67) The equation $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^n$ if A is an $n \times n$ matrix with rank n.

True . **[28,0,72]**

Comments:

(68) A matrix is linearly independent if its columns are different.

False. **[23,0,77]**

Comments: This question is nonsense! It makes sense to ask if a set of vectors is linearly independent but it doesn't make sense to ask if a matrix is linearly independent.

(69) If A is a 3×5 matrix, then the inverse of A is a 5×3 matrix.

False. **[26,5,69]**

Comments:

(70) There is a 3×3 matrix A such that $\mathcal{R}(A) = \mathcal{N}(A)$.

False. [**15,8,77**]

Comments:

(71) There is a 2×2 matrix A such that $\mathcal{R}(A) = \mathcal{N}(A)$.

True. [**18, 0, 82**]

Comments:

(72) If A is a 2×2 matrix it is possible for $\mathcal{R}(A)$ to be the parabola $y = x^2$.

False. [**18,0,82**]

Comments:

(73) Let A be the 2×2 matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

The null space of A is $\left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \;\middle|\; t \text{ is a real number} \right\}$.

True. [**15,3,82**]

Comments:

(74) Let A be the 2×2 matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

The null space of A is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

False. [**5,8,87**]

Comments:

(75) Let A be a 4×3 matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ x_2 \end{pmatrix}.$$

The range of A has many bases; one of them consists of the vectors

$$\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

True. **[5,3,92]**

Comments:

(76) A square matrix is invertible if and only if its nullity is zero.

True. **[20,3,77]**

Comments:

In the next 6 questions, A is a 4×4 matrix whose columns $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ have the property that $\underline{A}_1 + \underline{A}_2 = \underline{A}_3 + \underline{A}_4$.

False. **[18,5,77]**

Comments:

(77) The columns of A span $\mathbb{R}^4$.

True. **[23,3,74]**

Comments:

(78) A is not invertible.

True. **[23,3,74]**

Comments:

(79) The rows of A are linearly dependent.

True. **[23,5,72]**

Comments:

(80) The equation $A\underline{x} = 0$ has a non-trivial solution.

True. **[23,5,72]**

Comments:

(81) $A \begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix} = \underline{0}$.

False. [**15,13,72**]

Comments:

(82) $A \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \underline{0}$.

True. [**21,10,69**]

Comments: