Math 308 **Final** **June 10, 2013**

Instructions. No need to show your work. In Parts A and B you get 1 point per question or 1 point for each part of a multi-part question. Part C: use the scantron/bubble sheet with A =True and B =False: for Part C you will get • +1 for each correct answer, • -1 for each incorrect answer, and • 0 for no answer at all.

Part A.

(1) If you know a single solution, $\underline{w}$ say, to the equation $A\underline{x} = \underline{b}$ and all solutions to the equation $A\underline{x} = \underline{0}$, then the set of all solutions to $A\underline{x} = \underline{b}$ is $\{...|....\}$.

(2) [**2 points**] If $A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$ and the null space of A is spanned by $\begin{pmatrix}1\\1\\1\end{pmatrix}$ find two more solutions to the equation $A\underline{x} = \begin{pmatrix}3\\2\\1\end{pmatrix}$.

(3) Write $B\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$ as a linear combination of the columns $\underline{B}_1, \dots, \underline{B}_n$ of B.

(4) If $A\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}4\\3\\2\\1\end{pmatrix}$, then $2\underline{A}_1 + 4\underline{A}_2 + 6\underline{A}_3 + 8\underline{A}_4 =$.

(5) What matrix represents the linear transformation $T\begin{pmatrix}w\\x\\y\\z\end{pmatrix} = \begin{pmatrix}w+x\\x+y\\y+z\\z+w\end{pmatrix}$.

(6) Give a basis for the kernel of the linear transformation in Question 5.

(7) Give a basis for the range of the linear transformation in Question 5.

(8) If $A\begin{pmatrix}4\\3\\2\\-1\end{pmatrix} = \underline{0}$, the columns $\underline{A}_1, \dots, \underline{A}_4$ of A are linearly dependent because $\underline{A}_4 =$ _________

(9) The vectors $(1,1,1,3)^T$ and $(3,1,1,1)^T$ are a basis for the set of solutions to the homogeneous equations _________ and _________

(10) The vectors $(1,1,1,3)^T$ and $(3,1,1,1)^T$ are a basis for the range of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}?\\?\\?\\?\end{pmatrix}$.

(11) The vectors $(1,1,1,3)^T$ and $(3,1,1,1)^T$ are a basis for the kernel (null space) of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T\begin{pmatrix}w\\x\\y\\z\end{pmatrix} = \begin{pmatrix}?\\?\\?\end{pmatrix}$.

(12) Find a basis for the 2-plane $x_1 + x_2 - x_3 = x_2 + x_3 - x_4 = 0$ in $\mathbb{R}^4$.

(13) Give the equation of a 3-plane in $\mathbb{R}^4$ that contains the 2-plane in question 12.

(14) The matrix representing the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a vector by θ radians in the counter-clockwise direction is _________.

(15) What is the characteristic polynomial of the matrix $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$?

(16) **[2 points]** What are the eigenvalues of the matrix in question 15?

(17) **[2 points]** Find linearly independent eigenvectors for the matrix in question 15.

(18) **[2 points]** Give a precise description of the set of all solutions (a, b, c, d, e) to the system of linear equations $a + 2c + e = b + 3c + d + e = 0$.

Part B.

Complete the definitions and Theorems.
There is a difference between theorems and definitions.
Don't write the part of the question I have already written. Just fill in the blank.

(1) **Definition:** (Do not use the phrase *"linear combinations"* in your answer.) The <u>linear span</u> of $\underline{v}_1, \ldots, \underline{v}_n$ is the set of all vectors of the form _________

(2) To show that $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly independent I must show that the only solution to the equation _________

(3) To show that $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly dependent I must show that _________ $= \underline{0}$ for some _________

(4) Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$. To show that a function $T : V \to W$ is a linear transformation, I must show that _________.

(5) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. To show that $\underline{x}$ is in the kernel of T I must show that _________.

(6) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. To show that $\underline{v}$ is in the range of T I must show that _________.

(7) **Definition:** The <u>dimension</u> of a subspace V of $\mathbb{R}^n$ is ______________.

(8) **Theorem:** A linear transformation $T : V \to W$ has an inverse if and only if ______________.

(9) **Theorem:** The following conditions on an $n \times n$ matrix A are equivalent:
  (a) A is invertible
  (b) the rows of A are ___________
  (c) $\mathcal{R}(A) =$___________
  (d) $\mathcal{N}(A) =$___________
  (e) $\text{rank}(A) =$ ___________
  (f) $\det(A)$ is ___________

(10) **Definition:** Let λ be an eigenvalue for the $n \times n$ matrix A. The <u>λ-eigenspace</u> for A is

$$E_\lambda := \{\_\_ \mid \_\_\_\_\_\_\_\_\_\}$$

(11) **Theorem:** The λ-eigenspace of A is non-zero if and only if $\det(A - \lambda I) = 0$ because $E_\lambda =$ ______________.

(12) **Theorem:** The roots of the ______________ of a square matrix A are its______________.

(13) **Theorem:** Let $T : \mathbb{R}^p \to \mathbb{R}^q$ be a linear transformation. Then there is a unique ________ matrix A such that ________ for all ________. We call A the matrix that <u>represents</u> T.

(14) **Theorem:** The j^{th} column of the matrix representing T is _________.

(15) **Definition:** A set of non-zero vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is orthogonal if _________.

(16) **Theorem:** Let $\{\underline{u}_1, \dots, \underline{u}_k\}$ be an orthonormal basis for a subspace V of $\mathbb{R}^n$. The orthogonal projection of $\mathbb{R}^n$ onto V is the linear transformation $P : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula $P(\underline{x}) =$ ________. Furthermore,
 (a) $P(\underline{v}) = \underline{v}$ for all ________
 (b) $\mathcal{R}(P) =$ ________
 (c) $\ker(P) =$________
 (d) $\ker(P) \cap \mathcal{R}(P) =$________
 (e) $\ker(P) + \mathcal{R}(P) =$________

(17) **Definition:** Let A be an $m \times n$ matrix. A vector $\underline{x}^*$ in $\mathbb{R}^n$ is a least-squares solution to $A\underline{x} = \underline{b}$ if ________ is as close as possible to ________ .

(18) **Theorem:** If $A^T A\underline{x}^* = A^T\underline{b}$, then $\underline{x}^*$ is ________.

(19) **Definition:** An $n \times n$ matrix Q is orthogonal if ________.

(20) **Theorem:** The following 4 conditions on an $n \times n$ matrix Q are equivalent:
 (a) Q is orthogonal;
 (b) ________ for all $\underline{x} \in \mathbb{R}^n$;
 (c) ________ for all $\underline{u}, \underline{v} \in \mathbb{R}^n$.
 (d) the ________ an orthonormal basis ________

Part C.

True or False

Remember +1, 0, or -1.

(1) Let A and B be matrices having the same row-reduced echelon form, namely

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The equations $A\underline{x} = \underline{b}$ and $B\underline{x} = \underline{b}$ have the same solutions.

(2) Let A and B be as in question 1. Every solution to the equation $A\underline{x} = \underline{b}$ is a solution to the equation $(2A + 3B)\underline{x} = 5\underline{b}$.

(3) Let A be as in question 1. The set of solutions to the equation $A\underline{x} = \underline{b}$ is a subspace of $\mathbb{R}^4$.

(4) Let A be as in question 1. The set of solutions to the equation $A\underline{x} = \underline{0}$ is a subspace having dimension 1.

(5) Let A be as in question 1. The function $T(\underline{x}) = A\underline{x}$ is a linear transformation having rank 3 and nullity 1.

(6) The range of a non-zero linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ is a subspace of $\mathbb{R}^4$ whose dimension is either 1 or 2.

(7) Every non-zero linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^1$ is onto.

(8) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is not onto.

(9) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ that is one-to-one.

(10) Every non-zero linear transformation $T : \mathbb{R}^1 \to \mathbb{R}^6$ has rank one.

(11) Every non-zero linear transformation $T : \mathbb{R}^1 \to \mathbb{R}^6$ is one-to-one.

(12) If $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z-x\\z-y\\x+y\end{pmatrix}$, then $\ker(T) = \mathbb{R}\begin{pmatrix}1\\1\\1\end{pmatrix}$.
(13) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\mathcal{R}(T) = \mathcal{N}(T)$.
(14) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$. The null space of T is $\{(t, 0, 0, 0) \mid t \in \mathbb{R}\}$.
(15) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (0, x)$. The null space of T is $\{(0, z) \mid z \in \mathbb{R}\}$.
(16) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (0, x)$. Then $\{(0, 5)\}$ is a basis for the null space of T.
(17) The set $\{(-2, 0, 0, 0), (0, 0, 1, 1)\}$ is a basis for the range of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T(x, y, z) = (x, 0, y + z, y + z)$.
(18) The vector $(0, 0, 1)$ spans the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_2)$.
(19) The function from $\mathbb{R}^3$ to itself that interchanges the first and third coordinates is a linear transformation.
(20) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range is the set of solutions to the equation $xy = z$.
(21) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range is the set of solutions to the equation $x + y = z$.
(22) The linear transformation $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\x\\y\end{pmatrix}$ is one-to-one.
(23) The linear transformation in question 22 is onto.
(24) If T is the linear transformation in question 22, then $T^2 = T^{-1}$.
(25) 1 is an eigenvalue of the linear transformation in question 22.
(26) $\dim(E_1) = 1$ for the linear transformation in question 22.
(27) The linear transformation $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\\x\end{pmatrix}$ is one-to-one.
(28) The rank of the linear transformation in question 27 is 2.
(29) If $T(\underline{e}_1) = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $T(\underline{e}_2) = \begin{pmatrix}0\\-1\\-2\end{pmatrix}$, then $T\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix}$.
(30) Let S and T be the linear transformations $T(x, y) = (x + 3y, 3x + y)$ and $S(x, y) = (-x + 3y, 3x - y)$. Then $S \circ T = T \circ S$.
(31) Let A be any square matrix. Then A^2 commutes with $I - A$.
(32) If an $n \times n$ matrix has n distinct eigenvalues, then its columns form a basis for $\mathbb{R}^n$.
(33) Let V be a 4-dimensional subspace of $\mathbb{R}^5$. Every set of five vectors in V is linearly dependent.
(34) Let V be a 4-dimensional subspace of $\mathbb{R}^5$. Every set of four vectors in V spans V.
(35) There is a 3-plane and a 2-plane in $\mathbb{R}^5$ whose intersection is a line.
(36) Let V be a 3-dimensional subspace of $\mathbb{R}^5$. There is a basis for $\mathbb{R}^5$ consisting of 3 vectors in V and 2 vectors not in V.
(37) If V is a subspace of $\mathbb{R}^n$, then $(V^\perp)^\perp = V$.

(38) $\{3\underline{u}-2\underline{v}, 2\underline{v}-4\underline{w}, 4\underline{w}-3\underline{u}\}$ is linearly dependent for all choices of $\underline{u}$, $\underline{v}$, and $\underline{w}$.
(39) Every set of orthogonal vectors in $\mathbb{R}^n$ is linearly independent.
(40) If $A = BC$, then every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.
(41) If $A = BC$, then $\mathcal{R}(A) \subset \mathcal{R}(B)$.
(42) Let A be an $m \times n$ matrix and B an $n \times p$ matrix. If $C = AB$, then $\{C\underline{x} \mid \underline{x} \in \mathbb{R}^p\} \subset \{A\underline{w} \mid \underline{w} \in \mathbb{R}^n\}$.
(43) The linear span of the vectors $(4,0,0,1)$, $(0,2,0,-1)$ and $(4,3,2,1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.
(44) The linear span of the vectors $(4,0,0,1)$, $(0,3,2,0)$ and $(4,3,2,1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.
(45) Let $\underline{u}$, $\underline{v}$, and $\underline{w}$ be vectors in $\mathbb{R}^n$. Then $\mathsf{span}\{\underline{u}, \underline{v}, \underline{w}\} = \mathsf{span}\{\,\underline{u}+2\underline{v}, \underline{u}+3\underline{v}, \underline{u}+\underline{v}+\underline{w}\}$.
(46) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 = x_2 - x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.
(47) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 - x_2 + x_4 = 1\}$ is a subspace of $\mathbb{R}^4$.
(48) Every subspace of $\mathbb{R}^n$ has an orthogonal basis.
(49) Every subspace of $\mathbb{R}^n$ has an orthonormal basis.
(50) $\text{span}\left\{ \begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}5\\6\\7\\8\end{pmatrix}, \begin{pmatrix}9\\10\\11\\12\end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\end{pmatrix} \right\}$.
(51) Every subset of a linearly independent set is linearly independent.
(52) Let X and Y be subsets of $\mathbb{R}^n$. If $X \subset Y$ and X is linearly dependent, then Y is linearly dependent.
(53) The row reduced echelon form of a square matrix is the identity if and only if the matrix is invertible.
(54) The vectors $(2,2,-4,3,0)$ and $(0,0,0,0,1)$ are a basis for the subspace $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$ of $\mathbb{R}^5$.
(55) The vectors $(1,1)$, $(1,-2)$, $(2,-3)$ are a basis for the subspace of $\mathbb{R}^4$ that is the set of solutions to the equations $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$.
(56) $\{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = x_1 + 2x_4 = 1\}$ is a subspace of $\mathbb{R}^5$.
The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cup \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.
(57) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cap \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.
(58) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is linear combination of the columns of A.
(59) The dimension of the column space of a matrix is equal to the dimension of its row space.
(60) If U, V, and W are subspaces of $\mathbb{R}^n$ so is $\{\underline{u} + 2\underline{v} + 3\underline{w} \mid \underline{u} \in U, \underline{v} \in V \text{ and } \underline{w} \in W\}$.
(61) $\{(1,1,1),\ (1,0,1),\ (3,2,3)\}$ is a basis for $\mathbb{R}^3$.
(62) If $\{\underline{u}, \underline{v}, \underline{w}\}$ is an orthonormal set of vectors, then $\underline{u}+\underline{v}+\underline{w}$ and $\underline{u}+2\underline{v}-3\underline{w}$ are orthogonal.
(63) The length of the vector $(1,2,2,4)$ is 9.
(64) There is a 2×3 matrix A and a 3×2 matrix B such that AB is the identity matrix.
(65) If A is a 2×3 matrix and B is a 3×2 matrix, then BA can not be the identity matrix because $\mathcal{N}(A) \neq \{\underline{0}\}$.

(66) If A is a 2×3 matrix and B is a 3×2 matrix, then BA can not be the identity matrix because $\mathcal{R}(B) \neq \mathbb{R}^3$.

(67) Let $\underline{u}$ and $\underline{v}$ be linearly independent vectors that belong to the subspace of $\mathbb{R}^4$ consisting of solutions to the homogeneous system of equations

$$4x_1 - 3x_2 + 2x_3 - x_4 = 0$$
$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

The vector $(1, 2, 3, 4)$ is a linear combination of $\underline{u}$ and $\underline{v}$.

(68) Let $\underline{u}$ and $\underline{v}$ be linearly independent vectors that belong to the subspace of $\mathbb{R}^4$ consisting of solutions to the homogeneous system of equations

$$4x_1 - 3x_2 + 2x_3 - x_4 = 0$$
$$2x_1 + 3x_2 - 4x_3 + x_4 = 0$$

The vector $(1, 2, 3, 4)$ is a linear combination of $\underline{u}$ and $\underline{v}$.

(69) If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

(70) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

(71) If A is the matrix representing the linear transformation T, then A and T have the same range.

(72) If A is the matrix representing the linear transformation T, then the null space of A is equal to the kernel of T.

(73) The determinant of $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$ is zero.

(74) If $\underline{x}$ is an eigenvector for A with eigenvalue 5, then $4\underline{x}$ is an eigenvector for A with eigenvalue 20.

(75) If $\underline{x}$ is an eigenvector for A with eigenvalue 5, and an eigenvector for B with eigenvalue 4, then its is an eigenvector for AB with eigenvalue 20.

(76) The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ are a basis for the subspace of $\mathbb{R}^3$ that they span.

(77) The matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

(78) For every value of θ, the rows of the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ form an orthonormal basis for $\mathbb{R}^2$.

(79) The characteristic polynomial of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is $(1+t)^2(1-t)^2$.

(80) The matrix in the previous question has exactly two eigenspaces, each of dimension 2.