Math 308 **Spring Midterm Answers** **May 6, 2013**

Instructions. Part A consists of questions that require a short answer. There is no partial credit and no need to show your work. In Part A you get 2 points per question. Part B is about Theorems and Definitions. Each question is worth 1 point. Part C consists of true/false questions. Use the scantron/bubble sheet with the convention that $A =$True and $B =$False. You will get

- +1 for each correct answer,
- -1 for each incorrect answer, and
- 0 for no answer at all.

The maximum possible scores are 33 for Part A, 15 for Part B, 48 for Part C, for a total of 96. The best score was 82, the mean 37, and the median 30.

To save space I will often avoid write a column vector as the transpose of a row vector. For example, I will write $(1, 2, 3, 4)^\mathsf{T}$ **instead of**

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Part A.

Short answer questions

(1) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

Answer: $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$.
Comment: A couple of people wrote $x_1\underline{A}_1, \dots, x_n\underline{A}_n$.

(2) Write down the system of linear equations you need to solve in order to find the curve $y = ax^3 + bx^2 + cx + d$ passing through the points $(2, 0)$, $(1, -2)$, $(0, -2)$, $(-1, -6)$.

Answer:

$$\begin{aligned} 8a + 4b + 2c + d &= 0 \\ a + b + c + d &= -2 \\ d &= -2 \\ -a + b - c + d &= -6 \end{aligned}$$

(3) Write the system of linear equations in the previous question as a matrix equation $A\underline{x} = \underline{b}$. What are A, $\underline{x}$, and $\underline{b}$?

Answer:

$$A = \begin{pmatrix} 8 & 4 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 \\ -2 \\ -2 \\ -6 \end{pmatrix},$$

(4) I have chosen a 3×3 matrix A and a 3×1 matrix $\underline{b}$ but I am not going to tell you what they are. Find a point $\underline{u} \in \mathbb{R}^3$ such that the set of solutions to the equation

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a\underline{u} \;\middle|\; a \in \mathbb{R} \right\}.$$

I will give you two clues. The set of solutions to $A\underline{x} = 0$ is the line $x + y = x + z = 0$ in $\mathbb{R}^3$ and

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Answer: $\underline{u}$ can be any non-zero multiple of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Comments: This question is answered by using the following result: if $A\underline{v} = \underline{b}$, then $\{\underline{u} + \underline{v} \mid A\underline{u} = \underline{0}\}$ is the set of solutions to the equation $A\underline{x} = \underline{b}$. In this question $\underline{b} = (1, 2, 3)^{\mathsf{T}}$. The first clue says that $\{\underline{u} \mid A\underline{u} = \underline{0}\}$ consists of the points on the line $x + y = x + z = 0$; since $(-1, 1, 1)^{\mathsf{T}}$ lies on this line

$$\{\underline{u} \mid A\underline{u} = \underline{0}\} = \mathbb{R} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

the set of all scalar multiples of $(-1, 1, 1)^{\mathsf{T}}$. The second clue provides a particular $\underline{v}$, namely $(1, 1, 1)^{\mathsf{T}}$.

(5) [4 points] I want a geometric description of the solutions: the set of solutions to a 4×2 system of linear equations is either
 (a) the __________ set
 (b) or __________ in ____
 (c) or __________ in ____
 (d) or $\underline{\mathbb{R}^?}$

Answer: the empty set; a point in $\mathbb{R}^2$; a line in $\mathbb{R}^2$; $\mathbb{R}^2$.

Comments: It is not correct to call the empty set the *trivial set* or the *zero set*.

(6) [3 points] I want a geometric description of the possibilities for the range of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. The range of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. is either
 (a) __________
 (b) or __________ in ____
 (c) or $\underline{\mathbb{R}^?}$

Answer: the set $\{\underline{0}\}$; a line through the origin in $\mathbb{R}^2$; $\mathbb{R}^2$.

(7) Write down a matrix A such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$.

Answer:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Comment: Use the fact that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\underline{A}_1 + y\underline{A}_2 + z\underline{A}_3$ where $\underline{A}_j$ is the j^{th} column of A.

(8) Write down a matrix A such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 2y - z \\ 2x + y + z \\ x \end{pmatrix}$.

Answer: $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Comment: Use the fact that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\underline{A}_1 + y\underline{A}_2 + z\underline{A}_3$ where $\underline{A}_j$ is the j^{th} column of A.

(9) Let $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ be the columns of a matrix A and suppose that $3\underline{A}_1 + 2\underline{A}_4 - 2\underline{A}_3 = \underline{A}_2$. Write down a solution to the equation $A\underline{x} = \underline{0}$ of the form

$$\underline{x} = \begin{pmatrix} 6 \\ ? \\ ? \\ ? \end{pmatrix}$$

Answer: $\begin{pmatrix} 6 \\ -2 \\ -4 \\ 4 \end{pmatrix}$

Comment: Use the fact that $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1\underline{A}_1 + x_2\underline{A}_2 + x_3\underline{A}_3 + x_4\underline{A}_4$.

The question tells you that $3\underline{A}_1 - \underline{A}_2 - 2\underline{A}_3 + 2\underline{A}_4 = \underline{0}$. In other words the statement of the question is telling you that

$$A \begin{pmatrix} 3 \\ -1 \\ -2 \\ 2 \end{pmatrix} = \underline{0}.$$

If $\underline{v}$ is a solution to a homogeneous equation $A\underline{x} = \underline{0}$, then $\lambda\underline{v}$ is also a solution to $A\underline{x} = \underline{0}$ for all $\lambda \in \mathbb{R}$. Hence

$$2 \begin{pmatrix} 3 \\ -1 \\ -2 \\ 2 \end{pmatrix}$$

is a solution.

(10) $(3, 4, 4, 3)$ and $(1, 2, 2, 1)$ are a basis for the subspace of $\mathbb{R}^4$ that is the set of all solutions to the equations _________ and _________.

Answer: $x_1 = x_4,\ x_2 = x_3$

Comments: Mine is not the only correct answer, just the simplest. Another correct answer is $x_1 - x_2 + x_3 - x_4 = 0$ and $x_1 + x_2 - x_3 - x_4 = 0$. A correct answer must consist of two inequivalent equations that are satisfied by the points $(3, 4, 4, 3)$ and $(1, 2, 2, 1)$. A couple of people wrote

$$3x_1 + 4x_2 + 4x_3 + 3x_4 = 0$$
$$x_1 + 2x_2 + 2x_3 + x_4 = 0$$

but this is incorrect because neither $(3, 4, 4, 3)$ nor $(1, 2, 2, 1)$ satisfies these equations. Another answer was $A\underline{x} = \underline{b}$, but this equation has nothing to do with the question.

(11) Find a basis for the plane $x_1 + x_2 = x_2 - 2x_3 = 0$ in $\mathbb{R}^4$.

Answer: $\{(-2, 2, 1, 0), (0, 0, 0, 1)\}$.

Comments: There are lots of correct answers because every subspace except $\{\underline{0}\}$ has infinitely many bases. Another correct answer is $\{(1, 1, -1, 0), (2, 4, 0, 6)\}$. Someone gave the "answer" $\{(1\ 1\ 0),\ (0\ 1\ 0)\}$. When writing a row vector one must separate the entries by commas, just as you have been doing for the past several years when speaking of points in $\mathbb{R}^2$ or $\mathbb{R}^3$, e.g. $\{(1, 1, 0),\ (0, 1, 0)\}$. Even so, those two points belong to $\mathbb{R}^3$, not to $\mathbb{R}^4$.

A plane has dimension two so every basis for it consists of exactly *two* elements.

(12) Find two linearly independent vectors that lie on the plane in $\mathbb{R}^4$ given by the equations

$$x_1 + x_2 + 2x_3 - x_4 = 0$$
$$x_1 + x_2 + 3x_3 - x_4 = 0$$

Answer: $(-1, 1, 0, 0), (1, 0, 0, 1)$. Note for the next question that the plane has dimension 2 so these two vectors are a basis for it.

Comments: Two people wrote $\{(1,1,2,-1),(1,1,3,-1)\}$ but $(1,1,2,-1)$ is not a point on the the plane because it does not satisfy the equations. I think those people were just writing down the coefficients of the equations.

(13) $(2,1,0,3)$ is a linear combination of the vectors in your answer to the previous question because _________.

Answer: it satisfies the two equations in the previous question and therefore lies on the plane in question; also every point on the plane is a linear combination of the two linearly independent vectors in your answer to the previous question.

Comments: it is not enough to say "because $(2,1,0,3) = a(-1,1,0,0) + b(1,0,0,1)$ for some $a,b \in \mathbb{R}$" because such an answer is, in effect, saying that " $(2,1,0,3)$ is a linear combination of the vectors in the answer to the previous question because it is a linear combination of the vectors in the answer to the previous question". It would be OK to say "because $(2,1,0,3) = (-1,1,0,0) + 3(1,0,0,1)$".

(14) Let A be a 4×5 matrix and $\underline{b} \in \mathbb{R}^5$. Suppose the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & 3 & 0 & | & 2 \\ 0 & 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Write down all solutions to the equation $A\underline{x} = \underline{b}$.

Answer: the free variables are x_2 and x_4. They can be any real numbers. Then $x_1 = 2 - 2x_2 - x_4$, $x_3 = 2 - 3x_4$, and $x_5 = 0$. So, the solutions are the points

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 - 2x_2 - x_4 \\ x_2 \\ 2 - 3x_4 \\ x_4 \\ 0 \end{pmatrix}$$

where x_2 and x_4 are arbitrary real numbers (parameters). It is ok to say $(x_2, x_4) \in \mathbb{R}^2$.

(15) Write down all solutions to the equation $A\underline{x} = \underline{b}$ in the previous question in the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \ x_5 \end{pmatrix} = ?+?+?$$

where the free variables appear on the right-hand side of the $=$ sign.[1]

Answer:

[1] I meant to say "where the symbol $\mathbb{R}$ appears twice on the right-hand side of the $=$ sign."

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

where $(x_2, x_4) \in \mathbb{R}^2$. The answer should get across the idea that x_2 and x_4 can take on any values—they parametrize the solution set.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

Part B.

Complete the definitions and Theorems.
There is a difference between theorems and definitions.
Don't write the part of the question I have already written. Just fill in the blank.

(1) **Definition:** A vector $\underline{x}$ is a linear combination of $\underline{v}_1, \dots, \underline{v}_n$ if ________

Answer: if $\underline{x} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some $a_1, \dots, a_n \in \mathbb{R}$.
Comments: An answer like "if $\underline{x} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$" is incomplete.
It is not good to answer this question by saying "$\underline{x} \in \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_n\}$" and then answer the next question by saying "linear combinations of $\underline{v}_1, \dots, \underline{v}_n$" because you will not have shown me that you what a linear combination is.

(2) **Definition:** The linear span of $\underline{v}_1, \dots, \underline{v}_n$ is the set of all ________

Answer: linear combinations of $\underline{v}_1, \dots, \underline{v}_n$.
Comments:

(a) An answer like "$\{a_1\underline{v}_1 + \cdots + a_n\underline{v}_n\}$" is incomplete. To complete it to a correct answer one must say

$$\{a_1\underline{v}_1 + \cdots + a_n\underline{v}_n \mid a_1, \dots, a_n \in \mathbb{R}\}.$$

(b) "all linear combinations of $\underline{v}$" is incorrect. There is no $\underline{v}$ mentioned in the first half of the definition.

(c)

(3) **Definition:** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent if the only solution to the equation ________

Answer: $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ is $a_1 = \cdots = a_n = \underline{0}$.
Comments:

(a) An answer that begins with the words "if $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$" must be incorrect because it is not grammatically correct to say "the only solution to the equation if $3x = 6$ is $x = 2$". It *is* grammatically correct to say "the only solution to the equation $3x = 6$ is $x = 2$".

(b) Definitions must be precise if they are to be understood by others. Every grammatical error obscures your intended meaning. The reader should not be left to guess what you mean. A definition should be crystal clear: like a legal document that leaves no wiggle room.
(c) Close but no cigar: "$a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ where $a_1 = \cdots = a_n = \underline{0}$." The problem with this "answer" is that the word "where" has been substituted for "is". To see the effect of this small change compare the sentences "the only solution to the equation $3x = 6$ is $x = 2$" and "the only solution to the equation $3x = 6$ where $x = 2$". The first sentence is grammatically correct, the second is not.
(d) I am grading you on the basis of what you write, not what you mean, not on the sentence I write after replacing one of your words by another word, etc. If you are unsure what is a correct definition and what is not look at the old midterms and finals linked to the course webpage. You can also write out definitions and show them to another person taking the class—compare and contrast. Or ask me in class so everyone can benefit.
(e) It is incorrect to say "$a_1\underline{v}_1, \ldots, a_n\underline{v}_n = \underline{0}$".

(4) **Definition:** $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly dependent if _________ $= \underline{0}$ for some _________

Answer: $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ for some $a_1, \ldots, a_n \in \mathbb{R}$, not all equal to zero.

Comments:
(a) Close but no cigar: $a_1\underline{v}_1+\cdots+a_n\underline{v}_n = \underline{0}$ where at least one of $a_1, \ldots, a_n \neq 0$, $a \in \mathbb{R}$. A correct answer along these lines is: "$a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ where at least one of $a_1, \ldots, a_n$ is non-zero". The problem with the "no cigar answer" is it looks like $a_n \neq 0$. Also, I don't know how to interpret $a \in \mathbb{R}$. There are real numbers $a_1, \ldots, a_n$ involved in the definition but no a without a subscript.

(5) [3 points] **Definition:** In this question use R_i to denote the i^{th} row of a matrix.

The three elementary row operations are
(a) _________
(b) _________
(c) _________

Answer: replace R_i by R_j and R_j by R_i; replace R_i by cR_i for a non-zero $c \in \mathbb{R}$; replace R_i by $R_i + cR_j$ for some $i \neq j$ and some $c \in \mathbb{R}$.

Comment: it is good to use an active verb like "replace" to clearly convey what is happening. Saying something like "add one row to another row" is vague. One *can* add one row to another but you should say what to do with the result of that addition.

(6) **Definition:** A subset W of $\mathbb{R}^n$ is a subspace if
(a) _________
(b) _________
(c) _________

Answer: $\underline{0} \in W$; $\underline{u} + \underline{v} \in W$ whenever $\underline{u}$ and $\underline{v}$ are in W; $\lambda\underline{v} \in W$ for all $\lambda \in \mathbb{R}$ and all $\underline{v} \in W$.

Comments:

(a) "It has the $\underline{0}$" is clunky. "It contains $\underline{0}$" is smooth. Compare "the set of integers has the 2" with "the set of integers contains 2" or "2 is in the set of integers" or "$2 \in \mathbb{Z}$".

(b) Several people tried to save time with answers that were too brief to convey what was meant. For example, someone wrote "$\underline{u}, \underline{v} \in W$; $\underline{u}+\underline{v} \in W$". I would be happy with an answer like "if $\underline{u}, \underline{v} \in W$ so is $\underline{u} + \underline{v}$".

(7) **Definition:** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_d\}$ is a basis for a vector space V if ______________

Answer: it is linearly independent and spans V.

(8) **Definition:** The dimension of a subspace V of $\mathbb{R}^n$ is ______________.

Answer: the number of vectors in a basis for it.

Comment: I did not like that answer " the number of vectors in its basis" because the word "its" conveys the idea that there is only one basis for V. Every vector space other than the zero vector space has infinitely many bases. For the same reason it is not correct to say "the number elements in the basis" unless you have a specific basis you are discussing.

(9) **Definition:** The rank of a matrix is ______________.

Answer: the number of non-zero rows in its row reduced echelon form.

(10) **Theorem:** The rank of a matrix is equal to the dimension of _________.

Answer: its row space; or its range.

(11) **Theorem:** If the row reduced echelon forms of the augmented coefficient matrices for two systems of linear equations are the same, then __________.

Answer: the two systems have exactly the same solutions.

Comment: several people said "they are row equivalent" or "they are equivalent". However, those answers are a little uninformative and involve another definition. Much better to give the informative answer that I give. After all, the reason we perform elementary row operations is to get a simpler system of linear equations that has the same solutions as the original system.

(12) **Theorem:** Let A be an $n \times n$ matrix and $\underline{b} \in \mathbb{R}^n$. The equation $A\underline{x} = \underline{b}$ has a unique solution if and only if A is ___________.

Answer: A is invertible; or $\mathrm{rank}(A) = n$.

Comment: some said "$\underline{b}$ is a linear combination of the columns of A" but that does not take into account the word *"unique"* that appears in the statement of the theorem. There is a *different* theorem that says *The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.*

(13) **Theorem:** Let A be an $m \times n$ matrix and let E be the row-reduced echelon matrix that is row equivalent to the transpose A^{T}. Then the transposes of the non-zero rows of E are a basis for a certain subspace of $\mathbb{R}^m$, namely ________.

Answer: the column space of A.

(14) **Theorem:** A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.

Answer: a linear combination

(15) **Theorem:** An $n \times n$ matrix A is invertible if and only if the equation $A\underline{x} = \underline{b}$ _______________ for all _______________.

Answer: has a unique solution for all $\underline{b} \in \mathbb{R}^n$.

A comment on writing. The UW is frequently rated among the top 50 universities in the world. At such a university one is expected to write well. You should be able to write clear, precise, grammatically correct, sentences. I'm no expert at this myself. It is difficult to write well. But hold yourself to high standards. Clear thinking and clear writing go hand in hand. Especially in mathematics. Mathematics is difficult enough without adding another layer of difficulty. Practice, practice, practice,...

A comment on notation. I will deduct points if you use the symbol R for the real numbers. The correct symbol is $\mathbb{R}$. Write two parallel lines to the left of the symbol, just as I do in class.

Similarly, when a question asks about a vector $\underline{v}$ make sure you write it as $\underline{v}$ not as v. I try to be consistent when writing $\underline{A}_1, \dots, \underline{A}_n$ for the columns of a matrix A. When I use that notation for the columns of A in a question you should also use the notation $\underline{A}_1, \dots, \underline{A}_n$, not $A_1, \dots, A_n$.

Part C.

True or False

Remember +1, 0, or -1.

(1) Every set of six vectors in $\mathbb{R}^5$ is linearly dependent.

True. []

(2) Every set of five vectors in $\mathbb{R}^2$ spans $\mathbb{R}^2$.

False. []

(3) Every set of four vectors in $\mathbb{R}^4$ that spans $\mathbb{R}^4$ is a basis for $\mathbb{R}^4$.

True. []

(4) The matrix $\begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}$ never has an inverse.

True. []

(5) A homogeneous system of 7 linear equations in 8 unknowns always has a non-trivial solution.

True. []

(6) The set of solutions to a system of 5 linear equations in 6 unknowns can be a 3-plane in $\mathbb{R}^6$.

True. []

(7) If the dimension of $\text{span}(\underline{v}_1, \ldots, \underline{v}_k)$ is k, then $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is linearly independent.

True. []

(8) If $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is linearly independent, then the dimension of $\text{span}(\underline{v}_1, \ldots, \underline{v}_k)$ is k.

True. []

(9) $\mathbb{R}^1$ has exactly two subspaces.

True. []

(10) If $n \geq 2$, then $\mathbb{R}^n$ has infinitely many subspaces

True. []

(11) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.

True. []

(12) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

True. []

(13) There is a matrix whose inverse is $\begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 3 \\ 0 & 3 & 6 \end{pmatrix}$.

False. []

(14) If $B^3 = A$ and $A^3 = I$, then $B^{-1} = B^8$.

True. []

(15) If $A = BC$, then every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.

True. []

(16) If $A = BC$, then every solution to $A\underline{x} = \underline{0}$ is a solution to $C\underline{x} = \underline{0}$.

False. []

(17) Let A be an $m \times n$ matrix and B an $n \times p$ matrix. If $C = AB$, then $\{C\underline{x} \mid \underline{x} \in \mathbb{R}^p\} \subset \{A\underline{w} \mid \underline{w} \in \mathbb{R}^n\}$.

True. []

(18) The linear span of the vectors $(4, 0, 0, 1)$, $(0, 2, 0, -1)$ and $(4, 3, 2, 1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

True. []

(19) If $\{\underline{a}, \underline{b}, \underline{c}\} \subset \mathsf{span}\{\underline{u}, \underline{v}, \underline{w}\}$ and $\underline{u}, \underline{v}, \underline{w} \in \mathsf{span}\{\underline{x}, \underline{y}, \underline{z}\}$, then $\underline{a}$, $\underline{b}$, and $\underline{c}$ are linear combinations of $\underline{x}$, $\underline{y}$, and $\underline{z}$.

True. []

(20) The set $\{x_1, x_2, x_3, x_4) \mid x_1 + x_3 = x_2 - x_4 = 0\}$ is a 2-dimensional subspace of $\mathbb{R}^4$.

True. []

(21) $\{\underline{x} \in \mathbb{R}^4 \mid x_1 + x_3 = x_2 - x_4\}$ is a 3-dimensional subspace of $\mathbb{R}^4$.

True. []

(22) $\{\underline{x} \in \mathbb{R}^4 \mid x_1 + x_3 = x_2 - x_4\}$ is a 2-dimensional subspace of $\mathbb{R}^4$.

False. []

(23) $\{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a 3-dimensional subspace of $\mathbb{R}^5$.

False. []

(24) For every three vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$ in $\mathbb{R}^n$, $\mathsf{span}\{\underline{u}, \underline{v}, \underline{w}\} = \mathsf{span}\{\underline{u} + \underline{v}, \underline{v} + \underline{w}, \underline{w} - \underline{u}\}$.

False. []

(25) For any vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, $\{\underline{u}, \underline{v}, \underline{w}\}$ and $\{\underline{u} - \underline{v}, \underline{v}, \underline{w} - \underline{u}\}$ have the same linear span.

True. []

(26) $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$ have the same the linear span as $\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\begin{pmatrix}5\\7\\9\end{pmatrix}$.

True. []

(27) The dimension of a subspace is the number of elements in it.

False. []

(28) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5, \underline{v}_6\}$ is linearly independent so is $\{\underline{v}_2, \underline{v}_4, \underline{v}_6\}$.

True. []

(29) If $\{\underline{v}_2, \underline{v}_4, \underline{v}_6\}$ is linearly dependent so is $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5, \underline{v}_6\}$.

True. []

(30) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, then $\{\underline{v}_1 + 3\underline{v}_2, 3\underline{v}_2 + \underline{v}_3, \underline{v}_3 - \underline{v}_1\}$ is linearly dependent.

False. []

(31) If A is an invertible matrix, then $A^{-1}\underline{b}$ is the unique solution to the equation $A\underline{x} = \underline{b}$.

True. []

(32) $\text{span}\{\underline{u}_1, \dots, \underline{u}_r\} = \text{span}\{\underline{v}_1, \dots, \underline{v}_s\}$ if and only if every $\underline{u}_i$ is a linear combination of the $\underline{v}_j$s and every $\underline{v}_j$ is a linear combination of the $\underline{u}_i$s.

True. []

(33) The row reduced echelon form of a square matrix is the identity if and only if the matrix is invertible.

True. []

(34) If W is a subspace of $\mathbb{R}^n$ that contains $\underline{u} + \underline{v}$, then W contains $\underline{u}$ and $\underline{v}$.

False. []

(35) If W is a subspace of $\mathbb{R}^n$ that contains $3\underline{u}$, then W contains $\underline{u}$.

True. []

(36) If $\underline{u}$ and $\underline{v}$ are linearly independent vectors on the plane in $\mathbb{R}^4$ given by the equations $x_1 - x_2 + x_3 - 4x_4 = 0$ and $x_1 - x_2 + x_3 - 2x_4 = 0$, then every solution to the system of equations $x_1 - x_2 + x_3 - 4x_4 = 0$ and $x_1 - x_2 + x_3 - 2x_4 = 0$ is a linear combination of $\underline{u}$ and $\underline{v}$.

True. []

(37) The vectors $(2, 2, -4, 3, 0)$ and $(0, 0, 0, 0, 1)$ are a basis for the subspace $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$ of $\mathbb{R}^5$.

False. []

(38) The vectors $(1, 1)$, $(1, -2)$, $(2, -3)$ are a basis for the subspace of $\mathbb{R}^4$ give by the solutions to the equations $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$.

False. []

(39) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cap \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is a subspace of $\mathbb{R}^4$.

True. []

(40) The set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 = x_2\} \cap \{\underline{x} \in \mathbb{R}^4 \mid x_3 = x_4\}$ is equal to the set $\{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 - x_4 = 0\}$.

True. []

(41) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is linear combination of the rows of A.

True. []

(42) The equation $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^n$ if A is an $n \times n$ matrix with rank n.

True. []

(43) There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose range is the union of the subspaces $x = y$ and $x = -y$.

False. []

(44) Let A be the 2×2 matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix}.$$

The null space of A is $\left\{ \begin{pmatrix} t \\ t \end{pmatrix} \;\middle|\; t \text{ is a real number} \right\}$.

True. []

(45) Let A be a 4×3 matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_2 \\ x_2 \end{pmatrix}.$$

The range of A has many bases; one of them consists of the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

True. []

(46) The set of solutions to the system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -5 \\ x_3 - x_4 &= 1 \end{aligned}$$

is equal to $\{(1 + 2a + 3b, a, b + 2, b + 1 \mid (a, b) \in \mathbb{R}^2\}$.

True. []

(47) The set of solutions to the system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -5 \\ x_3 - x_4 &= 1 \end{aligned}$$

is equal to $(1, 0, 2, 1) + \mathbb{R}(2, 1, 0, 0) + \mathbb{R}(3, 0, 1, 1)$.

True. []

(48) The subsets $(1, 0, 2, 1) + \mathbb{R}(2, 1, 0, 0) + \mathbb{R}(3, 0, 1, 1)$ and $\{(1 + 2a + 3b, a, b + 2, b + 1 \mid (a, b) \in \mathbb{R}^2\}$ of $\mathbb{R}^4$ are equal.

True. []