Math 308 **Midterm Answers** **July 18, 2014**

- We will write $\underline{A}_1, \dots, \underline{A}_n$ for the columns of an $m \times n$ matrix A, and A_{ij} for the entry in row i and column j.
- Several questions involve an unknown vector $\underline{x} \in \mathbb{R}^n$. We will write $x_1, \dots, x_n$ for the entries of $\underline{x}$; thus $\underline{x} = (x_1, \dots, x_n)^T$.
- The linear span of a set of vectors is denoted by $\mathsf{span}(\underline{v}_1, \dots, \underline{v}_n)$.

A remark. In what follows, I sometimes say something is "obvious". If it isn't obvious to you, please ask me for more explanation.

Part A.

Short answer questions

Scoring: 2 points per question. No partial credit.

(1) Let A be a 4×5 matrix and $\underline{b} \in \mathbb{R}^4$. Suppose that the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & \mid & 2 \\ 0 & 0 & 1 & 3 & 0 & \mid & 2 \\ 0 & 0 & 0 & 0 & 1 & \mid & 0 \\ 0 & 0 & 0 & 0 & 0 & \mid & 0 \end{pmatrix}.$$

Which are the free, or independent, variables? Write down all solutions to the equation $A\underline{x} = \underline{b}$ in the form

$$\{\cdots\cdots \mid (s,t) \in \mathbb{R}^2\}.$$

Answer: The free variables are x_2 and x_4 and the set of solutions is

$$\big\{(2-2s-t, s, 2-3t, t, 0) \;\big|\; (s,t) \in \mathbb{R}^2\big\}.$$

This set is the same as $\big\{(2+2s+t, -s, 2+3t, -t, 0) \;\big|\; (s,t) \in \mathbb{R}^2\big\}$. Why is that?

It is also OK, but less desirable, to write

$$\big\{(2,0,2,0,0) + s(-2,1,0,0,0) + t(-1,0,-3,1,0) \;\big|\; (s,t) \in \mathbb{R}^2\big\}$$

Comments: The answer $\big\{2-2s-t.s.2-3t.t. \;\big|\; (s,t) \in \mathbb{R}^2\big\}$ is wrong. So is $\big\{2-2s-t, s, 2-3t, t, 0 \;\big|\; (s,t) \in \mathbb{R}^2\big\}$ because although $(2-2s-t, s, 2-3t, t, 0)$ is a point in $\mathbb{R}^5$, $2-2s-t, s, 2-3t, t, 0$ is not a point in $\mathbb{R}^5$. You need the parentheses: e.g., (1,3) denotes a point in $\mathbb{R}^2$ but 1,3 doesn't.

(2) If the system of equations

$$\begin{aligned} 2x + 3w - t &= -1 \\ t + 2 &= -x - w \\ 2x + 3t &= 1 + 2t \end{aligned}$$

is written as a matrix equation $A\underline{x} = \underline{b}$, what are A, $\underline{x}$, and $\underline{b}$? Do *not* change the names of the unknowns. Do not solve the system!

Answer:

Comments:

(3) Find points $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4 \in \mathbb{R}^5$ such that $\mathsf{span}\{(1,2,3,4,5), \underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\} = \mathbb{R}^5$. (Choose $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ so it is easy to see why your answer is correct; if it isn't easy for me to see why your answer is correct please show me why it is correct.)

Answer: There are infinitely many correct answers. The simplest is $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, $\underline{e}_4$. (Remember $\underline{e}_2 = (0,1,0,0,0)$, etc.)

Comments:

(4) Find integers a and b such that the columns of $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 4 & 2 \\ 3 & a & b \end{pmatrix}$ are linearly dependent.

Answer: There are infinitely many correct answers. Provided $a = b + 3$, your answer is correct, e.g., $(a, b) = (3, 0)$, or (4,1), or (5,2), or (0,-3), are all correct. The reason for this is that if $a = b + 3$, then

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ a \end{pmatrix},$$

i.e., the middle column is a linear combination of the other two columns.

Comments: I used the result that a set of vectors is linearly dependent if and only if one of them is a linear combination of the others. I noticed, after looking at the first and second rows of the matrix, that $1+3 = 4$ and $2+2 = 4$ so then I chose a and b so that $a = 3 + b$, thereby making the middle column is the sum of the other two columns.

(5) Find an equation for the plane in $\mathbb{R}^3$ that contains the points $(0,0,0)$, $(1,1,1)$, and $(1,2,3)$.

Answer: There are infinitely many correct answers. You just need an equation of the form $ax + by + cz = 0$ with the property that $(1,1,1)$, and $(1,2,3)$ are solutions to it, i.e., $a+b+c = 0$ and $a+2b+3c = 0$. For example, the points $(0,0,0)$, $(1,1,1)$, and $(1,2,3)$ all lie on the plane $x + z = 2y$.

Comments: One person gave the answer $(0,0,0) + \mathbb{R}(1,1,1) + \mathbb{R}(1,2,3)$. This is not correct but we can learn something from his error. First, the $(0,0,0)$ can be deleted because $(0,0,0) + \underline{u} = \underline{u}$; in other words,

$$(0,0,0) + \mathbb{R}(1,1,1) + \mathbb{R}(1,2,3) = \mathbb{R}(1,1,1) + \mathbb{R}(1,2,3).$$

Second, the answer <u>must</u> be wrong because the question asks for an equation and $\mathbb{R}(1,1,1) + \mathbb{R}(1,2,3)$ is not an equation, but a set of points in $\mathbb{R}^3$, in fact a plane. How does this plane relate to the question? It is the unique plane that contains the points $(0,0,0)$, $(1,1,1)$, and $(1,2,3)$. In other words, the question is asking for "the" equation of the plane $\mathbb{R}(1,1,1) + \mathbb{R}(1,2,3)$. I say "the" equation because there are infinitely many such equations, all multiples of one another. For example, one equation for this plane is the one I give in my answer above, namely $x + y - 2z = 0$. Another equation for the same plane is $3x + 3y - 6z$. An so on. Finally, notice that $(0,0,0)$ does belong to the plane $\mathbb{R}(1,1,1) + \mathbb{R}(1,2,3)$ because it is $0(1,1,1) + 0(1,2,3)$.

(6) This problem takes place in $\mathbb{R}^4$. Find two linearly independent vectors belonging to the plane that consists of the solutions to the system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 0 \\ x_1 - x_2 - x_3 + x_4 &= 0. \end{aligned}$$

Answer: There are infinitely many correct answers. The ones I found first were $(1,0,5,4)$ and $(5,4,1,0)$. It is obvious that these are linearly independent. I found these solutions by subtracting the second equation from the first to get the equation $-x_2 + 4x_3 - 5x_4 = 0$ or, equivalently, $x_2 = 4x_3 - 5x_4$. You can find a solution to $x_2 = 4x_3 - 5x_4$ by taking any values for x_3 and x_4, then computing $x_2 = 4x_3 - 5x_4$; for example, if I take $x_3 = 1$ and $x_4 = 0$, then $(x_2, x_3, x_4) = (4, 1, 0)$ is a solution to $x_2 = 4x_3 - 5x_4$; similarly, if $x_3 = 5$ and $x_4 = 4$, then $(x_2, x_3, x_4) = (0, 5, 4))$ is a solution to $x_2 = 4x_3 - 5x_4$. Now, to get a solution to the original system of equations, I compute $x_1 = x_2 + x_3 - x_4$. Doing so I obtain the solutions $(5,4,1,0)$ and $(1,0,5,4)$. It is obvious that $\{(5,4,1,0),(1,0,5,4)\}$ is linearly independent. Finally, because I am a cautious man, I check that $(5,4,1,0)$ and $(1,0,5,4)$ are solutions to $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$. They are. :)

Comments: A difficult problem for most people. You need to find the solutions to the system of equations so form the matrix A of coefficients then put it in RREF; there is no need to deal with the augmented matrix $(A \mid \underline{b})$ because in this case $\underline{b} = \underline{0}$. Since

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 1 & -1 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -4 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -5 & 6 \\ 0 & 1 & -4 & 5 \end{pmatrix}$$

The free variables are x_3 and x_4, and $x_1 = 5x_3 - 6x_4$ and $x_2 = 4x_3 - 5x_4$. The solutions given by $(x_3, x_4) = (1, 0)$ and $(x_3, x_4) = (0, 1)$ are $(5,4,1,0)$ and $(-6,-5,0,1)$. These two points are obviously linearly independent. So, $(5,4,1,0)$ and $(-6,-5,0,1)$ provide a solution to the problem.

The points/vectors in your answer must be solutions to both equations in order to lie on the plane of solutions, i.e., only solutions belong to the set of solutions! For example, (1,1,1,1) is not on the plane of solutions because it is not a solution to $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$. One student gave the points $(-4,-1,1,1)$ and $(-13,-6,1,2)$ but neither of these points is a solution to the equation $x_1 - x_2 - x_3 + x_4 = 0$. (You should check your answers.)

(7) The set of all solutions to the system of linear equations

$$\begin{cases} x_1 - x_2 + x_3 = & 2 \\ x_2 + x_3 + x_4 = & 9 \\ x_3 - x_4 - x_5 = & -6 \end{cases}$$

is

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \\ 5 \end{pmatrix}.$$

Give a geometric description of this set.

Answer: A plane in $\mathbb{R}^5$.

Comments: It's simpler to write the column vectors as row vectors so I will do that in these comments. There are two free variables, namely the coefficients of $(1,1,0,-1,0)$ and $(2,3,1,-4,5)$, so the set is a plane. More generally, if $\underline{u}$ and $\underline{v}$ are any two linearly independent points in $\mathbb{R}^n$ and $\underline{w}$ any point in $\mathbb{R}^n$, then $\underline{w}+\mathbb{R}\underline{u}+\mathbb{R}\underline{v}$ is a plane in $\mathbb{R}^n$.

A couple of people said *"the plane in $\mathbb{R}^5$ passing through the points $(1,2,3,4,5)$, $(1,1,0,-1,1)$, and $(2,3,1,-4,5)$"*. That isn't correct because the points $(1,1,0,-1,1)$, and $(2,3,1,-4,5)$ are not on the plane

$$(1,2,3,4,5)+\mathbb{R}(1,1,0,-1,1)+\mathbb{R}(2,3,1,-4,5).$$

It is good to understand why these points are not on the plane. The points on the plane are points that can be written as $(1,2,3,4,5)+s(1,1,0,-1,1)+t(2,3,1,-4,5)$ for some numbers s and t. If I say, for example, that $(1,3,4,2,8)$ lies on the plane that means there are numbers s and t such that

$$(1,3,4,2,8)=(1,2,3,4,5)+s(1,1,0,-1,1)+t(2,3,1,-4,5);$$

there are such numbers, namely $s=-2$ and $t=1$. You should check that $(1,3,4,2,8)=(1,2,3,4,5)-2(1,1,0,-1,1)+(2,3,1,-4,5)$. When someone says that the plane passes through $(1,1,0,-1,1)$ that person is making the claim that there are numbers s and t such that

$$(1,1,0,-1,1)=(1,2,3,4,5)+s(1,1,0,-1,1)+t(2,3,1,-4,5);$$

but there aren't such numbers. Let me prove that. Suppose to the contrary that $(1,1,0,-1,1)=(1,2,3,4,5)+s(1,1,0,-1,1)+t(2,3,1,-4,5)$; then $(1,1,0,-1,1)=(1+s+2t,2+s+3t,3+t,4-s-4t,5+s+5t)$; i.e.,

$$\begin{aligned}
1 &= 1+s+2t,\\
1 &= 2+s+3t,\\
0 &= 3+t,\\
-1 &= 4-s-4t,\\
1 &= 5+s+5t;
\end{aligned}$$

from the third equation we see that t must equal -3; replacing t by -3 in the other four equations gives

$$7=1+s, \quad 10=2+s, \quad -13=4-s-4t, \quad 16=5+s;$$

it is obvious that there is no number s that is a solution to these 4 equations; we conclude that $(1,1,0,-1,1)$ is not on the plane.

A couple of people said *"The plane in $\mathbb{R}^5$ passing through $(1,2,3,4,5)$."* That is wrong because "the" implies there is only one plane in $\mathbb{R}^5$ passing through $(1,2,3,4,5)$. There are infinitely many planes in $\mathbb{R}^5$ passing through $(1,2,3,4,5)$.

(8) Write down a solution to the system of equations in the previous question in which $x_5=0$.

Answer: There are infinitely many correct answers. But there is one obvious one. The statement of the question says that if s and t are any real numbers, then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \\ 5 \end{pmatrix}.$$

is a solution to the system of equations and that every solution is obtained as s and t vary over all real numbers. One can see at a glance that if $s = 0$ and $t = -1$, then $x_5 = 0$. Therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \\ 8 \\ 0 \end{pmatrix}$$

is a solution.

Some students gave

$$\begin{pmatrix} -4 \\ -3 \\ 3 \\ 9 \\ 0 \end{pmatrix}$$

as a solution. That's correct—it was obtained by taking $s = -5$ and $t = 0$.

Comments: If $s = -5(1 + t)$, then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \\ 5 \end{pmatrix}.$$

is a solution to the system of equations having the property that $x_5 = 0$. Explicitly,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} - 5(1+t) \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 - 3t \\ -3 - 2t \\ 3 + t \\ 9 + t \\ 0 \end{pmatrix}$$

and to check this is a solution observe that

$$\begin{cases} x_1 - x_2 + x_3 = & (-4 - 3t) - (-3 - 2t) + (3 + t) = 2, \\ x_2 + x_3 + x_4 = & (-3 - 2t) + (3 + t) + (9 + t) = 9, \\ x_3 - x_4 - x_5 = & (3 + t) - (9 + t) + 0 = -6. \end{cases}$$

(9) Provide one non-trivial solution to the system of linear equations

$$\begin{cases} x_1 + 2x_2 &= 0 \\ x_1 - 2x_3 &= 0 \\ x_2 - x_4 &= 0 \\ x_3 - x_5 &= 0. \end{cases}$$

Hint: it might help to think of x_3 as a free variable.

Answer: There are infinitely many correct answers. One is $(2, -1, 1, -1, 1)$. All other solutions are multiples of this one. So $(2t, -t, t, -t, t)$ is a non-trivial solution for all non-zero t in $\mathbb{R}$.

Comments: Several people performed a lot of computations before arriving at a solution. They did not know how to use the hint. The point of the hint is that if you pick any number, say 7, and look for a solution in which $x_3 = 7$ you will find a solution: if you look for a solution of the form $(x_1, x_2, 7, x_4, x_5)$, then x_1 must be 14 because we need $x_1 - 2x_3 = 0$; and x_2 must be -7 because we need $x_1 + 2x_2 = 0$; and x_4 must equal x_2 because we need $x_2 - x_4 = 0$; and x_5 must equal x_3 because we need $x_3 - x_5 = 0$; thus $(14, -7, 7, -7, 7)$.

(10) Provide a solution to the system of linear equations

$$\begin{cases} x_1 + 2x_2 &= 1 \\ x_1 - 2x_3 &= 2 \\ x_2 - x_4 &= 3 \\ x_3 - x_5 &= 4. \end{cases}$$

Answer: There are infinitely many correct answers. One is $(0, \frac{1}{2}, -1, -\frac{5}{2}, -5)$.

Comments:

(11) Find points $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^5$ such that $\underline{u} + \mathbb{R}\underline{v}$ is the set of all solutions to the system of linear equations

$$\begin{cases} x_1 + 2x_2 &= 1 \\ x_1 - 2x_3 &= 2 \\ x_2 - x_4 &= 3 \\ x_3 - x_5 &= 4. \end{cases}$$

Answer: There are infinitely many correct answers.

Comments: Look at the system of equations—once x_3 is assigned a value, then the values of the other x_is are determined: e.g., $x_1 = 2 + x_3$, $x_2 = \frac{1}{2}(1 - x_1) = \frac{1}{2}(1 - 2 - x_3)$, etc.

(12) [1 point] Give a geometric description of the set of solutions to the system of equations in Question 11.

Answer: A line in $\mathbb{R}^5$.

Comments: As the hint in Question 9 suggests, there is only one free variable, hence the set of solutions is a line.

(13) Let A, B, and C, be $n \times n$ matrices, and I the $n \times n$ identity matrix. Suppose $AB = BC = I$. Use the associative law for matrix multiplication to show that $A = C$ by writing a sequence of equalities

$$A = AI = \ldots = IC = C.$$

Make it clear where you are using the associative law.

Answer:

Comments:

(14) Suppose $\underline{v}$ is a solution to $A\underline{x} = \underline{b}$. Suppose also that $A\underline{u} = \underline{0}$. Then $\underline{u} + \underline{v}$ is a solution to $A\underline{x} = \underline{b}$ because ___________ (Your answer should involve a simple calculation.)

Answer:

Comments:

(15) Write down two linearly independent vectors/points in $\mathbb{R}^3$ that are on the plane $2x + 3y - z = 0$.

Answer: There are infinitely many correct answers. One is $(1, 0, 2)$ and $(0, 1, 3)$.

Comments: Although $(0, 0, 0)$ is a solution to the equation $2x+3y-z = 0$, an answer consisting of $(0, 0, 0)$ and $(1, 0, 2)$ is incorrect because every set that contains $\underline{0}$ is linearly dependent: $\{\underline{0}, \underline{v}_1, \ldots, \underline{v}_k\}$ is linearly dependent because

$$37\underline{0} + 0\underline{v}_1 + \cdots + 0\underline{v}_k = \underline{0}.$$

Part B.

Complete the following Definitions and Theorems. **You do not need to write the part I have already written.** Just complete the sentence.

Scoring: 2 points per question. No partial credit.

(1) **Definition.** Two matrices A and B are equal if ______________.

Answer: They have the same size and $A_{ij} = B_{ij}$ for all i and j.

Comments: The response *Each have the same number of entries and each entry* is wrong: a 6×2 and a 3×4 matrix have the same number of entries, 12; the student does not say that entries in the same position are the same.

Try to keep your answer short and to the point; e.g., *they have the same size* or Saying *"they have the same number of rows and columns"* is a little vague and could be interpreted as saying they are square matrices. Likewise, saying *"they contain the same m rows and n columns"* is vague and the reader must try to figure out what the writer means.

You must also be careful to say exactly what you want to say. For example, the answer *"every entry of A equals every entry of B"* is no good because if

$$A = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}$$

it is certainly true that every entry of A equals every entry of B, but $A \neq B$. It isn't easy to be precise.

(2) **Definition.** Let A be an $m \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$ and let

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The <u>product</u> $A\underline{x}$ is _______________.

Answer: $x_1\underline{A}_1 + \cdots x_n\underline{A}_n$.

Comments: One person said $A_1x_1 + \cdots + A_2x_2$; this is wrong because it suggests that $n = 2$. Also, one should underline the vectors $\underline{A}_1$, etc. Several people wrote $\underline{A}_1x_1 + \cdots + \underline{A}_nx_n$, but we usually write $x_1\underline{A}_1$ *not* $\underline{A}_1x_1$ for the same reason we write $23x$ rather than $x23$.

(3) **Definition.** Let A be an $m \times n$ matrix and B an $n \times p$ matrix with columns $\underline{B}_1, \dots, \underline{B}_p$. The <u>product</u> AB is _______________.

Answer: $AB = [A\underline{B}_1, \dots, A\underline{B}_p]$. This notation means that AB is the matrix whose first column is $A\underline{B}_1$, etc.

Comments: The response $AB = A\underline{B}_1 + \cdots + A\underline{B}_p$ is wrong because each $A\underline{B}_j$ is a column vector, a point in $\mathbb{R}^m$ so their sum is also a point in $\mathbb{R}^m$, but AB is an $m \times p$ matrix not an $m \times 1$ matrix.

Some people gave the answer *"an $m \times p$ matrix."* That is <u>not</u> the definition of the product. It is a property of the product.

(4) **Definition.** A vector $\underline{w}$ is a <u>linear combination</u> of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if __________

Answer: $\underline{w} = c_1\underline{v}_1 + \cdots + c_n\underline{v}_n$ for some $c_1, \dots, c_n \in \mathbb{R}$.

Comments: One person just wrote $c_1\underline{v}_1 + \cdots + c_n\underline{v}_n$ for some $c_1, \dots, c_n \in \mathbb{R}$. That is wrong. It is wrong because it has nothing to do with $\underline{w}$. It is wrong because the sentence would then read *"A vector $\underline{w}$ is a <u>linear combination</u> of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if $c_1\underline{v}_1 + \cdots + c_n\underline{v}_n$ for some $c_1, \dots, c_n \in \mathbb{R}$."*

The answer *"$\underline{w}$ is the sum of the $c_1\underline{v}_1 + \cdots + c_n\underline{v}_n$..."* is wrong for the same reason that saying *"9 is the sum of the $2 + 3 + 4$"* is incorrect. It is better to say $\underline{w}$ *is equal to* $c_1\underline{v}_1 + \cdots + c_n\underline{v}_n$....

(5) **Definition.** The <u>linear span</u> of $\{\underline{v}_1, \dots, \underline{v}_n\}$ consists of __________

Answer: all linear combinations of $\underline{v}_1, \dots, \underline{v}_n$.

Comments:

(6) **Definition.** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is <u>linearly independent</u> if the only solution to the equation _______________ is _______________.

Answer: $c_1\underline{v}_1 + \cdots c_n\underline{v}_n = \underline{0}$ is $c_1 = \cdots = c_n = 0$.

Comments: *"$A\underline{x} = 0$, trivial"* is wrong: the defintion does not involve a matrix A so an answer that mentions some A can't possibly be right, unless A is defined and introduced in a sensible way; it is possible to use the word "trivial" in the solution; e.g., *the only solution to the equation* $c_1\underline{v}_1 + \cdots c_n\underline{v}_n = \underline{0}$ *is the trivial solution,* $c_1 = \cdots = c_n = 0$.

Likewise, saying *"*$A\underline{x} = \underline{b}$ *is non-trivial"* must be wrong because the definition must state a property that $\{\underline{v}_1, \ldots, \underline{v}_n\}$ has in order to be linearly independent; the matrix A has nothing to do with $\{\underline{v}_1, \ldots, \underline{v}_n\}$.

(7) **Definition.** A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly dependent if the equation ______________ has a solution in which ______________.

Answer:

Comments: *"*$A\underline{x} = 0$*, nontrivial"* is wrong for the same reasons as in my comments on the previous question.

(8) **Definition.** Write R_i for the i^{th} row of the matrix A.

The elementary row operations are

(a) replace R_i by R_j and R_j by R_i;

(b) replace R_i by ______________

(c) replace R_i by ______________

Answer: (b) replace R_i by cR_i where c is a non-zero number, and (b) replace R_i by $R_i + cR_j$ where c is any number and $j \neq i$.

Comments: In (b) you *must* say $c \neq 0$. In (c) you *must* say $i \neq j$. It would be good if you understood why you need to do this.

One person wrote $R_i + R_jc \,|\, c \in \mathbb{R}$. One should only use | to mean "such that" when using set notation as in $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$, for example.

(9) **Theorem.** A homogeneous system of linear equations always has a non-zero solution if the number of unknowns is ______________.

Answer: greater than the number of equations.

Comments:

(10) **Theorem.** The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is ________

Answer: is in the linear span of the columns of A.

Comments: Several people said "if $\underline{b}$ is in the linear span of A". This is wrong because A does not have a span. Only a set of vectors, or points in some $\mathbb{R}^n$, has a span. Look at the definition of "span" or "linear span". It says "The linear span of $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is ...".

(11) **Theorem.** A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.

Answer: a linear combination

Comments:

(12) **Theorem.** Let $\underline{v}_1, \ldots, \underline{v}_k \in \mathbb{R}^n$. If $k < n$, then _________.

Answer: $\text{span}\{\underline{v}_1, \ldots, \underline{v}_k\} \neq \mathbb{R}^n$.

Comments:

(13) **Theorem.** Let $\underline{v}_1, \ldots, \underline{v}_k \in \mathbb{R}^n$. If $k > n$, then _________.

Answer: $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is linearly dependent.

Comments:

(14) State The Big Theorem, version 1.

Answer: Let A be the $n \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$. The following are equivalent:

(a) $\{\underline{A}_1, \dots, \underline{A}_n\}$ is linearly independent;
(b) $\{\underline{A}_1, \dots, \underline{A}_n\}$ spans $\mathbb{R}^n$;
(c) the equation $A\underline{x} = \underline{b}$ has a unique solution for every $\underline{b} \in \mathbb{R}^n$.

Comments: The following answer is wrong:

Let A be the $n \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$. The following are equivalent:

(a) A is linearly independent;
(b) A spans $\mathbb{R}^n$;
(c) the equation $A\underline{x} = \underline{b}$ has a solution.

First, it is incorrect to say that a matrix is linearly independent—only a set of vectors, or points in $\mathbb{R}^n$, can be linearly (in)dependent. Likewise, one can only speak of a set of vectors, or points in $\mathbb{R}^n$, as spanning, or not spanning, $\mathbb{R}^n$. It is a bit like the verb "bark". A dog can bark, but a flower can't. Finally, the statement that the equation $A\underline{x} = \underline{b}$ has a solution says nothing useful because it doesn't say anything about $\underline{b}$ (and the word "unique" is missing).

Part C.

True or False

Use your SCANTRON.

Make sure you have filled in the bubbles with your name and student ID.

Scoring: You get +1 for each correct answer, -1 for each incorrect answer, and 0 if you do not answer the question.

(1) Every set of five vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$. **(False)**
(2) Every set of four vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$. **(False)**
(3) Every set of four vectors in $\mathbb{R}^4$ is linearly independent. **(False)**
(4) If a subset of $\mathbb{R}^4$ spans $\mathbb{R}^4$ it is linearly independent. **(False)**
(5) $(1, 2, 1, 2, 1, 2)$ is a linear combination of $(3, 0, 3, 0, 3, 0)$ and $(1, 1, 1, 1, 1, 1)$. **(True)**
(6) $(0, 3, 0, 3, 0, 3)$ is a linear combination of $(1, 2, 1, 2, 1, 2)$ and $(1, 1, 1, 1, 1, 1)$. **(True)**
(7) $(1, 2, 3) + \mathbb{R}(1, 1, 1) = (1, 1, 1) + \mathbb{R}(1, 2, 3)$. **(False)**
(8) $(6, 7, 8) \in (1, 2, 3) + \mathbb{R}(1, 1, 1)$. **(True)**
(9) If $(1, 2, 3, 4)$ and $(4, 3, 2, 1)$ are solutions to a system of linear equations so is $\frac{1}{2}(5, 5, 5, 5)$. (I am NOT assuming that the equations are homogeneous.) **(True)**
(10) Let S and T be sets of points in $\mathbb{R}^n$. Suppose that $S \subseteq T$. If S spans $\mathbb{R}^n$ so does T. **(True)**
(11) Let S and T be sets of points in $\mathbb{R}^n$. Suppose that $S \subseteq T$. If T is linearly independent so is S. **(True)**
(12) If $\mathsf{span}(\underline{u}, \underline{v}, \underline{w}) = \mathsf{span}(\underline{v}, \underline{w})$, then $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$. **(True)**

(13) If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $\mathsf{span}(\underline{u}, \underline{v}, \underline{w}) = \mathsf{span}(\underline{v}, \underline{w})$. **(True)**

(14) The rows of $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{pmatrix}$ span $\mathbb{R}^3$. **(False)**

(15) $\mathsf{span}\{(1, -2, 1)\} = \mathsf{span}\{(-1, 2, -1)\}$. **(True)**

(16) $\mathsf{span}\{(1, 0, 2), (2, 0, 1)\} = \mathsf{span}\{(1, 0, -1), (-1, 0, 1)\}$. **(False)**

(17) Suppose the matrix product $C = AB$ makes sense. Every solution to the equation $B\underline{x} = \underline{b}$ is a solution to the equation $C\underline{x} = \underline{b}$. **(False)**

(18) Suppose the matrix product $C = AB$ makes sense. Every solution to the equation $B\underline{x} = \underline{0}$ is a solution to the equation $C\underline{x} = \underline{0}$. **(True)**

(19) If B is obtained from A by a sequence of elementary row operations, then the linear span of the rows of A equals the linear span of the rows of B. **(True)**

(20) If the augmented matrix $(B \,|\, \underline{c})$ is obtained from the augmented matrix $(A \,|\, \underline{b})$ by a sequence of elementary row operations, then the equations $A\underline{x} = \underline{b}$ and $B\underline{x} = \underline{c}$ have the same solutions. **(True)**

In the remaining questions, A is a 4×4 matrix whose columns are $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$. Suppose that $\underline{A}_1 + 2\underline{A}_2 = 3\underline{A}_3 + 4\underline{A}_4$.

(21) The columns of A span $\mathbb{R}^4$. **(False)**

(22) The columns of A are linearly dependent. **(True)**

(23) The equation $A\underline{x} = 0$ has a non-trivial solution. **(True)**

(24)
$$A \begin{pmatrix} 1 \\ 2 \\ -3 \\ -4 \end{pmatrix} = \underline{0}.$$

(True)