**Math 308** **Midterm Solutions** **February 7, 2014**

I haven't had time to write solutions for every question yet. Will try to do so but it would be helpful if you emailed me and asked for solutions to particular questions. Maybe there are some questions where I don't need to say anything.

## Part A.

Short answer questions

(1) Write down the system of linear equations you need to solve in order to decide whether $(1,4,9,16,25)$ is in the linear span of the vectors

$$(1,2,3,4,5), \quad (1,1,1,1,1), \quad (1,3,5,7,0), \quad (0,2,-1,1,3).$$

I am **not** asking you to decide whether $(1,4,9,16,25)$ is in the linear span.

**Solution:** The vector $(1,4,9,16,25)$ is in

$$\operatorname{span}\{(1,2,3,4,5),(1,1,1,1,1),(1,3,5,7,0),(0,2,-1,1,3)\}$$

if and only if there are numbers $a,b,c,d$ such that

$$(1,4,9,16,25) \;=\; a(1,2,3,4,5)+b(1,1,1,1,1)+c(1,3,5,7,0)+d(0,2,-1,1,3)$$

i.e., if and only if there is a solution to the system of equations

$$\begin{cases} a+b+c = 1 \\ 2a+b+3c+2d = 4 \\ 3a+b+5c-d = 9 \\ 4a+b+7c+d = 16 \\ 5a+b+3d = 25 \end{cases}$$

**Comments:** Well over 50% of the class got this correct but some got it so wrong I could not understand how those students arrived at the answers they gave.

(2) It is possible to express the system of linear equations in the previous question as a single matrix equation $A\underline{x} = \underline{b}$. What are $A$, $\underline{x}$, and $\underline{b}$?

**Solution:**

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 3 & 1 & 5 & -1 \\ 4 & 1 & 7 & 1 \\ 5 & 1 & 0 & 3 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix}.$$

**Comments:** One problem was that a some people used different notation in (1) and (2). If you used $a,b,c,d$ for your unknowns in question (1) you must write

$$\underline{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

in your answer to (2). Likewise, if you used $x_1, x_2, x_3, x_4$ for your unknowns in question (1) you must write

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

in your answer to (2).

A couple of people wrote down the augmented matrix for the problem but that wasn't what I asked for.

(3) Compute the product

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 3 \\ 3 & 1 \end{pmatrix}$$

**Answer:** $\begin{pmatrix} 13 & 11 \\ 13 & 15 \end{pmatrix}$.

**Comments:** A surprising number of arithmetic errors. Take care and check!

(4) Write down the transpose of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}.$$

**Answer:** $\begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}$.

**Comments:**

(5) I want a geometric description of the solutions: the set of solutions to a $2 \times 4$ system of linear equations is either
   (a) the _________ set
   (b) or _________ in ___
   (c) or _________ in ___
   (d) or $\underline{\mathbb{R}^?}$

**Answer:** The empty set; or a plane in $\mathbb{R}^4$,;or a 3-plane in $\mathbb{R}^4$; or $\mathbb{R}^4$.

**Comments:** Several people said *"An empty set"*, but there is only one empty set so they should say *"the empty set"*; of course, I had already given you the clue in (a) by writing *the* _________ *set*.

Some people said *"zero set"* or *"trivial set"*. That's not right. We call the empty set the empty set. I agree that the empty set is pretty trivial, and I agree that the number of elements in it is zero. Nevertheless, it is called the empty set. Compare with an *"empty box"*. It is not the same as the trivial box or the zero box, whatever those might be. The box is empty. The box is not zero. The box is not trivial.

A common error was to give an answer which said *"a plane through the origin"* and/or a *"a 3-plane containing the origin"* but the question does not say the system of equations is homogeneous so the set of solutions need not be a subspace. For example, the solutions to the equations $x_1 = 1$ and $x_2 = 2$ are the points on the plane $\{(1, 2, x_3, x_4) \,|x_3, x_4 \in \mathbb{R}\}$ and this is *not* a subspace because it does not contain $\underline{0}$.

One answer that appeared several times really puzzled me, namely *"a plane in $\mathbb{R}^3$"*. If anyone could help me understand what lies behind such a response I would be grateful. There are 4 unknowns so all solutions are points in $\mathbb{R}^4$. Similarly, a couple of people proposed *"a line in $\mathbb{R}^2$"* as an answer. Perhaps I just need to place a little more emphasis that a solution to a system of equations in $n$ unknowns is a point in $\mathbb{R}^n$. This is true even outside the realm of linear equations. For example every solution to the system of equations $x_1x_2x_3x_4 = 3$ and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 5$ is a point in $\mathbb{R}^4$.

In a similar vein, some wrote *"a line in $\mathbb{R}^2$*. People who wrote that forgot that a $2 \times 4$ system of linear equations means 2 equations in 4 unknowns. A solution to such a system consists of a point in $\mathbb{R}^4$, i,.e., each of the unknowns takes on a particular value and those 4 values are assembled into a single entity, $(1, 2, 3, 4)$ for example, thus giving a point in $\mathbb{R}^4$.

*A line in $\mathbb{R}^4$* can NOT be the solution to a $2 \times 4$ system of linear equations. You need 3 linear equations to specify a line in $\mathbb{R}^4$ (compare with a line in $\mathbb{R}^3$ which requires two equations to specify it). To get a line you need exactly 1 independent or free variable. Or, in more familiar terms, a line is given by a set of parametric equations with exactly *one* parameter, e.g., the line $(1 + 2t, 3 - t, 4 + 3t, 5t)$, $t \in \mathbb{R}$. The number of independent or free variables for an $m \times n$ system $A\underline{x} = \underline{b}$ is $n - \text{rank}(A)$. But $\text{rank}(A) \leq m$ and $\leq n$, so for a $2 \times 4$ system, $\text{rank}(A) \leq 2$, and the number of independent or free variables is $\geq 4 - 2 = 2$.

Another way to see this is to think in terms of dimension even though we haven't yet got a formal definition of that. One linear equation usually drops the dimension by 1 so, in $\mathbb{R}^4$, the solution to one linear equation is a 3-plane (except in exceptional situations), and the solution to two linear equations is a 2-plane (except in exceptional situations in which it is $\mathbb{R}^4$, or a 3-plane, or the empty set).

(6) The set of solutions to a $4 \times 4$ system of linear equations is one of the four possibilities in the previous answer
 (a) or a _________ in ___
 (b) or a _________ in ___

**Answer:** A point in $\mathbb{R}^4$, or a line in $\mathbb{R}^4$.

**Comments:** As explained in the comments on the previous question, the line need not contain $\underline{0}$.

In answering this question and the last one a few people called the range *" a 3d shape in $\mathbb{R}^4$"* and/or *" a 4D shape in $\mathbb{R}^4$"*. *"Shape"* is too vague and imprecise. The *only* geometric shapes that occur in linear algebra are points, lines, planes, 3-planes, and their higher dimensional analogues. Blobs, spheres, parabolas, ellipsoids, circles; *none* of these things turn up in linear algebra. It might sound circular, but the only "shapes" that arise in linear algebra are linear ones. By that I mean, a "shape" that has the property that if $p$ and $q$ belong to it so does the line though $p$ and $q$ and by *line* I mean the infinite line extending in both directions.

I can't overemphasize the importance of having a geometric view in linear algebra and in that geometric view having no visual images other than linear ones. Eliminate all other "shapes". Oh, and apart from points every geometric object in linear algebra has infinitely many points on it and is a union of lines.

One person said *"a 4-plane in $\mathbb{R}^4$"* in (a) and then said *"a 4-dimensional subspace of $\mathbb{R}^4$"* in (b). But the only 4-plane in $\mathbb{R}^4$ is $\mathbb{R}^4$ and the only 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$ so both these answers are the same as the answer to (d) in the previous question.

(7) Let $S$ denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and $T$ the set of solutions to the equation $A\underline{x} = \underline{b}$. I want you to describe the relation between $S$ and $T$: If $A\underline{u} = \underline{b}$, then

$$T = \{\ldots \mid \ldots\}.$$

Your answer should involve $\underline{u}$ and $S$ and the symbol $\in$ and some more.

$T = \{\underline{u} + \underline{v} \mid \underline{v} \in S\}$

**Comments:** This proved a difficult question for those who are not fluent in reading/understanding set notation.

The following answers are wrong:

(a) $T = \{u \in \mathbb{R}^n$ and $b \in \mathbb{R}^n \mid Au = \underline{\,}\}$.

(b) $T = \{\underline{u} - \underline{b} \mid \underline{u} \in S\}$.

(c) $T = \{\underline{u} + \underline{v} \mid \underline{v} S \in \mathbb{R}\}$.

I don't think that anyone who wrote those answers understood what they wrote. For example, in (c) the statement $S \in \mathbb{R}$ means $S$ is a real number. But one is told that $S$ is the set of solutions to the equation $A\underline{x} = \underline{0}$. If $A$ is an $m \times n$ matrix then $S$ is a subset of $\mathbb{R}^n$. Also the vertical line | means *"such that"* so (c) reads, in part, such that $\underline{v}$ $S$ in $\mathbb{R}$. It isn't clear to me whether the author means that $\underline{v}$ is also a real number. My guess is that the author of (c) had tried to memorize the correct answer rather than understand the correct answer—they remembered that the symbols $\in$ and $S$ and $\underline{v}$ appeared to the right of | so just put down those symbols in any order without regard to their meaning.

In (b), the statement $\underline{u} \in S$ says *"$\underline{u}$ is in $S$"* which means that $\underline{u}$ is a solution to $A\underline{x} = \underline{0}$, but the question says $\underline{u}$ is a solution to $A\underline{x} = \underline{b}$. Also, (b) says that if I subtract $\underline{b}$ from $\underline{u}$ one gets an element of $T$, i.e., a solution to $A\underline{x} = \underline{b}$; but $A(\underline{u} - \underline{b}) = A\underline{u} - A\underline{b} = \underline{b} - A\underline{b}$ which will not usually equal $\underline{b}$.

I don't have any idea what the author of (a) means—it is just a jumble of symbols and since $S$ is not mentioned in it the answer has nothing to do with $S$. The question asked you to describe the relation between $S$ and $T$; any answer to that question must mention of $S$.

(8) Let $A$ be a $5 \times 4$ matrix with columns $\underline{A}_1, \ldots, \underline{A}_4$. Let $\underline{x} = (a, b, c, d)^{\mathsf{T}}$. Express $A\underline{x}$ as a linear combination of the columns of $A$.

**Answer:** $A\underline{x} = a\underline{A}_1 + b\underline{A}_2 + c\underline{A}_3 + d\underline{A}_4$

**Comments:** It is important to take care. For example, whoever wrote "$\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ *where* $x \in \mathbb{R}$" could produce a correct answer with a little more care. He/she need only put an $A$ before the $\underline{x}$ on the left and, if they wish to say more about $\underline{x}$, could add "*where*

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

*and* $x_1, \ldots, x_n \in \mathbb{R}$." In fact, we should probably all tell the reader that $x_1, \ldots, x_n$ are the entries of $\underline{x}$. I have been a little lazy because I always adopt the convention that when $\underline{x}$ and $x_1, \ldots, x_n$ appear in the same paragraph the $x_i$s are assumed to be the entries of $\underline{x}$ and I don't say that explicitly.

One person said "where $\underline{A}_1, \ldots, \underline{A}_n \in \mathbb{R}$. Very wrong because $\underline{A}_i$ denotes the $i^{th}$ column of $A$ and will only be in $\mathbb{R}$, i.e., will only be a number, when $A$ is a $1 \times n$ matrix.

Some people wrote $A\underline{x} = \underline{A}_1 x_1 + \cdots + \underline{A}_n x_n$. Although not really incorrect it appears clunky and goes against convention. In high school you would write an equation as $y = 3x^2 + 2x - 7$ not as $y = x^2 3 + x2 - 7$. The latter is clunky and goes against convention. It is also open to some confusion when you write things like $x^3 3$ or $x7y$. Get in the habit of writing $x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$. The numbers, constants, coefficients, are almost always written to the left of other things like matrices, vectors, unknowns. Think about your street cred.

(9) Write down a $4 \times 3$ matrix $A$ such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

**Comments:** This question and the two after it are designed to see if you really understand what your answer to Question 8 means. About 30% of the class got this question and the next two wrong even though they got Question 8 correct. This tells me those people are content to memorize something without understanding what it means. Not good - intellectually lazy.

A few people just made careless errors but it was apparent they did understand the meaning of the Question 8. It is difficult to read carefully when time is short.

(10) Write down a $4 \times 3$ matrix $A$ such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y + z \\ x + y + z \\ 0 \end{pmatrix}$.

This question is "identical" to the previous one so the answer is the same, namely

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Comments:** Understand why this question is the same as the previous one.

(11) Let $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ be the columns of a $4 \times 4$ matrix $A$ and suppose that $2\underline{A}_1 + 2\underline{A}_4 = \underline{A}_3 - 3\underline{A}_2$. Write down a solution to the equation $A\underline{x} = \underline{0}$ of the form

$$\underline{x} = \begin{pmatrix} 6 \\ ? \\ ? \\ ? \end{pmatrix}$$

**Answer:** $\underline{x} = \begin{pmatrix} 6 \\ 9 \\ -3 \\ 6 \end{pmatrix}$

**Comments:** You are told that $2\underline{A}_1 + 3\underline{A}_2 - \underline{A}_3 + 2\underline{A}_4 = \underline{0}$. Since $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ the equation $2\underline{A}_1 + 3\underline{A}_2 - \underline{A}_3 + 2\underline{A}_4 = \underline{0}$ is telling you that

$$A\begin{pmatrix} 2 \\ 3 \\ -1 \\ 2 \end{pmatrix} = \underline{0}.$$

But you are asked for a solution to $A\underline{x}$ in which $x_1 = 6$. If $A\underline{x} = \underline{0}$, then $Ac\underline{x} = \underline{0}$ for all $c \in \mathbb{R}$. Hence

$$3\begin{pmatrix} 2 \\ 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -3 \\ 6 \end{pmatrix}$$

is a solution to $A\underline{x} = \underline{0}$.

(12) $(3,1,1,3)^T$ and $(1,2,2,1)^T$ are a basis for the set of solutions to the two (different!) homogeneous equations _________ and _________

**Answer:** There are infinitely many answers but one of the simplest is $x_1 - x_4 = 0$ and $x_2 - x_3 = 0$.

**Comments:** The question asks for two equations of the form $ax_1 + bx_2 + cx_3 + dx_4 = 0$ having the property that $(x_1, x_2, x_3, x_4) = (3,1,1,3)$ and $(x_1, x_2, x_3, x_4) = (1,2,2,1)$ are solutions to those two equations; i.e., $3a + b + c + 3d = a + 2b + 2c + d = 0$. You can certainly solve the system consisting of these two equations by forming the augmented matrix, doing EROs, etc., but that is a little tedious.

Since $(3,1,1,3)^T$ and $(1,2,2,1)^T$ are solutions to $x_1 - x_4 = x_2 - x_3 = 0$ they are also solutions to every equation of the form $\lambda(x_1 - x_4) + \mu(x_2 - x_3) = 0$.

This question is equivalent to the following geometric question: *find two different 3-planes in* $\mathbb{R}^4$ *(I could be more precise and say that these 3-planes are actually 3-dimensional subspaces of* $\mathbb{R}^4$*.) each of which contains the points* $(3,1,1,3)$ *and* $(1,2,2,1)$*?* The pattern I notice is that each of these points has its first and fourth components equal so lies on the 3-plane $x_1 = x_4$. Another pattern is that each of these points has its second and third digits equal so lies on the 3-plane $x_2 = x_3$. This gives me the homogeneous equations $x_1 - x_4 = 0$ and $x_2 - x_3 = 0$.

(13) Find two points on the line $x_1 + x_2 = x_2 + x_3 = 3x_1 - 2x_4 = 0$ in $\mathbb{R}^4$.

**Answer:** $(2,-2,2,3)$ and any multiple of that vector.

**Comments:** Regardless of the exact equations, I am asking for a basis for a line, a 1-dimensional subspace because it is a line that contains $\underline{0}$, in $\mathbb{R}^4$ so that basis must consist of *one* vector having *four* entries. So an answer having two or more vectors will be wrong. Likewise, an answer consisting of a single vector with 3 entries, for example, $(4,3,1)$, will be wrong.

When I write $x_1 + x_2 = x_2 + x_3 = 3x_1 - 2x_4 = 0$ I mean the system of equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \\ 3x_1 - 2x_4 &= 0. \end{aligned}$$

A non-trivial solution to the equation $3x_1 - 2x_4 = 0$ is given by taking $x_1 = 2$ and $x_4 = 3$. From the equation $x_1 + x_2 = 0$ one sees that $x_2 = -2$. From the equation $x_2 + x_3 = 0$ one sees that $x_3 = 2$. That's where $(2,-2,2,3)$ came from.

You might compare this with question 19 in part A of the 2009 midterm.

(14) Let $A$ be a $4 \times 5$ matrix and $\underline{b} \in \mathbb{R}^5$. Suppose the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & \mid & 2 \\ 0 & 0 & 1 & 3 & 0 & \mid & 2 \\ 0 & 0 & 0 & 0 & 1 & \mid & 0 \\ 0 & 0 & 0 & 0 & 0 & \mid & 0 \end{pmatrix}$$

Label the unknowns $x_1, x_2, x_3, x_4, x_5$.

(a) What are the independent or free variables?
(b) Write down the set of solutions to the equation $A\underline{x} = \underline{b}$ in parametric form.

**Comments:**

## Part B.

Complete the definitions and Theorems.
There is a difference between theorems and definitions.
Don't write the part of the question I have already written. Just fill in the blank.

(1) **Definition:** A vector $\underline{x}$ is a linear combination of $\underline{v}_1, \ldots, \underline{v}_n$ if _________
(2) **Definition:** The linear span of $\underline{v}_1, \ldots, \underline{v}_n$ is the set of all _________
(3) **Definition:** A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly independent if the only solution to the equation _________

**Answer:** $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ is $a_1 = a_2 = \cdots = a_n = 0$.
**Comments:** If you replace "is" by "where" in the above sentence the answer becomes wrong. The words "is" and "where" have completely different meanings. Compare "my dog is fierce" and "my dog where fierce". OR, "the only answer is $x = 3$" versus "the only answer where $x = 3$".

(4) **Definition:** $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly dependent if _________ $= \underline{0}$ for some _________
(5) **Definition:** The null space and range of an $m \times n$ matrix $A$ are
  (a) $\mathcal{N}(A) = \{\cdots \mid \cdots\}$ and
  (b) $\mathcal{R}(A) = \{\cdots \mid \cdots\}$.
(6) **Definition:** A subset $W$ of $\mathbb{R}^n$ is a subspace if
  (a) _________
  (b) _________
  (c) _________
(7) **Definition:** A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_d\}$ is a basis for a subspace $V$ of $\mathbb{R}^n$ if _____________
(8) **Definition:** The dimension of a subspace $V$ of $\mathbb{R}^n$ is _____________.
(9) **Definition:** The rank of a matrix $A$ is _____________.

**Answer:** The number of non-zero rows in a row echelon form of $A$ OR the number of non-zero rows in its row reduced echelon form.

**Comments:** It is incorrect to say "the number of non-zero rows in row echelon form" because you must say which matrix is being put in echelon form. It would be correct to say "the number of non-zero rows in the row echelon form of the given matrix".

(10) **Theorem:** Let $A$ be an $m \times n$ matrix and let $E$ be the row-reduced echelon matrix that is row equivalent to it. Then the non-zero rows of $E$ are a basis for ________.
(11) **Theorem:** A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.
(12) **Theorem:** Let $\underline{u}$ and $\underline{v}$ be different points in $\mathbb{R}^n$. Then $\{t\underline{u} + (1-t)\underline{v} \mid t \in \mathbb{R}\}$ is the set of all points ___________.

## Part C.

True or False

The three numbers after the answer are the % of students who were correct, wrong, did not answer. For example, **[40, 25, 35]** means that 40% gave the correct answer, 25% gave the wrong answer, and 35% did not answer the question.

(1) $\{(2,-2,2,3)\}$ is a basis for the subspace of $\mathbb{R}^4$ consisting of all solutions to the equations $x_1+x_2=x_2+x_3=3x_1-2x_4=0$.

True. **[65,20,15]**

**Comments:**

(2) $\{(2,-2,2,3)\}$ is a basis for the null space of the matrix $\begin{pmatrix}1&1&0&0\\0&1&1&0\\3&0&0&-2\end{pmatrix}$.

True. **[48,15,37]**

**Comments:**

(3) $\{(2,-2,2,3)\}$ is a basis for the null space of the matrix

$$\begin{pmatrix}3&0&0&-2\\1&1&0&0\\0&1&1&0\end{pmatrix}$$

True. **[50,10,40]**

**Comments:**

(4) $\{(0,1,1,0),(0,0,1,1)\}$ is a linearly independent set.

True. **[51,23,26]**

**Comments:**

(5) $(0,1,1,0)$ and $(0,0,1,1)$ lie on the plane in $\mathbb{R}^4$ given by the equations

$$2x_1-x_2+x_3-x_4=0$$
$$3x_1-x_2+x_3-x_4=0$$

True. **[95,2,3]**

**Comments:**

(6) $\{(0,1,1,0),(0,0,1,1)\}$ is a basis for the set of solutions to the system

$$\begin{cases}2x_1-x_2+x_3-x_4=0\\3x_1-x_2+x_3-x_4=0\end{cases}$$

True. **[52,25,23]**

**Comments:**

(7) $\{(0,1,1,0),(0,0,1,1)\}$ is a basis for the null space of the matrix

$$\begin{pmatrix}2&-1&1&-1\\3&-1&1&-1\end{pmatrix}$$

True. [**43,20,37**]

**Comments:**

(8) $(0,1,2,1)$ a linear combination of $(0,1,1,0)$ and $(0,0,1,1)$.

True. [**90,3,7**]

**Comments:**

(9) The set of solutions to the equation $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ is a 3-dimensional subspace of $\mathbb{R}^4$.

True. [**43,25,22**]

**Comments:**

(10) The linear span of the vectors $(4,0,0,1)$, $(4,2,0,0)$ and $(4,3,2,1)$ is the 3-plane $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ in $\mathbb{R}^4$.

True. [**57,5,38**]

**Comments:**

(11) Every set of four vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

False. [**65,20,15**]

**Comments:** The vectors $(1,0,0,0)$, $(2,0,0,0)$, $(3,0,0,0)$, $(4,0,0,0)$, do not span $\mathbb{R}^4$. They span a 1-dimensional subspace, the line $\mathbb{R}(1,0,0,0)$ that passes through the origin and $(1,0,0,0)$. The vectors $(1,0,0,0)$, $(2,0,0,0)$, $(3,0,0,0)$, $(4,0,0,0)$ are multiples of each other, so they are linearly dependent. Any fact any pair of them is a linearly dependent set.

(12) Every set of five vectors in $\mathbb{R}^4$ is linearly dependent.

True. [**63,17,20**]

**Comments:** We proved a theorem saying if $r > n$, then any set of $r$ vectors in $\mathbb{R}^n$ is linearly dependent. Look at the proof: it relies on the fact that a homogeneous system of $n$ equations in $r$ unknowns has a non-trivial solution if $r > n$.

(13) Every set of five vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

False. [**55,15,30**]

**Comments:** The vectors $(1,0,0,0)$, $(2,0,0,0)$, $(3,0,0,0)$, $(4,0,0,0)$, $(5,0,0,0)$, do not span $\mathbb{R}^4$.

(14) Every set of four vectors in $\mathbb{R}^4$ is linearly independent.

False. **67,10,23**]

**Comments:** The set $\{(1,0,0,0),(2,0,0,0),(3,0,0,0),(4,0,0,0)\}$ is linearly dependent.

(15) A system of linear equations can have exactly two different solutions.

False. [**73,15,12**] [**83,10,7**]

**Comments:** A system of linear equations has either no solutions, one solution, or infinitely many solutions. If $p$ and $q$ are different solutions so is every point on the line through $p$ and $q$. That line consists of the points $tp + (1-t)q$, $t \in \mathbb{R}$.

(16) A homogeneous system of 7 linear equations in 8 unknowns always has a non-trivial solution.

True. [**82,10,8**]

**Comments:** A homogeneous system of $n$ equations in $r$ unknowns has a non-trivial solution if $r > n$.

(17) Let $A$ be an $n \times n$ matrix. If $A^3 - A^2 = 3I + 2A$, then $\frac{1}{3}(A^2 - A - 2I)$ is the inverse of $A$.

True. [**50,7,43**]

**Comments:**

(18) Row equivalent matrices have the same rank.

True. [**70,10,20**]

**Comments:**

(19) If $A$ and $B$ are invertible then $(AB)^{-1} = A^{-1}B^{-1}$.

False. [**68,10,22**] Good

**Comments:**

(20) The system of linear equations with the following augmented matrix is inconsistent.

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 2 & 3| & 2 \\ 3 & 6 & 2 & 1 & 1 & 4| & 4 \\ 0 & 0 & -3 & -1 & 0 & 0| & 6 \\ 1 & 3 & 0 & 0 & 2 & 4| & 2 \\ 2 & 6 & 0 & 0 & 4 & 8| & 3 \end{pmatrix}$$

True. [**58,7,25**] Good.

**Comments:**

(21) $\mathbb{R}^3$ has infinitely many 2-dimensional subspaces

True. [**77,3,20**] Good.

**Comments:**

(22) $\mathbb{R}^3$ has infinitely many 3-dimensional subspaces

False. [**35,35,30**]

**Comments:**

(23) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of $A$.

True. **[75,3,22]**

**Comments:** It is good to remember how to prove this. Let $\underline{A}_1, \dots, \underline{A}_n$ be the columns of $A$. If you rewrite $A\underline{x} = \underline{b}$ as $x_1\underline{A}_1 + \cdots + x_n\underline{A}_n = \underline{b}$ you see that $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of $\underline{A}_1, \dots, \underline{A}_n$.

(24) If $A$ and $B$ are $m \times n$ matrices such that $B$ can be obtained from $A$ by elementary row operations, then $A$ can also be obtained from $B$ by elementary row operations.

True. **[87,0,13]**

**Comments:**

(25) If $A = BC$, then every solution to $C\underline{x} = \underline{0}$ is a solution to $A\underline{x} = \underline{0}$.

True. **[63,7,30]**

**Comments:**

(26) Let $A$ be an $m \times n$ matrix and $B$ an $n \times p$ matrix. If $C = AB$, then $\{C\underline{x} \mid \underline{x} \in \mathbb{R}^p\} \subset \{A\underline{w} \mid \underline{w} \in \mathbb{R}^n\}$.

True. **[45,5,50]**

**Comments:**

(27) The inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}$

False. **[57,18,25]**

**Comments:** You don't need to know how to compute the inverse to answer this question. Just multiply the two matrices and notice that their product is not the identity matrix.

(28) The inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is $\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$

False. **[35,40,25]**

**Comments:** You don't need to know how to compute the inverse to answer this question. Just multiply the two matrices and notice that their product in both orders is the identity matrix.

(29) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$ have the same the linear span as $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

True. **[65,3,32]**

**Comments:**

(30) The dimension of a subspace is the number of elements in it.

False. **[40,23,37]**

**Comments:**

(31) Every subset of a linearly independent set is linearly independent.

True. **[60,7,33]**

**Comments:**

(32) Every subset of a linearly dependent set is linearly dependent.

False. **[30,35,35]**

**Comments:**

(33) The rank of an $m \times n$ matrix is $\leq m$.

True. **[52,10,38]**

**Comments:**

(34) The rank of an $m \times n$ matrix is $\leq n$.

True. **[40,25,35]**

**Comments:**

(35) If the matrix $B$ can be obtained from $A$ by elementary row operations, then $A$ can also be obtained from $B$ by elementary row operations.

True. **[75,3,22]**

**Comments:**

(36) The column space of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is equal to $\mathbb{R}^3$.

True. **[47,10,43]**

**Comments:**

(37) For all matrices $A$ and $B$, $\mathcal{R}(AB) \subset \mathcal{R}(A)$

True. **[20,17,63]**

**Comments:**

(38) If $W$ is a subspace of $\mathbb{R}^n$ that contains $\underline{u} + \underline{v}$, then $W$ contains $\underline{u}$ and $\underline{v}$.

False. **[53,22,25]**

**Comments:**

(39) If $\underline{u}$ and $\underline{v}$ are linearly independent vectors on the plane in $\mathbb{R}^4$ given by the equations $x_1 - x_2 + x_3 - 4x_4 = 0$ and $x_1 - x_2 + x_3 - 2x_4 = 0$, then $(1, 2, 1, 0)$ a linear combination of $\underline{u}$ and $\underline{v}$.

True. **[35,10,55]**

**Comments:**

(40) The range of a matrix is its columns.

False. **[40,20,40]**

**Comments:**

(41) The vectors $(2, 2, -4, 3, 0)$ and $(0, 0, 0, 0, 1)$ are a basis for the subspace $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$ of $\mathbb{R}^5$.

True. **[35,15,50]**

**Comments:**

(42) The vectors $(1, 1)$, $(1, -2)$, $(2, -3)$ are a basis for the subspace of $\mathbb{R}^4$ give by the solutions to the equations $x_1 - x_2 = 2x_2 + x_3 = 3x_1 - 2x_4 = 0$.

False. **[33,12,55]**

**Comments:**

(43) $\{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4\}$ is a subspace of $\mathbb{R}^4$.

True. **[47,3,50]**

**Comments:**

(44) $\{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a subspace of $\mathbb{R}^5$.

False. **[35,12,52]**

**Comments:**

(45) The range of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is equal to its null space.

True. **[23,7,70]**

**Comments:**

(46) Let $A$ be the $2 \times 2$ matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

The null space of $A$ is $\left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} \middle| \; t \text{ is a real number} \right\}$.

True. **[33,2,65]**

**Comments:**

(47) Let $A$ be the $2 \times 2$ matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

The null space of $A$ is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

True. **[22,5,73]**

**Comments:**

(48) Let $A$ be the $2 \times 2$ matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}.$$

The null space of $A$ is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

True. **[20,7,73]**

**Comments:**

(49) Let $A$ be a $4 \times 3$ matrix such that

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ x_2 \end{pmatrix}.$$

The range of $A$ has many bases; one of them consists of the vectors

$$\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

True. **[22,7,71]**

**Comments:**

(50) If $A$ is an $m \times n$ matrix such that $A\underline{x} = \underline{0}$ for all $\underline{x} \in \mathbb{R}^n$, then $A = 0$.

True. **[35,22,43]**

**Comments:**

(51) Let $\underline{e}_1, \dots, \underline{e}_n$ be the standard basis for $\mathbb{R}^n$. If $A$ is an $m \times n$ matrix such that $A\underline{e}_i = \underline{0}$ for $i = 1, \dots, n$, then $A = 0$.

True. **[32,8,60]**

**Comments:**

(52) Every element of $\mathbb{R}^n$ is a linear combination of the standard basis vectors $\underline{e}_1, \dots, \underline{e}_n$.

True. **[47,3,50]**

**Comments:**

(53) If $A$ and $B$ are $n \times n$ matrices such that $\mathcal{R}(A) \subset \mathcal{N}(B)$, then $BA\underline{x} = \underline{0}$ for all $\underline{x} \in \mathbb{R}^n$.

True. **[25,0,75]**

**Comments:**

(54) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, then $\{\underline{v}_1 + 3\underline{v}_2, 3\underline{v}_2 + \underline{v}_3, \underline{v}_3 - \underline{v}_1\}$ is linearly dependent.

True. **[25,12,63]**

**Comments:**

(55) $\text{span}\{\underline{u}_1, \dots, \underline{u}_r\} = \text{span}\{\underline{v}_1, \dots, \underline{v}_s\}$ if and only if every $\underline{u}_i$ is a linear combination of the $\underline{v}_j$s and every $\underline{v}_j$ is a linear combination of the $\underline{u}_i$s.

True. **[42,2,56]**

**Comments:**

(56) If $\mathsf{span}\{\underline{u}, \underline{v}\} \subset \mathsf{span}\{\underline{x}, \underline{w}, \underline{z}\}$ then every linear combination of $\underline{u}$ and $\underline{v}$ is a linear combination of $\underline{x}$, $\underline{w}$, and $\underline{z}$.

True. **[47,0,53]**

**Comments:**