

# MATH 308 LINEAR ALGEBRA NOTES

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## CONTENTS

1. Introduction	2
2. Matrices	2
2.1. What is a matrix?	2
2.2. Row and column vectors	3
2.3. Matrix arithmetic: addition and subtraction	4
2.4. The zero matrix and the negative of a matrix	4
2.5. Matrix arithmetic: multiplication	5
2.6. Pitfalls and warnings	8
2.7. Transpose	9
2.8. Some special matrices	9
2.9. Solving an equation involving an upper triangular matrix	10
2.10. Some special products	11
3. Matrices and motions in $\mathbb{R}^2$ and $\mathbb{R}^3$	12
3.1. Linear transformations	12
3.2. Rotations in the plane	13
3.3. Projections in the plane	13
3.4. Contraction and dilation	13
3.5. Reflections in the plane	13
3.6. Reflections in $\mathbb{R}^3$	13
3.7. Projections from $\mathbb{R}^3$ to a plane	13
4. Systems of Linear Equations	13
4.1. General systems of equations	13
4.2. A single linear equation	16
4.3. Systems of linear equations	17
4.4. A system of linear equations is a single matrix equation	17
4.5. Specific examples	18
4.6. The number of solutions	20
4.7. A geometric view on the number of solutions	21
4.8. Homogeneous systems	22
5. Row operations and row equivalence	24
5.1. Equivalent systems of equations	24
5.2. Echelon Form	25
5.3. An example	26
5.4. You already did this in high school	26
5.5. The rank of a matrix	27

5.6.	Inconsistent systems	28
5.7.	Consistent systems	28
5.8.	Parametric equations for lines and planes	30
5.9.	The importance of rank	31
5.10.	The word “solution”	32
6.	The geometry of systems of linear equations	32
7.	The Vector space $\mathbb{R}^n$	33
7.1.	Arithmetic in $\mathbb{R}^n$	33
7.2.	The standard basis for $\mathbb{R}^n$	34
7.3.	Linear combinations and linear span	34
8.	Subspaces	36
8.1.	The definition and examples	36
8.2.	The row and column spaces of a matrix	39
8.3.	Lines, planes, and translations of subspaces	40
8.4.	Linguistic difficulties: algebra vs. geometry	42
9.	Linear dependence and independence	42
9.1.	The definition	42
9.2.	Criteria for linear (in)dependence	43
9.3.	Linear (in)dependence and systems of equations	45
10.	Non-singular matrices, invertible matrices, and inverses	47
10.1.	Singular and non-singular matrices	47
10.2.	Inverses	48
10.3.	Comments on the definition of inverse	49
10.4.	Making use of inverses	50
10.5.	The inverse of a $2 \times 2$ matrix	51
10.6.	If $A$ is non-singular how do we find $A^{-1}$ ?	53
11.	Bases, coordinates, and dimension	53
11.1.	Definitions	53
11.2.	A subspace of dimension $d$ is just like $\mathbb{R}^d$	54
11.3.	All bases for $W$ have the same number of elements	54
11.4.	Every subspace has a basis	55
11.5.	Properties of bases and spanning sets	56
11.6.	How to find a basis for a subspace	57
11.7.	How to find a basis for the range of a matrix	57
11.8.	Rank + Nullity	58
11.9.	How to compute the null space and range of a matrix	61
11.10.	Orthogonal and orthonormal bases	61
11.11.	The Gram-Schmidt process	63
11.12.	Gazing into the distance: Fourier series	64
12.	Linear transformations	65
12.1.	First observations	65
12.2.	Linear transformations and matrices	67
12.3.	How to find the matrix representing a linear transformation	68
12.4.	Invertible matrices and invertible linear transformations	68
12.5.	How to find the formula for a linear transformation	68

12.6.	Rotations in the plane	69
12.7.	Reflections in $\mathbb{R}^2$	69
12.8.	Orthogonal projections	70
12.9.	Invariant subspaces	70
12.10.	The one-to-one and onto properties	71
12.11.	Gazing into the distance: differential operators as linear transformations	72
13.	Determinants	74
13.1.	The definition	74
13.2.	Elementary row operations and determinants	77
13.3.	The determinant and invertibility	79
13.4.	Properties	79
13.5.	Elementary column operations and determinants	79
14.	Eigenvalues	82
14.1.	Definitions and first steps	82
14.2.	Reflections in $\mathbb{R}^2$ , revisited	83
14.3.	The $2 \times 2$ case	83
14.4.	The equation $A^2 + I$	85
14.5.	The characteristic polynomial	86
14.6.	How to find eigenvalues and eigenvectors	87
14.7.	The utility of eigenvectors	91
15.	Complex vector spaces and complex eigenvalues	91
15.1.		92
15.2.	The complex numbers	92
15.3.	Linear algebra over $\mathbb{C}$	95
15.4.	The complex norm	96
16.	Similar matrices	98
16.1.		98
16.2.	Definition and first properties	98
16.3.	Example	98
16.4.		99
16.5.	Diagonalizable matrices	100
16.6.	Orthogonal matrices	104
17.	Applications of linear algebra and vector spaces	105
17.1.	Simple electrical circuits	106
17.2.	Least-squares solutions to inconsistent systems	107
17.3.	Approximating data by polynomial curves	109
17.4.	Magic squares	110
18.	Last rites	112
18.1.	A summary of notation	112
18.2.	A sermon	113

## 1. INTRODUCTION

The practical problem, solving systems of linear equations, that motivates the material in this course, Linear Algebra, is introduced in section 4. Although the problem is concrete and easily understood, the methods and theoretical framework required for a deeper understanding of it are abstract. Linear algebra can be approached in a completely abstract fashion, untethered from the problems that give rise to the subject.

Such an approach is daunting for the average student so we will strike a balance between the practical, specific, abstract, and general. We begin in section 2 with material about matrices that will be familiar to some students. Very little from section 2 is required to understand the first few subsections of section 4 so you might want to skip section 2 and return to it as you need it.

Linear algebra plays a central role in almost all parts of modern technology. Systems of linear equations involving hundreds, thousands, even hundreds of millions of variables are solved every second of every day in all corners of the globe. One of the more fantastic uses is the way in which Google prioritizes pages on the web. All web pages are assigned a *page rank* that measures its importance. The page ranks are the unknowns in an enormous system of linear equations. To find the page rank one must solve the system of linear equations. In order to handle such large systems of linear equations one must use sophisticated techniques that are developed first as abstract results about linear algebra.

Systems of linear equations are rephrased in terms of matrix equations, equations involving matrices. The translation is straightforward but after mastering the basics of “matrix arithmetic” one must interpret those basics in geometric terms. That leads to linear geometry and the language of vectors and vector spaces.

We begin in section 2 with matrices, then introduce systems of linear equations, the motivating problem, in section 4.

## 2. MATRICES

You can skip this section if you want, start reading at section 4, and return to this section whenever you need to.

Matrices are an essential part of the language of linear algebra and linear equations. This chapter isolates this part of the language so you can easily refer back to it when you need to.

**2.1. What is a matrix?** An  $m \times n$  **matrix** (we read this as “an  $m$ -by- $n$  matrix”) is a rectangular array of  $mn$  numbers arranged into  $m$  rows and  $n$  columns. For example,

$$(2-1) \quad \begin{pmatrix} 1 & 3 & 0 \\ -4 & 5 & 2 \end{pmatrix}$$

is a  $2 \times 3$  matrix. The prefix  $m \times n$  is called the size of the matrix.

2.1.1. We use upper case letters to denote matrices. The numbers in the matrix are called its entries. The entry in row  $i$  and column  $j$  is called the  $ij^{\text{th}}$  entry, and if the matrix is denoted by  $A$  we often write  $A_{ij}$  for its  $ij^{\text{th}}$  entry. If  $A$  is the matrix in (2-1) its entries are

$$\begin{aligned} A_{11} &= 1, & A_{12} &= 3, & A_{13} &= 0 \\ A_{21} &= -4, & A_{22} &= 5, & A_{23} &= 2. \end{aligned}$$

In this example, we call 5 the 22-entry; read this as *two-two entry*. Likewise, the 21-entry (two-one entry) is 4.

We sometimes write  $A = (a_{ij})$  for the matrix whose  $ij^{\text{th}}$  entry is  $a_{ij}$ . For example, we might write

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Notice that  $a_{ij}$  is the entry in row  $i$  and column  $j$

2.1.2. *Equality of matrices.* Two matrices  $A$  and  $B$  are equal if and only if they have the same size and  $A_{ij} = B_{ij}$  for all  $i$  and  $j$ .

2.1.3. *Square matrices.* An  $n \times n$  matrix is called a square matrix. For example,

$$(2-2) \quad A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

is a  $2 \times 2$  square matrix.

2.2. **Row and column vectors.** Matrices having just one row or column are of particular importance and are often called vectors.

A matrix having a single row is called a row matrix or a row vector.

A matrix having a single column is called a column matrix or a column vector.

For example, the  $1 \times 4$  matrix  $(1 \ 2 \ 3 \ 4)$  is a row vector, and the  $4 \times 1$  matrix

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

is a column vector.

2.2.1. *Underlining row and column vectors.* Matrices are usually denoted by upper case letters but this convention is usually violated for row and column vectors. We usually denote a row or column vector by a lower case letter *that is underlined*. The underlining is done to avoid confusion between a lower case letters denoting numbers and lower case letters denoting row or

column vectors. For example, I will write

$$\underline{u} = (1 \ 2 \ 3 \ 4) \quad \text{and} \quad \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

**2.3. Matrix arithmetic: addition and subtraction.** We *add* two matrices of the same size by adding the entries of one to those of the other, e.g.,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} -1 & 2 & 0 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 3 \\ 7 & 3 & 7 \end{pmatrix}.$$

The result of adding two matrices is called their sum. Subtraction is defined in a similar way, e.g.,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} - \begin{pmatrix} -1 & 2 & 0 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 7 & 5 \end{pmatrix}.$$

Abstractly, if  $(a_{ij})$  and  $(b_{ij})$  are matrices of the same size, then

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \quad \text{and} \quad (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}).$$

**2.3.1. Matrices of different sizes can not be added.** The sum of two matrices of different sizes is not defined. Whenever we write  $A + B$  we tacitly assume that  $A$  and  $B$  have the same size.

**2.3.2. Addition of matrices is commutative:**

$$A + B = B + A.$$

The commutativity is a simple consequence of the fact that addition of numbers is commutative.

**2.3.3. Addition of matrices, like addition of numbers, is associative:**

$$(A + B) + C = A + (B + C).$$

This allows us to dispense with the parentheses: the expression  $A + B + C$  is unambiguous—you get the same answer whether you first add  $A$  and  $B$ , then  $C$ , or first add  $B$  and  $C$ , then  $A$ . Even more, because  $+$  is commutative you also get the same answer if you first add  $A$  and  $C$ , then  $B$ .

**2.4. The zero matrix and the negative of a matrix.** The zero matrix is the matrix consisting entirely of zeroes. Of course, we shouldn't really say *the* zero matrix because there is one zero matrix of size  $m \times n$  for each choice of  $m$  and  $n$ . We denote the zero matrix of any size by  $0$ . Although the symbol  $0$  now has many meanings it should be clear from the context which zero matrix is meant.

The zero matrix behaves like the number 0:

- (1)  $A + 0 = A = 0 + A$  for all matrices  $A$ , and
- (2) given a matrix  $A$ , there is a unique matrix  $A'$  such that  $A + A' = 0$ .

The matrix  $A'$  in (2) is denoted by  $-A$  and its entries are the negatives of the entries in  $A$ .

After we define multiplication of matrices we will see that the zero matrix shares another important property with the number zero: the product of a zero matrix with any other matrix is zero.

**2.5. Matrix arithmetic: multiplication.** Let  $A$  be an  $m \times n$  matrix and  $B$  a  $p \times q$  matrix. The product  $AB$  is only defined when  $n = p$  and in that case the product is an  $m \times q$  matrix. For example, if  $A$  is a  $2 \times 3$  matrix and  $B$  a  $3 \times 4$  matrix we can form the product  $AB$  but there is no product  $BA$ ! In other words,

the product  $AB$  can only be formed if the number of columns in  $A$  is equal to the number of rows in  $B$ .

The rule for multiplication will strike you as complicated and arbitrary at first though you will later understand why it is natural. Let  $A$  be an  $m \times n$  matrix and  $B$  an  $n \times q$  matrix. Before being precise, let me say that the

the  $ij^{\text{th}}$  entry of  $AB$  is obtained by multiplying the  $i^{\text{th}}$  row of  $A$  by the  $j^{\text{th}}$  column of  $B$ .

What we mean by this can be seen by reading the next example carefully. The product  $AB$  of the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 4 & 3 \\ 2 & 1 & 2 \end{pmatrix}$$

is the  $2 \times 3$  matrix whose

$$11\text{-entry is (row 1)} \times (\text{column 1}) = (1 \ 2) \times \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 1 \times 5 + 2 \times 2 = 9$$

$$12\text{-entry is (row 1)} \times (\text{column 2}) = (1 \ 2) \times \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1 \times 4 + 2 \times 1 = 6$$

$$13\text{-entry is (row 1)} \times (\text{column 3}) = (1 \ 2) \times \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1 \times 3 + 2 \times 2 = 7$$

$$21\text{-entry is (row 2)} \times (\text{column 1}) = (0 \ -1) \times \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0 \times 5 - 1 \times 2 = -2$$

$$22\text{-entry is (row 2)} \times (\text{column 2}) = (0 \ -1) \times \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 0 \times 4 - 1 \times 1 = -1$$

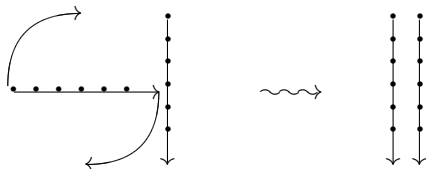
$$23\text{-entry is (row 2)} \times (\text{column 3}) = (0 \ -1) \times \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 0 \times 3 - 1 \times 2 = -2$$

In short,

$$AB = \begin{pmatrix} 9 & 6 & 7 \\ -2 & -1 & -2 \end{pmatrix}.$$

The product  $BA$  does not make sense in this case.

My own mental picture for remembering how to multiply matrices is encapsulated in the diagram



2.5.1. *Powers of a matrix.* A square matrix, i.e., an  $n \times n$  matrix, has the same number of rows as columns so we can multiply it by itself. We write  $A^2$  rather than  $AA$  for the product. For example, if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{then } A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

Please check those calculations to test your understanding.

2.5.2. *The formal definition of multiplication.* Let  $A$  be an  $m \times n$  matrix and  $B$  an  $n \times q$  matrix. Then  $AB$  is the  $m \times q$  matrix whose  $ij^{\text{th}}$  entry is

$$(2-3) \quad (AB)_{ij} = \sum_{t=1}^n A_{it}B_{tj}.$$

*Do you understand this formula?* Is it compatible with what you have understood above? If  $A$  is the  $3 \times 4$  matrix with  $A_{ij} = |4 - i - j|$  and  $B$  is the  $4 \times 3$  matrix with  $B_{ij} = ij - 4$ , what is  $(AB)_{23}$ ? If you can't answer this question there is something about the definition of multiplication or something about the notation I am using that you do not understand. Do yourself and others a favor by asking a question in class.

2.5.3. *The zero matrix.* For every  $m$  and  $n$  there is an  $m \times n$  matrix we call zero. It is the  $m \times n$  matrix with all its entries equal to 0. The product of any matrix with the zero matrix is equal to zero. Why?

2.5.4. *The identity matrix.* For every  $n$  there is an  $n \times n$  matrix we call the  $n \times n$  identity matrix. We often denote it by  $I_n$ , or just  $I$  if the  $n$  is clear from the context. It is the matrix with entries

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

In other words every entry on the NW-SE diagonal is 1 and all other entries are 0.

The key property of the  $n \times n$  identity matrix  $I$  is that if  $A$  is any  $m \times n$  matrix, then  $AI = A$  and, if  $B$  is any  $n \times p$  matrix  $IB = B$ .

2.5.5. *Multiplication is associative.* Another way to see whether you understand the definition of the product in (2-3) is to try using it to prove that matrix multiplication is associative, i.e., that

$$(2-4) \quad (AB)C = A(BC)$$

for any three matrices for which this product makes sense. This is an important property, just as it is for numbers: if  $a$ ,  $b$ , and  $c$ , are real numbers then  $(ab)c = a(bc)$ ; this allows us to simply write  $abc$ , or when there are four numbers  $abcd$ , because we know we get the same answer no matter how we group the numbers in forming the product. For example

$$(2 \times 3) \times (4 \times 5) = 6 \times 20 = 120$$

and

$$((2 \times 3) \times 4) \times 5 = (6 \times 4) \times 5 = 24 \times 5 = 120.$$

Formula (2-3) only defines the product of *two* matrices so to form the product of three matrices  $A$ ,  $B$ , and  $C$ , of sizes  $k \times \ell$ ,  $\ell \times m$ , and  $m \times n$ , respectively, we must use (2-3) twice. But there are two ways to do that: first compute  $AB$ , then multiply on the right by  $C$ ; or first compute  $BC$ , then multiply on the left by  $A$ . The formula  $(AB)C = A(BC)$  says that those two alternatives produce the same matrix. Therefore we can write  $ABC$  for that product (no parentheses!) without ambiguity. Before we can do that we must *prove* (2-4) by using the definition (2-3) of the product of two matrices.

2.5.6. Try using (2-3) to prove (2-4). To show two matrices are equal you must show their entries are the same. Thus, you must prove the  $ij^{\text{th}}$  entry of  $(AB)C$  is equal to the  $ij^{\text{th}}$  entry of  $A(BC)$ . To begin, use (2-3) to compute  $((AB)C)_{ij}$ . I leave you to continue.

This is a test of your desire to pass the course. I know you want to pass the course, but do you want that enough to do this computation?<sup>1</sup>

Can you use the associative law to prove that if  $J$  is an  $n \times n$  matrix with the property that  $AJ = A$  and  $JB = B$  for all  $m \times n$  matrices  $A$  and all  $n \times p$  matrices  $B$ , then  $J = I_n$ ?

2.5.7. *The distributive law.* If  $A$ ,  $B$ , and  $C$ , are matrices of sizes such that the following expressions make sense, then

$$(A + B)C = AC + BC.$$

---

<sup>1</sup>Perhaps I will ask you to prove that  $(AB)C = A(BC)$  on the midterm—would you prefer to discover whether you can do that now or then?

2.5.8. *The columns of the product  $AB$ .* Suppose the product  $AB$  exists. It is sometimes useful to write  $\underline{B}_j$  for the  $j^{\text{th}}$  column of  $B$  and write

$$B = [\underline{B}_1, \dots, \underline{B}_n].$$

The columns of  $AB$  are then given by the formula

$$AB = [A\underline{B}_1, \dots, A\underline{B}_n].$$

You should check this assertion.

2.5.9. *The product  $A\underline{x}$ .* Suppose  $A = [\underline{A}_1, \underline{A}_2, \dots, \underline{A}_n]$  is a matrix with  $n$  columns and

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then

$$A\underline{x} = x_1\underline{A}_1 + x_2\underline{A}_2 + \dots + x_n\underline{A}_n.$$

This last equation is *one of the most important equations in the course*. Tattoo it on your body. If you don't know this equation you will probably fail the course.

Try to prove it yourself. First do two simple examples,  $A$  a  $2 \times 3$  matrix, and  $A$  a  $3 \times 2$  matrix. Then try the general case,  $A$  an  $m \times n$  matrix.

2.5.10. *Multiplication by a scalar.* There is another kind of multiplication we can do. Let  $A$  be an  $m \times n$  matrix and let  $c$  be any real number. We define  $cA$  to be the  $m \times n$  matrix obtained by multiplying every entry of  $A$  by  $c$ . Formally,  $(cA)_{ij} = cA_{ij}$  for all  $i$  and  $j$ . For example,

$$3 \begin{pmatrix} 1 & 3 & 0 \\ -4 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 9 & 0 \\ -12 & 15 & 6 \end{pmatrix}.$$

We also define the product  $Ac$  by declaring that it is equal to  $cA$ , i.e.,  $Ac = cA$ .

## 2.6. Pitfalls and warnings.

2.6.1. *Warning: multiplication is not commutative.* If  $a$  and  $b$  are numbers, then  $ab = ba$ . However, if  $A$  and  $B$  are matrices  $AB$  need not equal  $BA$  even if both products exist. For example, if  $A$  is a  $2 \times 3$  matrix and  $B$  is a  $3 \times 2$  matrix, then  $AB$  is a  $2 \times 2$  matrix whereas  $BA$  is a  $3 \times 3$  matrix.

Even if  $A$  and  $B$  are square matrices of the same size, which ensures that  $AB$  and  $BA$  have the same size,  $AB$  need not equal  $BA$ . For example,

$$(2-5) \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

but

$$(2-6) \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

2.6.2. *Warning: a product of non-zero matrices can be zero.* The calculation (2-6) shows  $AB$  can be zero when both  $A$  and  $B$  are non-zero.

2.6.3. *Warning: you can't always cancel.* If  $A$  is not zero and  $AB = AC$  it need not be true that  $B = C$ . For example, in (2-6) we see that  $AB = 0 = A0$  but the  $A$  cannot be cancelled to deduce that  $B = 0$ .

2.7. **Transpose.** The transpose of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix  $A^T$  defined by

$$(A^T)_{ij} := A_{ji}.$$

Thus the transpose of  $A$  is “the reflection of  $A$  in the diagonal”, and the rows of  $A$  become the columns of  $A^T$  and the columns of  $A$  become the rows of  $A^T$ . An example makes it clear:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Notice that  $(A^T)^T = A$ .

The transpose of a column vector is a row vector and vice versa. We often use the transpose notation when writing a column vector—it saves space<sup>2</sup> and looks better to write  $\underline{v}^T = (1\ 2\ 3\ 4)$  rather than

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Check that

$$(AB)^T = B^T A^T.$$

A lot of students get this wrong: they think that  $(AB)^T$  is equal to  $A^T B^T$ . (If the product  $AB$  makes sense  $B^T A^T$  need not make sense.)

2.8. **Some special matrices.** We have already met two special matrices, the identity and the zero matrix. They behave like the numbers 1 and 0 and their great importance derives from that simple fact.

2.8.1. *Symmetric matrices.* We call  $A$  a symmetric matrix if  $A^T = A$ . A symmetric matrix must be a square matrix. A symmetric matrix is symmetric about its main diagonal. For example

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 4 & 5 & 6 \\ 2 & 5 & 7 & 8 \\ 3 & 6 & 8 & 9 \end{pmatrix}$$

is symmetric.

---

<sup>2</sup>Printers dislike blank space because it requires more paper. They also dislike black space, like  $\blacktriangle$ ,  $\star$ ,  $\clubsuit$ ,  $\blacksquare$ , because it requires more ink.

A square matrix  $A$  is symmetric if  $A_{ij} = A_{ji}$  for all  $i$  and  $j$ . For example, the matrix

$$\begin{pmatrix} 1 & -2 & 5 \\ -2 & 0 & 4 \\ 5 & 4 & -9 \end{pmatrix}$$

is symmetric. The name comes from the fact that the entries below the diagonal are the same as the corresponding entries above the diagonal where “corresponding” means, roughly, the entry obtained by reflecting in the diagonal. By the diagonal of the matrix above I mean the line from the top left corner to the bottom right corner that passes through the numbers 1, 0, and  $-9$ .

If  $A$  is any square matrix show that  $A + A^T$  is symmetric.

Is  $AA^T$  symmetric?

2.8.2. *Skew-symmetric matrices.* A square matrix  $A$  is skew-symmetric if  $A^T = -A$ .

If  $B$  is a square matrix show that  $B - B^T$  is skew symmetric.

What can you say about the entries on the diagonal of a skew-symmetric matrix.

2.8.3. *Upper triangular matrices.* A square matrix is upper triangular if all its entries below the diagonal are zero. For example,

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

is upper triangular.

2.8.4. *Lower triangular matrices.* A square matrix is lower triangular if all its entries above the diagonal are zero. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

is lower triangular. The transpose of a lower triangular matrix is upper triangular and vice-versa.

## 2.9. Solving an equation involving an upper triangular matrix.

Here is an easy problem: find numbers  $x_1, x_2, x_3, x_4$  such that

$$A\underline{x} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \underline{b}.$$

Multiplying the bottom row of the  $4 \times 4$  matrix by the column  $\underline{x}$  gives  $2x_4$  and we want that to equal  $b_4$  which is 4, so  $x_4 = 2$ . Multiplying the third row of the  $4 \times 4$  matrix by  $\underline{x}$  gives  $-x_3 + x_4$  and we want that to equal  $b_3$

which is 3. We already know  $x_4 = 2$  so we must have  $x_3 = -1$ . Repeating this with the second row of  $A$  gives  $2x_2 - x_3 = 2$ , so  $x_2 = \frac{1}{2}$ . Finally the first row of  $A$  gives  $x_1 + 2x_2 + 3x_3 + 4x_4 = 1$ ; plugging in the values we have found for  $x_4$ ,  $x_3$ , and  $x_2$ , we get  $x_1 + 1 - 3 + 8 = 1$  so  $x_1 = -5$ . We find that a solution is given by

$$\underline{x} = \begin{pmatrix} -5 \\ \frac{1}{2} \\ -1 \\ 4 \end{pmatrix}.$$

Is there any other solution to this equation? No. Our hand was forced at each stage.

Here is a hard problem:

**2.10. Some special products.** Here we consider an  $m \times n$  matrix  $B$  and the effect of multiplying  $B$  on the left by some special  $m \times m$  matrices. In what follows, let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

2.10.1. Let  $D$  denote the  $m \times m$  matrix on the left in the product.

$$(2-7) \quad \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{m-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \vdots \\ \lambda_m x_m \end{pmatrix}$$

Since  $DB = [D\underline{B}_1, \dots, D\underline{B}_n]$  it follows from (2-7) that the  $i^{\text{th}}$  row of  $DB$  is  $\lambda_i$  times the  $i^{\text{th}}$  row of  $B$ .

A special case of this appears when we discuss elementary row operations in section 5. There we take  $\lambda_i = c$  and all other  $\lambda_j$ s equal to 1. In that case  $DB$  is the same as  $B$  except that the  $i^{\text{th}}$  row of  $DB$  is  $c$  times the  $i^{\text{th}}$  row of  $B$ .

2.10.2. Fix two different integers  $i$  and  $j$  and let  $E$  be the  $m \times m$  matrix obtained by interchanging the  $i^{\text{th}}$  and  $j^{\text{th}}$  rows of the identity matrix. If  $X$  is an  $m \times 1$  matrix, then  $EX$  is the same as  $X$  except that the entries in the  $i^{\text{th}}$  and  $j^{\text{th}}$  positions of  $X$  are switched.

Thus, if  $B$  is an  $m \times n$  matrix, then  $EB$  is the same as  $B$  except that the  $i^{\text{th}}$  and  $j^{\text{th}}$  rows are interchanged. (Check that.) This operation, switching two rows, will also appear in section 5 when we discuss elementary row operations.

2.10.3. Fix two different integers  $i$  and  $j$  and let  $F$  be the  $m \times m$  matrix obtained from the identity matrix by changing the  $ij^{\text{th}}$  entry from zero to  $c$ . If  $X$  is an  $m \times 1$  matrix, then  $FX$  is the same as  $X$  except that the entry  $x_i$  has been replaced by  $x_i + cx_j$ . Thus, if  $B$  is an  $m \times n$  matrix, then  $FB$  is the same as  $B$  except that the  $i^{\text{th}}$  row is now the sum of the  $i^{\text{th}}$  row and  $c$  times the  $j^{\text{th}}$  row. (Check that.) This operation too appears when we discuss elementary row operations.

2.10.4. The matrices  $D$ ,  $E$ , and  $F$  all have inverses, i.e., there are matrices  $D'$ ,  $E'$ , and  $F'$ , such that  $I = DD' = D'D = EE' = E'E = FF' = F'F$ , where  $I$  denotes the  $m \times m$  identity matrix. (Oh, I need to be careful – here I mean  $D$  is the matrix in (2-7) with  $\lambda_i = c$  and all other  $\lambda_j$ s equal to 1.)

2.10.5. *Easy question.* Is there a positive integer  $n$  such that

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}?$$

If so what is the smallest such  $n$ ? Can you describe all other  $n$  for which this equality holds?

### 3. MATRICES AND MOTIONS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

3.1. **Linear transformations.** The geometric aspect of linear algebra involves lines, planes, and their higher dimensional analogues. It involves lines in the plane, lines in 3-space, lines in 4-space, planes in 3-space, planes in 4-space, and so on, ad infinitum. We might call the study of such things linear geometry.

Curvy things play no role in linear algebra and geometry. There are no circles, spheres, ellipses, parabolas, etc. All is linear.

Functions play a central role in all branches of mathematics. The functions relevant to linear algebra are *linear functions* or *linear transformations*.

We write  $\mathbb{R}^n$  for the set of  $n \times 1$  column vectors. This has a linear structure:

- (1) if  $\underline{u}$  and  $\underline{v}$  are in  $\mathbb{R}^n$  so is  $\underline{u} + \underline{v}$ ;
- (2) if  $c \in \mathbb{R}$  and  $\underline{u} \in \mathbb{R}^n$ , then  $c\underline{u} \in \mathbb{R}^n$ .

A non-empty subset  $W$  of  $\mathbb{R}^n$  is called a subspace if it has these two properties: i.e.,

- (1) if  $\underline{u}$  and  $\underline{v}$  are in  $W$  so is  $\underline{u} + \underline{v}$ ;
- (2) if  $c \in \mathbb{R}$  and  $\underline{u} \in W$ , then  $c\underline{u} \in W$ .

A linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

- (1)  $f(\underline{u} + \underline{v}) = f(\underline{u}) + f(\underline{v})$  for all  $\underline{u}, \underline{v} \in \mathbb{R}^n$ , and
- (2)  $f(c\underline{u}) = cf(\underline{u})$  for all  $c \in \mathbb{R}$  and all  $\underline{u} \in \mathbb{R}^n$ .

In colloquial terms, these two requirements say that linear transformations preserve the linear structure.

Left multiplication by an  $m \times n$  matrix  $A$  is a linear function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . If  $\underline{x} \in \mathbb{R}^n$ , then  $A\underline{x} \in \mathbb{R}^m$ .

**3.2. Rotations in the plane.****3.3. Projections in the plane.****3.4. Contraction and dilation.****3.5. Reflections in the plane.****3.6. Reflections in  $\mathbb{R}^3$ .****3.7. Projections from  $\mathbb{R}^3$  to a plane.****4. SYSTEMS OF LINEAR EQUATIONS**

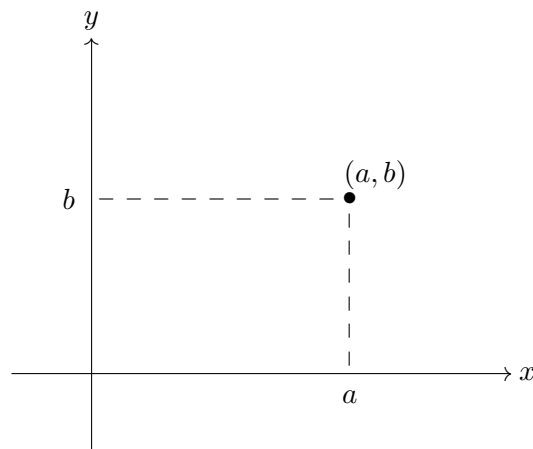
The subject matter of this section begins in section 4.2 below. However, some of the terminology and framework for systems of linear equations applies to other systems of equations so we begin in section 4.1 with a discussion of the general framework. Nevertheless, it is probably best to go directly to section 4.2, read through to the end of the section then return to section 4.1.

**4.1. General systems of equations.** You have been solving equations for many years. Sometimes you have considered the solutions to a single equation. For example, you are familiar with the fact that the solutions to the equation  $y = x^2 - 4x + 1$  form a parabola. Likewise, the solutions to the equation  $(x - 2)^2 + (y - 1)^2 + z^2 = 1$  form a sphere of radius 1 centered at the point  $(2, 1, 0)$ .

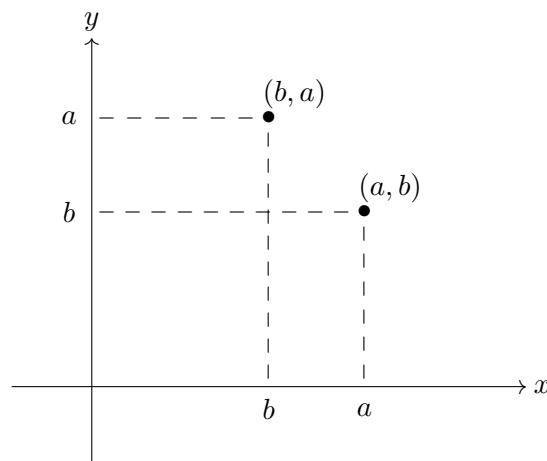
4.1.1. Let's consider these statements carefully. By a solution to the equation  $y = x^2 - x - 4$  we mean a pair of numbers  $(a, b)$  that give a true statement when they are plugged into the equation, i.e.,  $(a, b)$  is called a solution if  $b$  *does* equal  $a^2 - 4a + 1$ .

4.1.2. Pairs of numbers  $(a, b)$  also label points in the plane. More precisely, we can draw a pair of axes perpendicular to each other, usually called the  $x$ - and  $y$ -axes, and  $(a, b)$  denotes the point obtained by going  $a$  units along the  $x$ -axis and  $b$ -units along the  $y$ -axis. The picture below illustrates the

situation:



When we label a point by a pair of numbers the *order* of the numbers matters. For example, if  $a \neq b$ , the points  $(a, b)$  and  $(b, a)$  are different. In the above example, we have



The order also matters when we say  $(3, 2)$  is a solution to the equation  $y = x^2 - x - 4$ . Although  $(3, 2)$  is a solution to  $y = x^2 - x - 4$ ,  $(2, 3)$  is not.

4.1.3.  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . We write  $\mathbb{R}^2$  for the set of all ordered pairs of numbers  $(a, b)$ . Here  $\mathbb{R}$  denotes the set of real numbers and the superscript 2 in  $\mathbb{R}^2$  indicates *pairs* of real numbers. The general principle is that a solution to an equation in 2 unknowns is an ordered pair of real numbers, i.e., a point in  $\mathbb{R}^2$ .

A solution to an equation in 3 unknowns,  $(x - 2)^2 + (y - 1)^2 + z^2 = 1$  for example, is an ordered triple of numbers, i.e., a triple  $(a, b, c)$  such that  $(a - 2)^2 + (b - 1)^2 + c^2$  *does* equal 1. The set of all ordered triples of numbers is denoted by  $\mathbb{R}^3$ .

4.1.4. *A system of equations.* We can consider more than one equation at a time. When we consider several equations at once we speak of a *system* of equations. For example, we can ask for solutions to the system of equations

$$(4-1) \quad \begin{cases} y = x^2 - x - 4 \\ y = 3x - 7. \end{cases}$$

A solution to this system of equations is an ordered pair of numbers  $(a, b)$  with the property that when we plug in  $a$  for  $x$  and  $b$  for  $y$  *both equations are true*. That is,  $(a, b)$  is a solution to the system (4-1) if  $b = a^2 - a - 4$  and  $b = 3a - 7$ . For example,  $(3, 2)$  is a solution to the system (4-1). Similarly,  $(1, -4)$  is a solution to the system (4-1).

On the other hand,  $(4, 8)$  is *not* a solution to the system (4-1): it is a solution to  $y = x^2 - x - 4$  but not a solution to  $y = 3x - 7$  because  $8 \neq 3 \times 4 - 7$ .

4.1.5. *More unknowns and  $\mathbb{R}^n$ .* The demands of our modern complex world are such that we often encounter equations with many unknowns, sometimes millions of unknowns. Let's think about a modest situation, a system of 3 equations in 5 unknowns. For example, a solution to the system of equations

$$(4-2) \quad \begin{cases} x_1 + x_2^2 + x_3^3 + x_4^4 + x_5^5 = 100 \\ x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0 \\ x_1x_2 + x_3x_4 - x_5^2 = -20 \end{cases}$$

is an ordered 5-tuple<sup>3</sup> of numbers,  $(s_1, s_2, s_3, s_4, s_5)$ , having the property that when we plug in  $s_1$  for  $x_1$ ,  $s_2$  for  $x_2$ ,  $s_3$  for  $x_3$ ,  $s_4$  for  $x_4$ , and  $s_5$  for  $x_5$ , all three of the equations in (4-2) become true. We use the notation  $\mathbb{R}^5$  to denote the set of all ordered 5-tuples of real numbers. Thus, the solutions to the system (4-2) are elements of  $\mathbb{R}^5$ .

We often call elements of  $\mathbb{R}^5$  *points* just as we call points in the plane *points*. A point in  $\mathbb{R}^5$  has 5 coordinates, whereas a point in the plane has 2 coordinates.

More generally, solutions to a system of equations with  $n$  unknowns are, or can be thought of as, points in  $\mathbb{R}^n$ . We sometimes call  $\mathbb{R}^n$  *n-space*. For example, we refer to the physical world around us as 3-space. If we fix an origin and  $x$ -,  $y$ -, and  $z$ -axes, each point in our physical world can be labelled in a unique way by an ordered triple of numbers. If we take time as well as position into account we need 4-coordinates and then speak of  $\mathbb{R}^4$ . Physicists like to call  $\mathbb{R}^4$  *space-time*. It has 3 space-like coordinates and 1 time coordinate.

4.1.6. *Non-linear equations.* The equations  $y = x^2 - x - 4$  and  $(x - 2)^2 + (y - 1)^2 + z^2 = 1$  are *not* linear equations.

<sup>3</sup>We run out of words after *pair*, *triple*, *quadruple*, *quintuple*, *sextuple*, ... so invent the word *n-tuple* to refer to an ordered sequence of  $n$  numbers where  $n$  can be any positive integer.

**4.2. A single linear equation.** A linear equation is an equation of the form

$$(4-3) \quad a_1x_1 + \cdots + a_nx_n = b$$

in which the  $a_i$ s and  $b$  belong to  $\mathbb{R}$  and  $x_1, \dots, x_n$  are unknowns. We call the  $a_i$ s the coefficients of the unknowns.

A solution to (4-3) is an ordered  $n$ -tuple  $(s_1, \dots, s_n)$  of real numbers that when substituted for the  $x_i$ s makes equation (4-3) true, i.e.,

$$a_1s_1 + \cdots + a_ns_n = b.$$

When  $n = 2$  a solution is a pair of numbers  $(s_1, s_2)$  which we can think of as the coordinates of a point in the plane. When  $n = 3$  a solution is a triple of numbers  $(s_1, s_2, s_3)$  which we can think of as the coordinates of a point in 3-space. We will use the symbol  $\mathbb{R}^2$  to denote the plane and  $\mathbb{R}^3$  to denote 3-space. The idea for the notation is that the  $\mathbb{R}$  in  $\mathbb{R}^3$  denotes the set of real numbers and the 3 means a triple of real numbers. The notation continues:  $\mathbb{R}^4$  denotes quadruples  $(a, b, c, d)$  of real numbers and  $\mathbb{R}^4$  is referred to as 4-space.<sup>4</sup>

**A geometric view.** If  $n \geq 2$  and at least one  $a_i$  is non-zero, equation (4-3) has infinitely many solutions: for example, when  $n = 2$  the solutions are the points on a line in the plane  $\mathbb{R}^2$ ; when  $n = 3$  the solutions are the points on a plane in 3-space  $\mathbb{R}^3$ ; when  $n = 4$  the solutions are the points on a 3-plane in  $\mathbb{R}^4$ ; and so on.

It will be important in this course to have a geometric picture of the set of solutions. We begin to do this in earnest in chapter 7 but it is useful to keep this in mind from the outset. Solutions to equation (4-3) are ordered  $n$ -tuples of real numbers  $(s_1, \dots, s_n)$  and we think of the numbers  $s_i$  as the coordinates of a point in  $n$ -space, i.e., in  $\mathbb{R}^n$ .

The collection of *all* solutions to  $a_1x_1 + \cdots + a_nx_n = b$  is a special kind of subset in  $n$ -space: it has a *linear structure* and that is why we call (4-3) a linear equation. By a “linear structure” we mean this:

**Proposition 4.1.** *If the points  $p = (s_1, \dots, s_n)$  and  $q = (t_1, \dots, t_n)$  are different solutions to (4-3), then all points on the line through  $p$  and  $q$  are also solutions to (4-3).*

This result is easy to see once we figure out how to describe the points that lie on the line through  $p$  and  $q$ .

Let’s write  $\overline{pq}$  for the line through  $p$  and  $q$ . The prototypical line is the real number line  $\mathbb{R}$  so we want to associate to each real number  $\lambda$  a point on  $\overline{pq}$ . We do that by presenting  $\overline{pq}$  in parametric form:  $\overline{pq}$  consists of all points of the form

$$\lambda p + (1 - \lambda)q = (\lambda s_1 + (1 - \lambda)t_1, \dots, \lambda s_n + (1 - \lambda)t_n)$$

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<sup>4</sup>Physicists think of  $\mathbb{R}^4$  as space-time with coordinates  $(x, y, z, t)$ , 3 spatial coordinates, and one time coordinate.

as  $\lambda$  ranges over all real numbers. Notice that  $p$  is obtained when  $\lambda = 1$  and  $q$  is obtained when  $\lambda = 0$ .

**Proof of Proposition 4.1.** The result follows from the calculation

$$\begin{aligned} a_1(\lambda s_1 + (1 - \lambda)t_1) + \cdots + a_n(\lambda s_n + (1 - \lambda)t_n) = \\ \lambda(a_1 s_1 + \cdots + a_n s_n) + (1 - \lambda)(a_1 t_1 + \cdots + a_n t_n) \end{aligned}$$

which equals  $\lambda \underline{b} + (1 - \lambda)\underline{b}$ , i.e., equals  $\underline{b}$ , if  $(s_1, \dots, s_n)$  and  $(t_1, \dots, t_n)$  are solutions to (4-3).  $\diamond$

**4.3. Systems of linear equations.** An  $m \times n$  system of linear equations is a collection of  $m$  linear equations in  $n$  unknowns. We usually write out the general  $m \times n$  system like this:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The unknowns are  $x_1, \dots, x_n$ , and the aim of the game is to find them when the  $a_{ij}$ s and  $b_i$ s are specific real numbers. The  $a_{ij}$ s are called the coefficients of the system. We often arrange the coefficients into an  $m \times n$  array

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

called the coefficient matrix.

**4.4. A system of linear equations is a single matrix equation.** We can assemble the  $b_i$ s and the unknowns  $x_j$  into column vectors

$$\underline{b} := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad \underline{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The system of linear equations at the beginning of section 4.3 can now be written as a single matrix equation

$$A\underline{x} = \underline{b}.$$

You should multiply out this equation and satisfy yourself that it is the same system of equations as at the beginning of section 4.3. This matrix interpretation of the system of linear equations will be center stage throughout this course.

We often arrange all the data into a single  $m \times (n + 1)$  matrix

$$(A|\underline{b}) = \left( \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

that we call the augmented matrix of the system.

#### 4.5. Specific examples.

4.5.1. *A unique solution.* The only solution to the  $2 \times 2$  system

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &= 1 \end{aligned}$$

is  $(x_1, x_2) = (1, 1)$ . You can think of this in geometric terms. Each equation determines a line in  $\mathbb{R}^2$ , the points  $(x_1, x_2)$  on each line corresponding to the solutions of the corresponding equation. The points that lie on *both* lines therefore correspond to simultaneous solutions to the pair of equations, i.e., solutions to the  $2 \times 2$  system of equations.

4.5.2. *No solutions.* The  $2 \times 2$  system

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 6 \end{aligned}$$

has no solution at all: if the numbers  $x_1$  and  $x_2$  are such that  $x_1 + x_2 = 2$  then  $2(x_1 + x_2) = 4$ , *not* 6. It is enlightening to think about this system geometrically. The points in  $\mathbb{R}^2$  that are solutions to the equation  $x_1 + x_2 = 2$  lie on the line of slope  $-1$  passing through  $(0, 2)$  and  $(2, 0)$ . The points in  $\mathbb{R}^2$  that are solutions to the equation  $2(x_1 + x_2) = 6$  lie on the line of slope  $-1$  passing through  $(0, 3)$  and  $(3, 0)$ . Thus, the equations give two parallel lines: there is no point lying on both lines and therefore no common solution to the pair of equations, i.e., no solution to the given system of linear equations.

4.5.3. *No solutions.* The  $3 \times 2$  system

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &= 1 \\ x_1 - x_2 &= 3 \end{aligned}$$

has no solutions because the only solution to the  $2 \times 2$  system consisting of the first two equations is  $(1, 1)$  and that is *not* a solution to the third equation in the  $3 \times 2$  system. Geometrically, the solutions to each equation lie on a line in  $\mathbb{R}^2$  and the three lines do not pass through a common point.

4.5.4. *A unique solution.* The  $3 \times 2$  system

$$\begin{aligned}x_1 + x_2 &= 2 \\2x_1 - x_2 &= 1 \\x_1 - x_2 &= 0\end{aligned}$$

has a unique solution, namely  $(1, 1)$ . The three lines corresponding to the three equations all pass through the point  $(1, 1)$ .

4.5.5. *Infinitely many solutions.* It is obvious that the system consisting of the single equation

$$x_1 + x_2 = 2$$

has infinitely many solutions, namely all the points lying on the line of slope  $-1$  passing through  $(0, 2)$  and  $(2, 0)$ .

4.5.6. *Infinitely many solutions.* The  $2 \times 2$  system

$$\begin{aligned}x_1 + x_2 &= 2 \\2x_1 + 2x_2 &= 4\end{aligned}$$

has infinitely many solutions, namely all the points lying on the line of slope  $-1$  passing through  $(0, 2)$  and  $(2, 0)$  because the two equations actually give the *same* line in  $\mathbb{R}^2$ . A solution to the first equation is also a solution to the second equation in the system.

4.5.7. *Infinitely many solutions.* The  $2 \times 3$  system

$$\begin{aligned}x_1 + x_2 + x_3 &= 3 \\2x_1 + x_2 - x_3 &= 4\end{aligned}$$

also has infinitely many solutions. The solutions to the first equation are the points on the plane  $x_1 + x_2 + x_3 = 3$  in  $\mathbb{R}^3$ . The solutions to the second equation are the points on the plane  $2x_1 + x_2 - x_3 = 4$  in  $\mathbb{R}^3$ . The two planes meet one another in a line. That line can be described parametrically as the points

$$(1 + 2t, 2 - 3t, t)$$

as  $t$  ranges over all real numbers. You should check that when  $x_1 = 1 + 2t$ ,  $x_2 = 2 - 3t$ , and  $x_3 = t$ , both equations in the  $2 \times 3$  system are satisfied.

4.5.8. *A unique solution.* The  $3 \times 3$  system

$$\begin{aligned}x_1 + x_2 + x_3 &= 3 \\2x_1 + x_2 - x_3 &= 4 \\4x_1 + 3x_2 - 2x_3 &= 13\end{aligned}$$

has a unique solution, namely  $(x_1, x_2, x_3) = (-1, 5, -1)$ . The solutions to each equation are the points in  $\mathbb{R}^3$  that lie on the plane given by the equation. There is a unique point lying on all three planes, namely  $(-1, 5, -1)$ . This is typical behavior: two planes in  $\mathbb{R}^3$  meet in a line (unless they are parallel), and that line will (usually!) meet a third plane in a point. The next two

example show this doesn't always happen: in Example 4.5.9 the third plane is *parallel* to the line that is the intersection of the first two planes; in Example 4.5.10 the third plane *contains* the line that is the intersection of the first two planes.

4.5.9. *No solution.* We will now show that the  $3 \times 3$  system

$$\begin{aligned}x_1 + x_2 + x_3 &= 3 \\2x_1 + x_2 - x_3 &= 4 \\3x_1 + x_2 - 3x_3 &= 0\end{aligned}$$

has a no solution. In Example 4.5.7 we showed that all solutions to the system consisting of the first two equations in the  $3 \times 3$  system we are now considering are of the form  $(1 + 2t, 2 - 3t, t)$  for some  $t$  in  $\mathbb{R}$ . However, if  $(x_1, x_2, x_3) = (1 + 2t, 2 - 3t, t)$ , then

$$(4-4) \quad 3x_1 + x_2 - 3x_3 = 3(1 + 2t) + (2 - 3t) - 3t = 5,$$

*not* 0, so no solution to the first two equations is a solution to the third equation. Thus the  $3 \times 3$  system has no solution.

4.5.10. *Infinitely many solutions.* The  $3 \times 3$  system

$$\begin{aligned}x_1 + x_2 + x_3 &= 3 \\2x_1 + x_2 - x_3 &= 4 \\3x_1 + x_2 - 3x_3 &= 5\end{aligned}$$

has infinitely many solutions because, as equation (4-4) showed, every solution to the system consisting of the first two equations in the  $3 \times 3$  system we are currently considering is a solution to the equation third equation  $3x_1 + x_2 - 3x_3 = 5$ .

**4.6. The number of solutions.** The examples show that a system of linear equations can have no solution, or a unique solution, or infinitely many solutions. The next proposition shows these are the only possibilities.

**Proposition 4.2.** *If a system of linear equations has  $\geq 2$  solutions, it has infinitely many.*

**Proof.** Suppose  $(s_1, \dots, s_n)$  and  $(t_1, \dots, t_n)$  are different solutions to the  $m \times n$  system

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i, \quad (1 \leq i \leq m).$$

Then

$$(4-5) \quad (\lambda s_1 + (1 - \lambda)t_1, \dots, \lambda s_n + (1 - \lambda)t_n)$$

is also a solution for every  $\lambda \in \mathbb{R}$  because

$$\begin{aligned} a_{i1}(\lambda s_1 + (1 - \lambda)t_1) + \cdots + a_{in}(\lambda s_n + (1 - \lambda)t_n) &= \\ \lambda(a_{i1}s_1 + \cdots + a_{in}s_n) + (1 - \lambda)(a_{i1}t_1 + \cdots + a_{in}t_n) &= \\ = \lambda b_i + (1 - \lambda)b_i &= \\ = b_i. \end{aligned}$$

Since there are infinitely many  $\lambda$ s in  $\mathbb{R}$  there are infinitely many solutions to the system.  $\square$

If we think of the solutions to an  $m \times n$  system as points in  $\mathbb{R}^n$ , the points in (4-5) are the points on the line through the points  $(s_1, \dots, s_n)$  and  $(t_1, \dots, t_n)$ . Thus, the proof of Proposition 4.2 says that if two points in  $\mathbb{R}^n$  are solutions to  $A\mathbf{x} = \mathbf{b}$ , so are all points on the line through them. It is this linear nature of the set of solutions that gives rise to the name *linear* equation. It is also why we call this subject linear algebra. I think *Linear algebra and linear geometry* would be a better name.

The following result is an immediate consequence of Proposition 4.2.

**Corollary 4.3.** *A system of linear equations has either*

- (1) *a unique solution or*
- (2) *no solutions or*
- (3) *infinitely many solutions.*

A system of linear equations is consistent if it has a solution, and inconsistent if it does not.

**4.7. A geometric view on the number of solutions.** Two lines in the plane intersect at either

- (1) a unique point or
- (2) no point (they are parallel) or
- (3) infinitely many points (they are the same line).

Lines in the plane are “the same things” as equations  $a_1x + a_1'y = b_1$ . The points on the line are the solutions to the equation. The solutions to a pair (system) of equations  $a_1x + a_1'y = b_1$  and  $a_2x + a_2'y = b_2$  are the points that lie on both lines, i.e., their intersection.

In 3-space, which we will denote by  $\mathbb{R}^3$ , the solutions to a single equation form a plane. Two planes in  $\mathbb{R}^3$  intersect at either

- (1) no point (they are parallel) or
- (2) infinitely many points (they are not parallel).

In the second case the intersection is a line if the planes are different, and a plane if they are the same.

Three planes in  $\mathbb{R}^3$  intersect at either

- (1) a unique point or
- (2) no point (all three are parallel) or

- (3) infinitely many points (all three planes are the same or the intersection of two of them is a line that lies on the third plane).

I leave you to consider the possibilities for three lines in the plane and four planes in  $\mathbb{R}^3$ . Discuss with friends if necessary.

**4.8. Homogeneous systems.** A system of linear equations of the form  $A\mathbf{x} = 0$  is called a homogeneous system. A homogeneous system always has a solution, namely  $\mathbf{x} = 0$ , i.e.,  $x_1 = \cdots = x_n = 0$ . This is called the trivial solution because little brainpower is needed to find it. Other solutions to  $A\mathbf{x} = 0$  are said to be non-trivial.

For a homogeneous system the issue is to describe the set of *non-trivial* solutions.

4.8.1. *A trivial but important remark.* If  $(s_1, \dots, s_n)$  is a solution to a homogeneous system so is  $(\lambda s_1, \dots, \lambda s_n)$  for all  $\lambda \in \mathbb{R}$ , i.e., all points on the line through the origin  $0 = (0, \dots, 0)$  and  $(s_1, \dots, s_n)$  are solutions.

4.8.2. *A geometric view.* The next result, Proposition 4.4, shows that solutions to the system of equations  $A\mathbf{x} = \mathbf{b}$  are closely related to solutions to the homogeneous system of equations  $A\mathbf{x} = 0$ .<sup>5</sup> We can see this relationship in the setting of a single equation in 2 unknowns.

The solutions to the equation  $2x + 3y = 0$  are points on a line in the plane  $\mathbb{R}^2$ . Draw that line: it is the line through the origin with slope  $-\frac{2}{3}$ . The origin is the trivial solution. Other points on the line such as  $(3, -2)$  are non-trivial solutions. The solutions to the non-homogeneous equation  $2x + 3y = 1$  are also points on a line in  $\mathbb{R}^2$ . Draw it. This line has slope  $-\frac{2}{3}$  too. It is parallel to the line  $2x + 3y = 0$  but doesn't go through the origin.

Pick any point on the second line, i.e., any solution to  $2x + 3y = 1$ . Let's pick  $(-1, 1)$ . Now take a solution to the first equation, say  $(3, -2)$  and add it to  $(-1, 1)$  to obtain  $(2, -1)$ . The point  $(2, -1)$  is on the line  $2x + 3y = 1$ . In fact, if we take any solution to  $2x + 3y = 0$ , say  $(a, b)$ , and add it to  $(-1, 1)$  we obtain a solution to  $2x + 3y = 1$  because

$$(a, b) + (-1, 1) = (a - 1, b + 1)$$

and

$$2(a - 1) + 3(b + 1) = 2a + 3b - 2 + 3 = 0 + 1 = 1.$$

*Every solution to  $2x + 3y = 1$  can be obtained this way.* That is what Proposition 4.4 says, though in much greater generality.

There was nothing special about our choice of  $(-1, 1)$ . Everything would work in the same way if we had chosen  $(5, -3)$  for example. Check it out!

The line  $2x + 3y = 1$  is obtained by translating (moving) the line  $2x + 3y = 0$  to the parallel line through  $(-1, 1)$ . Alternatively, the line  $2x + 3y = 1$  is obtained by translating the line  $2x + 3y = 0$  to the parallel line through

---

<sup>5</sup>Students seem to have a hard time understanding Proposition 4.4. Section 4.8.3 discusses an analogy to Proposition 4.4 that will be familiar to you from your calculus courses.

(5, -3). The picture in section 8.3 might increase your understanding of this important point (though a different equation is used there). The discussion in section 8.3 might also help.

If I had taken  $2x + 3y = -3$  instead of  $2x + 3y = 1$  the same sort of thing happens. The solutions to  $2x + 3y = -3$  lie on a line parallel to  $2x + 3y = 0$ . In fact, every line parallel to  $2x + 3y = 0$  is the set of solutions to  $2x + 3y = c$  for a suitable  $c$ .

**Proposition 4.4.** *Suppose  $\underline{u}$  is a solution to the equation  $A\underline{x} = \underline{b}$ . Then the set of all solutions to  $A\underline{x} = \underline{b}$  consists of all vectors  $\underline{u} + \underline{v}$  in which  $\underline{v}$  ranges over all solutions to the homogeneous equation  $A\underline{x} = \underline{0}$ .*

**Proof.** Fix a solution  $\underline{u}$  to the equation  $A\underline{x} = \underline{b}$ ; i.e.,  $A\underline{u} = \underline{b}$ . If  $\underline{u}'$  is a solution to  $A\underline{x} = \underline{b}$ , then  $\underline{u}' - \underline{u}$  is a solution to the homogeneous equation  $A\underline{x} = \underline{0}$ , and  $\underline{u}' = \underline{u} + (\underline{u}' - \underline{u})$ .

Conversely, if  $A\underline{v} = \underline{0}$ , i.e., if  $\underline{v}$  is a solution to the homogeneous equation  $A\underline{x} = \underline{0}$ , then  $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{b} + \underline{0} = \underline{b}$ , so  $\underline{u} + \underline{v}$  is a solution to  $A\underline{x} = \underline{b}$ .  $\square$

As we shall later see, the set of solutions to a system of homogeneous linear equations is a subspace. Subspaces are fundamental objects in linear algebra. You should compare Proposition 4.4 with the statement of Proposition 8.9. The two results say the same thing in slightly different words.

**Corollary 4.5.** *Suppose the equation  $A\underline{x} = \underline{b}$  has a solution. Then  $A\underline{x} = \underline{b}$  has a unique solution if and only if  $A\underline{x} = \underline{0}$  has a unique solution.*

**Proof.** Let  $\underline{u}$  be a solution to the equation  $A\underline{x} = \underline{b}$ ; i.e.,  $A\underline{u} = \underline{b}$ . Let  $\underline{v} \in \mathbb{R}^n$ . Then  $A(\underline{u} + \underline{v}) = \underline{b} + A\underline{v}$  so  $\underline{u} + \underline{v}$  is a solution to the equation  $A\underline{x} = \underline{b}$  if and only if  $\underline{v}$  is a solution to the equation  $A\underline{x} = \underline{0}$ .  $\square$

You might find it easier to understand what Proposition 4.4 is telling you by looking at it in conjunction with section 5.7.1.

4.8.3. *An analogy for understanding Proposition prop.homog.non-homog.* You probably know that every solution to the differential equation  $\frac{dy}{dx} = 2x$  is of the form  $x^2 + c$  where  $c$  is an arbitrary constant. One particular solution to the equation  $\frac{dy}{dx} = 2x$  is  $x^2$ . Now consider the equation  $\frac{dy}{dx} = 0$ ; the solutions to this are the constant functions  $c$ ,  $c \in \mathbb{R}$ . Thus every solution to  $\frac{dy}{dx} = 2x$  is obtained by adding the particular solution  $x^2$  to a solution to  $\frac{dy}{dx} = 0$ .

Compare the previous paragraph with the statement of Proposition 4.4:  $2x$  is playing the role of  $\underline{b}$ ,  $x^2$  is playing the role of  $\underline{u}$ , and  $c$  is playing the role of  $\underline{v}$ .

In Proposition 4.4 it doesn't matter which  $\underline{u}$  we choose; all solutions to  $A\underline{x} = \underline{b}$  are given by  $\underline{u} + \underline{v}$  where  $\underline{v}$  runs over all solutions to  $A\underline{x} = \underline{0}$ .

Similarly, we can take any solution to  $\frac{dy}{dx} = 2x$ ,  $x^2 + 17\pi$  for example, and still all solutions to  $\frac{dy}{dx} = 2x$  are obtained by adding  $x^2 + 17\pi$  to a solution

to  $\frac{dy}{dx} = 0$ . That is, every solution to  $\frac{dy}{dx} = 2x$  is of the form  $x^2 + 17\pi + c$  for some  $c \in \mathbb{R}$ .

4.8.4. *A natural bijection between solutions to  $A\underline{x} = 0$  and solutions to  $A\underline{x} = \underline{b}$ .* A **bijection** between two sets  $X$  and  $Y$  is a rule that sets up a one-to-one correspondence between the elements on  $X$  and the elements of  $Y$ . We sometimes write  $x \leftrightarrow y$  to denote the fact that an element  $x$  in  $X$  corresponds to the element  $y$  in  $Y$ . For example, there is a bijection between the set of all integers and the set of even integers given by  $n \leftrightarrow 2n$ . There is also a bijection between the even numbers and the odd numbers given by  $2m \leftrightarrow 2m + 1$ .

Suppose the equation  $A\underline{x} = \underline{b}$  has a solution, say  $\underline{u}$ . Then Proposition 4.4 says there is a bijection between the set of solutions to  $A\underline{x} = \underline{b}$  and the set of solutions to  $A\underline{x} = 0$  given by

$$\underline{v} \longleftrightarrow \underline{u} + \underline{v}.$$

In other words,  $\underline{v}$  is a solution to  $A\underline{x} = 0$  if and only if  $\underline{u} + \underline{v}$  is a solution to  $A\underline{x} = \underline{b}$ . In plainer language, if  $A\underline{v} = 0$ , then  $A(\underline{u} + \underline{v}) = \underline{b}$  and conversely.

This is an arithmetic explanation, and generalization, of what was said in the discussion prior to Proposition 4.4. There

$$(a, b) \longleftrightarrow (a, b) + (-1, 1).$$

As I said before there is nothing special about our choice of  $(-1, 1)$ . We could take any point on  $2x + 3y = 1$ . For example, if we pick  $(5, -3)$  we obtain another bijection between the solutions to  $2x + 3y = 0$  and the solutions to  $2x + 3y = 1$ , namely

$$(a, b) \longleftrightarrow (a, b) + (5, -3).$$

## 5. ROW OPERATIONS AND ROW EQUIVALENCE

**5.1. Equivalent systems of equations.** Two systems of linear equations are **equivalent** if they have the same solutions. An important strategy for solving a system of linear equations is to repeatedly perform operations on the system to get simpler but equivalent systems, eventually arriving at an equivalent system that is trivial to solve. The simplest systems to solve are those in row-reduced echelon form. We discuss those in the next section. In this section we discuss the three “row operations” you can perform on a system to obtain a simpler but equivalent system.

We will write  $R_i$  to denote the  $i^{\text{th}}$  row in a matrix.

The three **elementary row operations** on a matrix are the following:

- (1) switch the positions of  $R_i$  and  $R_j$ ;
- (2) replace  $R_i$  by a non-zero multiple of  $R_i$ ;
- (3) replace  $R_i$  by  $R_i + cR_j$  for any number  $c$ .

If  $A'$  can be obtained from  $A$  by a single row operation, then  $A$  can be obtained from  $A'$  by a single row operation. We say that two matrices

are equivalent or row equivalent if one can be obtained from the other by a *sequence* of elementary row operations.

**Proposition 5.1.** *Two  $m \times n$  systems of linear equations  $A\mathbf{x} = \mathbf{b}$  and  $A'\mathbf{x} = \mathbf{b}'$  are equivalent if and only if the augmented matrices  $(A|\mathbf{b})$  and  $(A'|\mathbf{b}')$  are row equivalent.*

You might take a look back at section 2.10 to better understand the elementary row operations.

5.1.1. *Exercise.* Find conditions on  $a, b, c, d$  such that the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

are row equivalent. (Use the fact that two matrices are row equivalent if they have the same sets of solutions.)

5.2. **Echelon Form.** A matrix  $E$  is in echelon form if

- (1) all rows consisting entirely of zeroes are at the bottom of  $E$ ; and
- (2) the left-most entry in each non-zero row is a 1; and
- (3) the left-most 1 in a non-zero row is to the right of the left-most 1 in the row above it.

A matrix  $E$  is in row-reduced echelon form (RREF) if it is in echelon form and each column that contains the left-most 1 of some row has all its other entries equal to zero.

For example,

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is in RREF.

**Theorem 5.2.** *Given a matrix  $A$ , there is a unique row-reduced echelon matrix that is equivalent to  $A$ .*

**Proof.** □

The uniqueness result in Theorem 5.2 allows us to speak of *the* row-reduced echelon matrix that is equivalent to  $A$ . We will denote it by

$$\mathbf{rref} A.$$

5.2.1. *Reminder.* It is important to know there are infinitely many sequences of elementary row operations that can get you from  $A$  to  $\mathbf{rref} A$ .

In an exam, if you are given a specific matrix  $A$  and asked to *put  $A$  in reduced echelon form* there is a unique matrix,  $\mathbf{rref} A$ , that you must produce, but the row operations that one student uses to get from  $A$  to  $\mathbf{rref} A$  will probably not be the same as the row operations used by another student

to get from  $A$  to  $\text{rref } A$ . There is *no best way* to proceed in doing this and therefore no guiding rule I can give you to do this. Experience and practice will teach you how to minimize the number and complexity of the operations and how to keep the intermediate matrices reasonably “nice”. I often don’t swap the order of the rows until the very end. I begin by trying to get zeroes in the columns, i.e., all zeroes in some columns, and only one non-zero entry in other columns, and I don’t care which order of columns I do this for.

**5.3. An example.** Doing elementary row operations by hand to put a matrix in row-reduced echelon form can be a pretty tedious task. It is an ideal task for a computer! But, let’s take a deep breath and plunge in. Suppose the system is

MORE TO DO

**5.4. You already did this in high school.** You probably had to solve systems of equations like

$$\begin{aligned}x_1 + x_2 + x_3 &= 3 \\2x_1 + x_2 - x_3 &= 4 \\4x_1 + 3x_2 - 2x_3 &= 13\end{aligned}$$

in high school. (This is the system in section 4.5.8.) I want to persuade you that the approach you used then is pretty much identical to the method we are using here, i.e., putting the matrix  $(A | \underline{b})$  in row-reduced echelon form.

A typical high school approach to the problem is to add the first two equations (to eliminate the  $x_3$  variable) and to add twice the first equation to the third equation to get another equation without an  $x_3$  term. The two new equations are

$$\begin{aligned}3x_1 + 2x_2 &= 7 \\6x_1 + 5x_2 &= 19\end{aligned}$$

Now subtract twice the first of these two equations from the second to get  $x_2 = 5$ . Substitute this into  $3x_1 + 2x_2 = 7$  and get  $x_1 = -1$ . Finally, from the first of the original equations,  $x_1 + x_2 + x_3 = 3$ , we obtain  $x_3 = -1$ . Thus, the system has a unique solution,  $(x_1, x_2, x_3) = (-1, 5, -1)$ .

Let’s rewrite what we did in terms of row operations. The first two operations we performed are, in matrix notation,

$$(A | \underline{b}) = \left( \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & -1 & 4 \\ 4 & 1 & 1 & 13 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 2 & 0 & 7 \\ 6 & 5 & 0 & 19 \end{array} \right).$$

Then we subtracted twice the second row from the third to get

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 0 & 5 \end{array} \right).$$

The last row tells us that  $x_2 = 5$ . Above, we substituted  $x_2 = 5$  into  $3x_1 + 2x_2 = 7$  and solved for  $x_1$ . That is the same as subtracting twice the third row from the second, then dividing by three, i.e.,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 0 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \end{array}\right)$$

which gives  $x_1 = -1$ . Now subtract the second and third rows from the top row to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \end{array}\right)$$

which gives  $x_3 = -1$ . Admittedly the last matrix is not in row-reduced echelon form because the left-most ones do not proceed downwards in the southeast direction, but that is fixed by one more row operation, moving the top row to the bottom, i.e.,

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

**5.5. The rank of a matrix.** The rank of an  $m \times n$  matrix  $A$  is the number

$$\text{rank}(A) := \text{the number of non-zero rows in } \underline{\text{rref}} A.$$

5.5.1. *A very important remark about rank.* Later on we will see that the rank of  $A$  is equal to the dimension of the range of  $A$ . After proving that equality it will be important to keep in mind these two different interpretations of rank  $A$ , i.e., as the number of non-zero rows in rref  $A$  and as the dimension of the range of  $A$ .

5.5.2. *Remark.* If rref  $A = \underline{\text{rref}} B$ , then  $\text{rank } A = \text{rank } B$ .

**Proposition 5.3.** *If  $A$  is an  $m \times n$  matrix, then*

$$\text{rank } A \leq m$$

and

$$\text{rank } A \leq n.$$

**Proof.** The rank of  $A$  is at most the number of rows in rref  $A$ . But rref  $A$  and  $A$  have the same size so  $\text{rank } A \leq m$ .

Each non-zero row of rref  $A$  has a left-most 1 in it so the rank of  $A$  is the number of left-most 1s. But a column of rref  $A$  contains at most one left-most 1, so the rank of  $A$  is at most the number of columns in  $A$ , i.e.,  $\text{rank } A \leq n$ .  $\square$

**Proposition 5.4.** *Let  $A$  be an  $n \times n$  matrix. Then*

- (1)  $\text{rank } A = n$  if and only if rref  $A = I$ , the identity matrix;
- (2) if  $\text{rank } A < n$ , then rref  $A$  has a row of zeroes.

**Proof.** (1) If  $\text{rank } A = n$ , then  $\text{rref } A$  has  $n$  non-zero rows and each of those rows has a left-most 1, i.e., every row of  $\text{rref } A$  contains a left-most 1. Suppose the left-most 1 in row  $i$  appears in column  $c_i$ . The 1s move at least one column to the right each time we go from each row to the one below it so

$$1 \leq c_1 < c_2 < \cdots < c_n \leq n.$$

It follows that  $c_1 = 1, c_2 = 2, \dots, c_n = n$ . In other words, the left-most 1 in row  $j$  appears in column  $j$ ; i.e., all the left-most 1s lie on the diagonal. Therefore  $\text{rref } A = I$ .

(2) This is clear. □

**5.6. Inconsistent systems.** A system of linear equations is inconsistent if it has no solution (section 4.6). If a row of the form

$$(5-1) \quad (0 \ 0 \ \cdots \ 0 \ | \ b)$$

with  $b \neq 0$  appears after performing some row operations on the augmented matrix  $(A \ | \ \underline{b})$ , then the system is inconsistent because row (5-1) says the equation  $0x_1 + \cdots + 0x_n = b$  appears in a system that is equivalent to the original system of equations, and it is utterly clear that there is no solution to the equation  $0x_1 + \cdots + 0x_n = b$  when  $b \neq 0$ , and therefore no solution to the original system of equations.

**5.7. Consistent systems.** Suppose  $A\underline{x} = \underline{b}$  is a consistent  $m \times n$  system of linear equations .

- Form the augmented matrix  $(A \ | \ \underline{b})$ .
- Perform elementary row operations on  $(A \ | \ \underline{b})$  until you obtain  $(\text{rref } A \ | \ \underline{d})$ .
- If column  $j$  contains the left-most 1 that appears in some row of  $\text{rref } A$  we call  $x_j$  a **dependent variable**.
- The other  $x_j$ s are called **independent variables**.
- The number of dependent variables equals  $\text{rank } A$  and, because  $A$  has  $n$  columns, the number of independent variables is  $n - \text{rank } A$ .
- All solutions are now obtained by allowing each independent variable to take any value and the dependent variables are now completely determined by the equations corresponding to  $(\text{rref } A \ | \ \underline{d})$  and the values of the independent variables.

An example will clarify my meaning. If

$$(5-2) \quad (\text{rref } A \ | \ \underline{d}) = \left( \begin{array}{cccccc|c} 1 & 0 & 2 & 0 & 0 & 3 & 4 & 1 \\ 0 & 1 & 3 & 0 & 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

the dependent variables are  $x_1, x_2, x_4$ , and  $x_5$ . The independent variables  $x_3, x_6$ , and  $x_7$ , may take any values and then the dependent variables are

determined by the equations

$$\begin{aligned}x_1 &= 1 - 2x_3 - 3x_6 - 4x_7 \\x_2 &= 3 - 3x_3 - 4x_6 \\x_3 &= x_3 \\x_4 &= 2x_6 + x_7 \\x_5 &= 5 - x_6 + 2x_7 \\x_6 &= x_6 \\x_7 &= x_7\end{aligned}$$

In other words, the set of all solutions to this system is given by

$$\underline{x} = \begin{pmatrix} 1 - 2x_3 - 3x_6 - 4x_7 \\ 3 - 3x_3 - 4x_6 \\ x_3 \\ 2x_6 + x_7 \\ 5 - x_6 + 2x_7 \\ x_6 \\ x_7 \end{pmatrix}$$

or, equivalently,

$$(5-3) \quad \underline{x} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -3 \\ -4 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

as  $x_3, x_6, x_7$  range over all real numbers. We call (5-3) the general solution to the system.<sup>6</sup>

5.7.1. *Particular vs. general solutions.* If we make a particular choice of  $x_3, x_6,$  and  $x_7,$  we call the resulting  $\underline{x}$  a particular solution to the system. For

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<sup>6</sup>You will later see that the vectors obtained from the three right-most terms of (5-3) as  $x_3, x_6,$  and  $x_7,$  vary over all of  $\mathbb{R}$  form a subspace, namely the null space of  $A$ . Thus (5-3) is telling us that the solutions to  $A\underline{x} = \underline{b}$  belong to a translate of the null space of  $A$  by a particular solution of the equation, the particular solution being the term immediately to the right of the equal sign in (5-3). See section 8.3 and Proposition 8.9 below for more about this.

example, when  $(x_3, x_6, x_7) = (0, 0, 0)$  we obtain the *particular* solution

$$\begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \\ 0 \\ 0 \end{pmatrix}.$$

Notice that

$$\begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ -4 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

are solutions to the homogeneous system  $A\underline{x} = 0$ . The general solution is obtained by taking one particular solution, say  $\underline{u}$ , perhaps that given by setting all the independent variables to zero, and adding to it solutions to the homogeneous equation  $A\underline{x} = 0$ . That is exactly what Proposition 4.4 below says.

The number of independent variables, in this case 3, which is  $n - \text{rank } A$ , gives a measure of “how many” solutions there are to the equation  $A\underline{x} = \underline{b}$ . The fact that  $x_3$ ,  $x_6$ , and  $x_7$ , can be any real numbers says there are 3 degrees of freedom for the solution space. Later we will introduce more formal language which will allow us to say that the space of solutions is a translate of a 3-dimensional subspace of  $\mathbb{R}^7$ . That 3-dimensional subspace is called the *null space* of  $A$ . You will understand this paragraph later. I hope you read it again after you have been introduced to the null space,

The discussion in section 8.3 is also relevant to what we have discussed in this section. In the language of section 8.3, the solutions to  $A\underline{x} = \underline{b}$  are obtained by translating the solutions to  $A\underline{x}$ , i.e., the null space of  $A$ , by a particular solution to  $A\underline{x} = \underline{b}$ .

Sections 4.8.2 and 4.8.4 are also related to the issue of particular versus general solutions. Section 4.8.2 discusses the particular solution  $(-1, 1)$  to the equation  $2x + 3y = 1$  and the general solution  $(-1, 1) + \lambda(3, -2)$  where  $\lambda$  runs over all real numbers. As  $\lambda$  runs over all real numbers the points  $\lambda(3, -2)$  give all the points on the line  $2x + 3y = 0$ .

**5.8. Parametric equations for lines and planes.** You should review the material about parametric equations for curves and surfaces that you encountered in a course prior to this. The only curves and surfaces relevant to this course are lines and planes. These have particularly simple parametric descriptions.

For example, the line  $2x - 3y = 1$  in the plane consists of all points of the form  $(3t + 2, 2t + 1)$ ,  $t \in \mathbb{R}$ . Another way of saying this is the line is given by the equations

$$\begin{aligned}x &= 3t + 2 \\y &= 2t + 1.\end{aligned}$$

In other words,  $t$  is allowed to be any real number, each choice of  $t$  gives exactly one point in the plane, and the totality of all points obtained is exactly all points that lie on the line  $2x - 3y = 1$ . Another way of looking at this is that the points in the plane that are solutions to the equation  $2x - 3y = 1$  are the points on the line. For example, when  $t = 0$  we get the solution  $x = 2$  and  $y = 1$  or, equivalently, the point  $(2, 1)$ ; when  $t = -5$  we get the solution  $(x, y) = (-13, -9)$ ; when  $t = 4$  we get the solution  $(x, y) = (14, 9)$ ; and so on.

In a similar fashion, the points on the plane  $2x - y - z = 3$  are given by the parametric equations

$$\begin{aligned}x &= \frac{1}{2}(3 + s + t) \\y &= s \\z &= t\end{aligned}$$

or, if you prefer,

$$\begin{aligned}x &= \frac{3}{2} + s + t \\y &= 2s \\z &= 2t.\end{aligned}$$

As  $(s, t)$  ranges over all points of  $\mathbb{R}^2$ , i.e., as  $s$  and  $t$  range over  $\mathbb{R}$ , the points  $(\frac{3}{2} + s + t, 2s, 2t)$  are all points on the plane  $2x - y - z = 3$ .

In a similar fashion, the points on the line that is the intersection of the planes  $2x - y - z = 3$  and  $x + y + z = 4$  are given by the parametric equations

$$\begin{aligned}x &= \\y &= \\z &= \end{aligned}$$

**5.9. The importance of rank.** We continue to discuss a consistent  $m \times n$  system  $A\underline{x} = \underline{b}$ .

- We have seen that  $\text{rank } A \leq m$  and  $\text{rank } A \leq n$  (Proposition 5.3) and that the number of independent variables is  $n - \text{rank } A$ .
- If  $\text{rank } A = n$  there are no independent variables, so the system has a *unique solution*.
- If  $\text{rank } A < n$  there is at least one independent variable, so the system has *infinitely many solutions*.

**Theorem 5.5.** *Let  $A$  be an  $n \times n$  matrix and consider the equation  $A\underline{x} = \underline{b}$ . If  $\text{rank } A = n$ , and the elementary row operations on the augmented matrix  $(A | \underline{b})$  produce  $(\text{rref } A | \underline{d}) = (I | \underline{d})$  where  $I$  is the identity matrix, then  $\underline{d}$  is the unique matrix such that  $A\underline{d} = \underline{b}$ .*

**Proof.** By Proposition 5.4,  $\text{rref } A = I$ . Hence the elementary row operations change  $(A | \underline{b})$  to  $(I | \underline{d})$  where  $I$  is the identity matrix. Therefore the equations  $A\underline{x} = \underline{b}$  and  $I\underline{x} = \underline{d}$  have the same set of solutions. But  $I\underline{x} = \underline{x}$  so the only solution to  $I\underline{x} = \underline{d}$  is  $\underline{x} = \underline{d}$ .  $\square$

**Proposition 5.6.** *Let  $A\underline{x} = \underline{0}$  be an  $m \times n$  system of homogeneous linear equations. If  $m < n$ , the system has a non-trivial solution.*

**Proof.** By Proposition 5.3,  $\text{rank } A \leq m$ . The number of independent variables for the equation  $A\underline{x} = \underline{0}$  is  $n - \text{rank } A \geq n - m \geq 1$ . Hence there are infinitely many solutions by the remarks in section 5.9.  $\square$

A shorter, simpler version of Proposition 5.6 is the following.

**Corollary 5.7.** *A homogeneous system of equations with more unknowns than equations always has a non-trivial solution.*

5.9.1. *A simple proof of Corollary 5.7.* Let  $A\underline{x} = \underline{0}$  be a homogeneous system of  $m$  equations in  $n$  unknowns and suppose  $m < n$ . Since  $\text{rref}(A)$  has more columns than rows there will be a column that does not have a leading 1. Hence there will be at least one independent variable. That variable can be anything; in particular, it can be given a non-zero value which leads to a non-trivial solution.

5.10. **The word “solution”.** A midterm contained the following question: *Suppose  $\underline{u}$  is a solution to the equation  $A\underline{x} = \underline{b}$  and that  $\underline{v}$  is a solution to the equation  $A\underline{x} = \underline{0}$ . Write out a single equation, involving several steps, that shows  $\underline{u} + \underline{v}$  is a solution to the equation  $A\underline{x} = \underline{b}$ .*

The correct answer is

$$A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{b} + \underline{0} = \underline{b}.$$

Done! I talked to some students who could not answer the question. They did not understand the meaning of the phrase “ $\underline{u}$  is a solution to the equation  $A\underline{x} = \underline{b}$ ”. The meaning is that  $A\underline{u} = \underline{b}$ . In principle, it is no more complicated than the statement “2 is a solution to the equation  $x^2 + 3x = 10$ ” which means that when you plug in 2 for  $x$  you obtain the true statement  $4 + 6 = 10$ .

## 6. THE GEOMETRY OF SYSTEMS OF LINEAR EQUATIONS

It is very useful to have a geometric picture of linear equations and their solutions.

The solutions to a single linear equation in  $n$  unknowns are points in  $n$ -space, or  $\mathbb{R}^n$ . By such a point we simply mean an ordered sequence of  $n$

real numbers,  $(s_1, \dots, s_n)$ . We will often call points in  $n$ -space *vectors* and denote them by an underlined letter. For example,  $\underline{v} = (1, 2, 3, 4)$  is a point in  $\mathbb{R}^4$ .

We also talk about *lines* in  $\mathbb{R}^n$ . The simplest way to describe a line is to describe it parametrically. For example, the set of all points of the form  $(1 + \lambda, 2 - 3\lambda, 3 - \lambda, 4)$ ,  $\lambda \in \mathbb{R}$ , is a line in  $\mathbb{R}^4$ . The line just described passes through the point  $(1, 2, 3, 4)$ ; that is the point obtained when the parameter  $\lambda$  is given the value zero. When we give  $\lambda$  the value 1 we obtain the point  $(2, -1, 2, 4)$ . When we give  $\lambda$  the value -3 we obtain the point  $(-2, 11, 6, 4)$ .

There are infinitely many different parametric descriptions of a line. Let  $L$  be the line in the previous paragraph. The line through

### 7. THE VECTOR SPACE $\mathbb{R}^n$

We write  $\mathbb{R}^n$  for the set of all  $n \times 1$  column vectors. Elements of  $\mathbb{R}^n$  are called vectors or points in  $\mathbb{R}^n$ . We call them points to encourage you to think of  $\mathbb{R}^n$  as a geometric object, a space, in the same sort of way as you think of the number line  $\mathbb{R}$ , the case  $n = 1$ , and the plane  $\mathbb{R}^2$ , in which the points of  $\mathbb{R}^2$  are labelled by row vectors  $(a, b)$  after choosing a coordinate system, and 3-space  $\mathbb{R}^3$ , the space in which we live or, when you include time,  $\mathbb{R}^4$ .

Solutions to an  $m \times n$  system  $A\underline{x} = \underline{b}$  of linear equations are therefore points in  $\mathbb{R}^n$ . The totality of all solutions is a certain subset of  $\mathbb{R}^n$ . As we said just after Proposition 4.2, the set of solutions has a linear nature: if  $p$  and  $q$  are solutions to  $A\underline{x} = \underline{b}$  so are all points on the line  $\overline{pq}$ .

**7.1. Arithmetic in  $\mathbb{R}^n$ .** Because the elements of  $\mathbb{R}^n$  are matrices, we can add them to one another, and we can multiply them by scalars (numbers). These operations have the following properties which we have been using for some time now:

- (1) if  $\underline{u}, \underline{v} \in \mathbb{R}^n$  so is  $\underline{u} + \underline{v}$ ;
- (2) if  $\underline{u} \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ , then  $\lambda\underline{u} \in \mathbb{R}^n$ ;
- (3)  $\underline{u} + \underline{v} = \underline{v} + \underline{u}$  for all  $\underline{u}, \underline{v} \in \mathbb{R}^n$ ;
- (4)  $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$  for all  $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$ ;
- (5) there is an element  $0 \in \mathbb{R}^n$  having the property that  $0 + \underline{u} = \underline{u}$  for all  $\underline{u} \in \mathbb{R}^n$ ;<sup>7</sup>
- (6) if  $\underline{u} \in \mathbb{R}^n$  there is an element  $\underline{u}' \in \mathbb{R}^n$  such that  $\underline{u} + \underline{u}' = 0$ ;<sup>8</sup>
- (7)  $(\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$  for all  $\alpha, \beta \in \mathbb{R}$  and all  $\underline{u} \in \mathbb{R}^n$ ;
- (8)  $\lambda(\underline{u} + \underline{v}) = \lambda\underline{u} + \lambda\underline{v}$  for all  $\lambda \in \mathbb{R}$  and all  $\underline{u}, \underline{v} \in \mathbb{R}^n$ ;

<sup>7</sup>It is important to observe that there is only one element with this property—if  $0' \in \mathbb{R}^n$  has the property that  $0' + \underline{u} = \underline{u}$  for all  $\underline{u} \in \mathbb{R}^n$ , then  $0 = 0' + 0 = 0 + 0' = 0'$  where the first equality holds because  $0' + \underline{u} = \underline{u}$  for all  $\underline{u}$ , the second equality holds because of the commutativity property in (3), and the third equality holds because  $0 + \underline{u} = \underline{u}$  for all  $\underline{u}$ .

<sup>8</sup>We denote  $\underline{u}'$  by  $-\underline{u}$  after showing that it is unique—it is unique because if  $\underline{u} + \underline{u}' = \underline{u} + \underline{u}'' = 0$ , then

$$\underline{u}' = \underline{u}' + 0 = \underline{u}' + (\underline{u} + \underline{u}'') = (\underline{u}' + \underline{u}) + \underline{u}'' = (\underline{u} + \underline{u}') + \underline{u}'' = 0 + \underline{u}'' = \underline{u}''.$$

- (9)  $(\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$  for all  $\alpha, \beta \in \mathbb{R}$  and all  $\underline{u} \in \mathbb{R}^n$ ;  
 (10)  $1.\underline{u} = \underline{u}$  for all  $\underline{u} \in \mathbb{R}^n$ .

**7.2. The standard basis for  $\mathbb{R}^n$ .** The vectors

$$\begin{aligned}\underline{e}_1 &:= (1, 0, \dots, 0, 0), \\ \underline{e}_2 &:= (0, 1, 0, \dots, 0), \\ &\vdots \\ \underline{e}_n &:= (0, 0, \dots, 0, 1)\end{aligned}$$

in  $\mathbb{R}^n$  will appear often from now on. We call  $\{\underline{e}_1, \dots, \underline{e}_n\}$  the standard basis for  $\mathbb{R}^n$ . The notion of a basis is defined and discussed in section 11 below. From now on the symbols

$$\underline{e}_1, \dots, \underline{e}_n$$

will be reserved for the special vectors just defined. There is some potential for ambiguity since we will use  $\underline{e}_1$  to denote the vector  $(1, 0)$  in  $\mathbb{R}^2$ , the vector  $(1, 0, 0)$  in  $\mathbb{R}^3$ , the vector  $(1, 0, 0, 0)$  in  $\mathbb{R}^4$ , etc. Likewise for the other  $\underline{e}_i$ s. It will usually be clear from the context which one we mean.

One reason for the importance of the  $\underline{e}_j$ s is that

$$(a_1, \dots, a_n) = a_1\underline{e}_1 + a_2\underline{e}_2 + \dots + a_n\underline{e}_n.$$

In other words, every vector in  $\mathbb{R}^n$  can be written as a sum of multiples of the  $\underline{e}_j$ s in a unique way.

**7.3. Linear combinations and linear span.** A linear combination of vectors  $\underline{v}_1, \dots, \underline{v}_n$  in  $\mathbb{R}^m$  is a vector that can be written as

$$a_1\underline{v}_1 + \dots + a_n\underline{v}_n$$

for some numbers  $a_1, \dots, a_n \in \mathbb{R}$ . We say the linear combination is non-trivial if at least one of the  $a_i$ s is non-zero.

The set of all linear combinations of  $\underline{v}_1, \dots, \underline{v}_n$  is called the linear span of  $\underline{v}_1, \dots, \underline{v}_n$  and denoted

$$\text{Sp}(\underline{v}_1, \dots, \underline{v}_n) \quad \text{or} \quad \langle \underline{v}_1, \dots, \underline{v}_n \rangle.$$

For example, the linear span of  $\underline{e}_1, \dots, \underline{e}_n$  is  $\mathbb{R}^n$  itself: if  $(a_1, a_2, \dots, a_n)$  is any vector in  $\mathbb{R}^n$ , then

$$(a_1, a_2, \dots, a_n) = a_1\underline{e}_1 + a_2\underline{e}_2 + \dots + a_n\underline{e}_n.$$

**7.3.1.** Note that the linear span of  $\underline{v}_1, \dots, \underline{v}_n$  has the following properties:

- (1)  $0 \in \text{Sp}(\underline{v}_1, \dots, \underline{v}_n)$ ;
- (2) if  $\underline{u}$  and  $\underline{v}$  are in  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_n)$  so are  $\underline{u} + \underline{v}$ ;
- (3) if  $a \in \mathbb{R}$  and  $\underline{v} \in \text{Sp}(\underline{v}_1, \dots, \underline{v}_n)$  so is  $a\underline{v}$ .

You should check these three properties hold. We will return to them later on – they are the defining properties of a subspace of  $\mathbb{R}^n$  (see section 8.1).

**Proposition 7.1.** *Let  $A$  be any  $m \times n$  matrix and  $\underline{x} \in \mathbb{R}^n$ . Then  $A\underline{x}$  is a linear combination of the columns of  $A$ . Explicitly,*

$$(7-1) \quad A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$$

where  $\underline{A}_j$  denotes the  $j^{\text{th}}$  column of  $A$ .

**Proof.** Let's write  $A = (a_{ij})$  and  $\underline{x} = (x_1, \dots, x_n)^T$ . Then

$$\begin{aligned} x_1\underline{A}_1 + \cdots + x_n\underline{A}_n &= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{pmatrix} \\ &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= A\underline{x}. \end{aligned}$$

The above calculation made use of scalar multiplication (2.5.10): if  $r \in \mathbb{R}$  and  $B$  is a matrix, then  $rB$  is the matrix whose  $ij^{\text{th}}$  entry is  $rB_{ij}$ ; since  $r$  and  $B_{ij}$  are numbers  $rB_{ij} = B_{ij}r$ .  $\square$

A useful way of stating Proposition 7.1 is that for a fixed  $A$  the set of values  $A\underline{x}$  takes as  $\underline{x}$  ranges over all of  $\mathbb{R}^n$  is  $\text{Sp}(\underline{A}_1, \dots, \underline{A}_n)$ , the linear span of the columns of  $A$ . Therefore, if  $\underline{b} \in \mathbb{R}^m$ , the equation  $A\underline{x} = \underline{b}$  has a solution if and only if  $\underline{b} \in \text{Sp}(\underline{A}_1, \dots, \underline{A}_n)$ . This is important enough to state as a separate result.

**Corollary 7.2.** *Let  $A$  be an  $m \times n$  matrix and  $\underline{b} \in \mathbb{R}^m$ . The equation  $A\underline{x} = \underline{b}$  has a solution if and only if  $\underline{b}$  is a linear combination of the columns of  $A$ .*

## 8. SUBSPACES

Let  $W$  denote the set of solutions to a system of homogeneous equations  $A\underline{x} = \underline{0}$ . It is easy to see that  $W$  has the following properties: (a)  $\underline{0} \in W$ ; (b) if  $\underline{u}$  and  $\underline{v}$  are in  $W$  so is  $\underline{u} + \underline{v}$ ; (c) if  $\underline{u} \in W$  and  $\lambda \in \mathbb{R}$ , then  $\lambda\underline{u} \in W$ . Subsets of  $\mathbb{R}^n$  having this property are called subspaces and are of particular importance in all aspects of linear algebra. It is also important to know that every subspace *is* the set of solutions to some system of homogeneous equations.

Another important way in which subspaces turn up is that the linear span of a set of vectors is a subspace.

Now we turn to the official definition.

**8.1. The definition and examples.** A subset  $W$  of  $\mathbb{R}^n$  is called a subspace if

- (1)  $\underline{0} \in W$ , and
- (2)  $\underline{u} + \underline{v} \in W$  for all  $\underline{u}, \underline{v} \in W$ , and
- (3)  $\lambda\underline{u} \in W$  for all  $\lambda \in \mathbb{R}$  and all  $\underline{u} \in W$ .

The book gives a different definition but then proves that a subset  $W$  of  $\mathbb{R}^n$  is a subspace if and only if it satisfies these three conditions. We will use our definition not the book's.

We now give numerous examples.

8.1.1. *The zero subspace.* The set  $\{\underline{0}\}$  consisting of the zero vector in  $\mathbb{R}^n$  is a subspace of  $\mathbb{R}^n$ . We call it the zero subspace. It is also common to call it the *trivial subspace*.

8.1.2.  $\mathbb{R}^n$  *itself is a subspace.* This is clear.

8.1.3. *The null space and range of a matrix.* Let  $A$  be an  $m \times n$  matrix. The null space of  $A$  is

$$\mathcal{N}(A) := \{\underline{x} \mid A\underline{x} = \underline{0}\}.$$

The range of  $A$  is

$$\mathcal{R}(A) := \{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}.$$

**Proposition 8.1.** *Let  $A$  be an  $m \times n$  matrix. Then  $\mathcal{N}(A)$  is a subspace of  $\mathbb{R}^n$  and  $\mathcal{R}(A)$  is a subspace of  $\mathbb{R}^m$ .*

**Proof.** First, we consider  $\mathcal{N}(A)$ . Certainly  $\underline{0} \in \mathcal{N}(A)$  because  $A\underline{0} = \underline{0}$ . now suppose that  $\underline{u}$  and  $\underline{v}$  belong to  $\mathcal{N}(A)$  and that  $a \in \mathbb{R}$ . Then

$$A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} = \underline{0}$$

and

$$A(a\underline{u}) = aA\underline{u} = a\underline{0} = \underline{0}$$

so both  $\underline{u} + \underline{v}$  and  $a\underline{u}$  belong to  $\mathcal{N}(A)$ . Hence  $\mathcal{N}(A)$  is a subspace of  $\mathbb{R}^n$ .

Now consider  $\mathcal{R}(A)$ . Certainly  $\underline{0} \in \mathcal{R}(A)$  because  $A\underline{0} = \underline{0}$ . Now suppose that  $\underline{u}$  and  $\underline{v}$  belong to  $\mathcal{R}(A)$  and that  $a \in \mathbb{R}$ . Then there exist  $\underline{x}$  and  $\underline{y}$  in  $\mathbb{R}^n$  such that  $\underline{u} = A\underline{x}$  and  $\underline{v} = A\underline{y}$ . Therefore

$$\underline{u} + \underline{v} = A\underline{x} + A\underline{y} = A(\underline{x} + \underline{y})$$

and

$$a\underline{u} = aA\underline{x} = Aa\underline{x}$$

so both  $\underline{u} + \underline{v}$  and  $a\underline{u}$  belong to  $\mathcal{R}(A)$ . Hence  $\mathcal{R}(A)$  is a subspace of  $\mathbb{R}^m$ .  $\square$

It is sometimes helpful to think of  $\mathcal{R}(A)$  as all “multiples” of  $A$  where a “multiple of  $A$ ” means a vector of the form  $A\underline{x}$ . It is clear that a multiple of  $AB$  is a multiple of  $A$ ; explicitly,  $AB\underline{x} = A(B\underline{x})$ . It follows that

$$\mathcal{R}(AB) \subset \mathcal{R}(A).$$

The following analogy might be helpful: every multiple of 6 is a multiple of 2 because  $6 = 2 \times 3$ . In set-theoretic notation

$$\{6x \mid x \in \mathbb{Z}\} \subset \{2x \mid x \in \mathbb{Z}\}.$$

Similarly,

$$\{AB\underline{x} \mid \underline{x} \in \mathbb{R}^n\} \subset \{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}$$

which is the statement that  $\mathcal{R}(AB) \subset \mathcal{R}(A)$ .

There is a similar inclusion for null spaces. If  $B\underline{x} = \underline{0}$ , then  $AB\underline{x} = \underline{0}$ , so

$$\mathcal{N}(B) \subset \mathcal{N}(AB).$$

8.1.4. *Lines through the origin are subspaces.* Let  $\underline{u} \in \mathbb{R}^n$ . We introduce the notation

$$\mathbb{R}\underline{u} := \{\lambda\underline{u} \mid \lambda \in \mathbb{R}\}$$

for the set of all real multiples of  $\underline{u}$ . We call  $\mathbb{R}\underline{u}$  a line if  $\underline{u} \neq \underline{0}$ . If  $\underline{u} = \underline{0}$ , then  $\mathbb{R}\underline{u}$  is the zero subspace.

**Lemma 8.2.** *Let  $\underline{u} \in \mathbb{R}^n$ . Then  $\mathbb{R}\underline{u}$  is a subspace of  $\mathbb{R}^n$ .*

**Proof.** The zero vector is a multiple of  $\underline{u}$  so is in  $\mathbb{R}\underline{u}$ . A sum of two multiples of  $\underline{u}$  is again a multiple of  $\underline{u}$ , and a multiple of a multiple of  $\underline{u}$  is again a multiple of  $\underline{u}$ .  $\square$

8.1.5. *Planes through the origin are subspaces.* Suppose that  $\{\underline{u}, \underline{v}\}$  is linearly independent subset of  $\mathbb{R}^n$ . We introduce the notation

$$\mathbb{R}\underline{u} + \mathbb{R}\underline{v} := \{a\underline{u} + b\underline{v} \mid a, b \in \mathbb{R}\}.$$

We call  $\mathbb{R}\underline{u} + \mathbb{R}\underline{v}$  the plane through  $\underline{u}$  and  $\underline{v}$ .

**Lemma 8.3.**  *$\mathbb{R}\underline{u} + \mathbb{R}\underline{v}$  is a subspace of  $\mathbb{R}^n$ .*

**Proof.** Exercise. It is also a special case of Proposition 8.4 below.  $\square$

You already know something about the geometry of  $\mathbb{R}^3$ . For example, if you are given three points in  $\mathbb{R}^3$  that do not lie on a single line, then there is a unique plane containing them. In the previous paragraph the hypothesis that  $\{\underline{u}, \underline{v}\}$  is linearly independent implies that  $0$ ,  $\underline{u}$ , and  $\underline{v}$ , do not lie on a single line. (In fact, the linear independence is equivalent to that statement—why?) The unique plane containing  $0$ ,  $\underline{u}$ , and  $\underline{v}$  is  $\mathbb{R}\underline{u} + \mathbb{R}\underline{v}$ .

You also know, in  $\mathbb{R}^3$ , that given two lines meeting at a point, there is a unique plane that contains those lines. Here, the lines  $\mathbb{R}\underline{u}$  and  $\mathbb{R}\underline{v}$  meet at  $0$ , and  $\mathbb{R}\underline{u} + \mathbb{R}\underline{v}$  is the unique plane containing them.

8.1.6. *The linear span of any set of vectors is a subspace.*

**Proposition 8.4.** *If  $\underline{v}_1, \dots, \underline{v}_d$  are vectors in  $\mathbb{R}^n$ , then their linear span  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$  is a subspace of  $\mathbb{R}^n$ . It is the smallest subspace of  $\mathbb{R}^n$  containing  $\underline{v}_1, \dots, \underline{v}_d$ .*

**Proof.** We leave the reader to show that  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$  is a subspace of  $\mathbb{R}^n$ . That is a straightforward exercise that you should do in order to check that you not only understand the meanings of *linear combination*, *linear span*, and *subspace*, but can put those ideas together in a simple proof.

Suppose we have shown that  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$  is a subspace of  $\mathbb{R}^n$ . It certainly contains  $\underline{v}_1, \dots, \underline{v}_d$ . However, if  $V$  is a subspace of  $\mathbb{R}^n$ , it contains all linear combinations of any set of elements in it. Thus if  $V$  is a subspace containing  $\underline{v}_1, \dots, \underline{v}_d$  it contains all linear combinations of those vectors, i.e.,  $V$  contains  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$ . Hence  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$  is the smallest subspace of  $\mathbb{R}^n$  containing  $\underline{v}_1, \dots, \underline{v}_d$ .  $\square$

**Proposition 8.5.** *The range of a matrix is equal to the linear span of its columns.<sup>9</sup>*

**Proof.** Let  $A$  be an  $m \times n$  matrix with columns  $\underline{A}_1, \dots, \underline{A}_n$ . If  $\underline{x} \in \mathbb{R}^n$ , then

$$A\underline{x} = x_1\underline{A}_1 + \dots + x_n\underline{A}_n$$

so

$$\mathcal{R}(A) = \{x_1\underline{A}_1 + \dots + x_n\underline{A}_n \mid x_1, \dots, x_n \in \mathbb{R}\}.$$

But the right hand side is the set of all linear combinations of the columns of  $A$ , i.e., the linear span of those columns.  $\square$

<sup>9</sup>This result can be viewed as a restatement of Corollary 7.2 which says that  $A\underline{x} = \underline{b}$  has a solution if and only if  $\underline{b}$  is a linear combination of the columns of  $A$  because the range of an  $m \times n$  matrix can be described as

$$\begin{aligned} \mathcal{R}(A) &= \{\underline{b} \in \mathbb{R}^m \mid \underline{b} = A\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n\} \\ &= \{\underline{b} \in \mathbb{R}^m \mid \text{the equation } \underline{b} = A\underline{x} \text{ has a solution}\}. \end{aligned}$$

**8.2. The row and column spaces of a matrix.** The row space of a matrix  $A$  is the linear span of its rows. The column space of a matrix  $A$  is the linear span of its columns.

**Lemma 8.6.** *If  $A$  and  $B$  are equivalent matrices they have the same row space.*

**Proof.** Since equivalent matrices are obtained from one another by a sequence of elementary row operations it suffices to show that the row space does not change when we perform a single elementary row operation on  $A$ .

Let  $R_1, \dots, R_m$  denote the rows of  $A$ . If  $R_i$  and  $R_j$  are switched the row space certainly stays the same. If  $R_i$  is replaced by  $\lambda R_i$  for any non-zero number  $\lambda$  the row space stays the same. Since  $aR_i + bR_j = a(R_i + \lambda R_j) + (b - \lambda a)R_j$  the row space does not change when  $R_i$  is replaced by  $R_i + \lambda R_j$ .  $\square$

**8.2.1. Sums of subspaces are subspaces.** If  $S$  and  $T$  are subsets of  $\mathbb{R}^n$ , their sum is the set

$$S + T := \{\underline{u} + \underline{v} \mid \underline{u} \in S \text{ and } \underline{v} \in T\}.$$

**Lemma 8.7.** *If  $U$  and  $V$  are subspaces of  $\mathbb{R}^n$  so is  $U + V$ .*

**Proof.** Since  $U$  and  $V$  are subspaces, each contains  $0$ . Hence  $0 + 0 \in U + V$ ; i.e.,  $U + V$  contains  $0$ .

If  $\underline{x}, \underline{x}' \in U + V$ , then  $\underline{x} = \underline{u} + \underline{v}$  and  $\underline{x}' = \underline{u}' + \underline{v}'$  for some  $\underline{u}, \underline{u}' \in U$  and  $\underline{v}, \underline{v}' \in V$ . Using the fact that  $U$  and  $V$  are subspaces, it follows that

$$\underline{x} + \underline{x}' = (\underline{u} + \underline{v}) + (\underline{u}' + \underline{v}') = (\underline{u} + \underline{u}') + (\underline{v} + \underline{v}') \in U + V$$

and, if  $a \in \mathbb{R}$ , then

$$a\underline{x} = a(\underline{u} + \underline{v}) = a\underline{u} + a\underline{v} \in U + V.$$

Hence  $U + V$  is a subspace.  $\square$

The result can be extended to sums of more than subspaces using an induction argument. You'll see the idea by looking at the case of three subspaces  $U$ ,  $V$ , and  $W$ , say. First, we define

$$U + V + W = \{\underline{u} + \underline{v} + \underline{w} \mid \underline{u} \in U, \underline{v} \in V, \text{ and } \underline{w} \in W\}.$$

It is clear that  $U + V + W = (U + V) + W$ . By the lemma,  $U + V$  is a subspace. Now we may apply the lemma to the subspaces  $U + V$  and  $W$ ; the lemma then tells us that  $(U + V) + W$  is a subspace. Hence  $U + V + W$  is a subspace.

As an example of the sum of many subspaces, notice that  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d) = \mathbb{R}\underline{v}_1 + \dots + \mathbb{R}\underline{v}_d$  so repeated application of the lemma tells us that the linear span of any set of vectors is a subspace. This gives an alternative proof of Proposition 8.4.

8.2.2. *Intersections of subspaces are subspaces.*

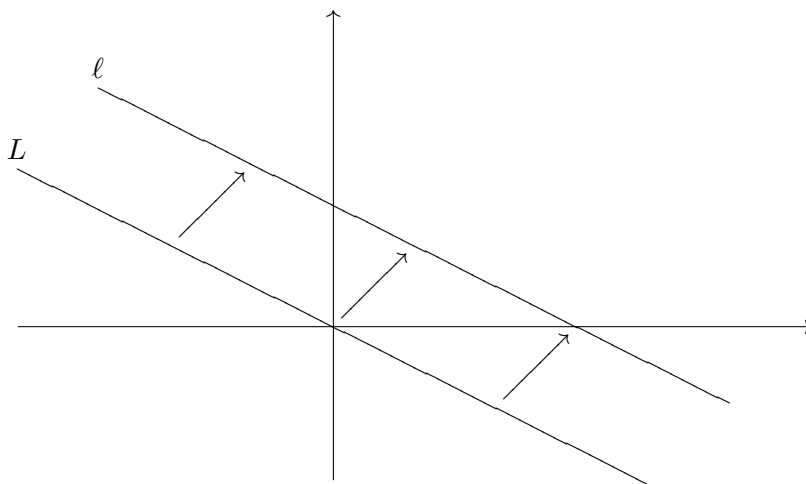
**Lemma 8.8.** *If  $U$  and  $V$  are subspaces of  $\mathbb{R}^n$  so is  $U \cap V$ .*

**Proof.** Because  $U$  and  $V$  are subspaces each of them contains the zero vector. Hence  $\underline{0} \in U \cap V$ . Now suppose  $\underline{u}$  and  $\underline{v}$  belong to  $U \cap V$ , and that  $a \in \mathbb{R}$ . Because  $\underline{u}$  and  $\underline{v}$  belong to the subspace  $U$  so does  $\underline{u} + \underline{v}$ ; similarly,  $\underline{u}$  and  $\underline{v}$  belong to  $V$  so  $\underline{u} + \underline{v} \in V$ . Hence  $\underline{u} + \underline{v} \in U \cap V$ . Finally if  $\underline{w} \in U \cap V$  and  $a \in \mathbb{R}$ , then  $a\underline{w} \in U$  because  $U$  is a subspace and  $a\underline{w} \in V$  because  $V$  is a subspace so  $a\underline{w} \in U \cap V$ .  $\square$

8.2.3. *Planes through the origin are subspaces.* If the set  $\{\underline{u}, \underline{v}\}$  is linearly independent we call the linear span of  $\underline{u}$  and  $\underline{v}$  a **plane**.

8.3. **Lines, planes, and translations of subspaces.** Let  $m$  and  $c$  be non-zero real numbers. Let  $\ell$  be the line  $y = mx + c$  and  $L$  the line  $y = mx$ . Both lines lie in  $\mathbb{R}^2$ . Now  $L$  is a subspace but  $\ell$  is not because it does not contain  $(0, 0)$ . However,  $\ell$  is parallel  $L$ , and can be obtained by sliding  $L$  sideways, not changing the slope, so that it passes through  $(0, c)$ . We call the line  $y = mx + c$  a *translation* of the line  $y = mx$ ; i.e.,  $\ell$  is a translation of the subspace  $L$ . The picture below is an example in which

$$\ell \text{ is } y = -\frac{1}{2}(x - 3) \text{ and } L \text{ is } y = -\frac{1}{2}x$$



Returning to the general case, every point on  $\ell$  is the sum of  $(0, c)$  and a point on the line  $y = mx$  so, if we write  $L$  for the subspace  $y = mx$ , then

$$\begin{aligned} \ell &= (0, c) + L \\ &= \{(0, c) + \underline{v} \mid \underline{v} \in L\} \\ &= \{(0, c) + (x, mx) \mid x \in \mathbb{R}\} \\ &= (0, c) + \mathbb{R}(1, m). \end{aligned}$$

Let  $V$  be a subspace of  $\mathbb{R}^n$  and  $\underline{z} \in \mathbb{R}^n$ . We call the subset

$$\underline{z} + V := \{\underline{z} + \underline{v} \mid \underline{v} \in V\}$$

the translation of  $V$  by  $\underline{z}$ . We also call  $\underline{z} + V$  a translate of  $V$ .

**Proposition 8.9.** *Let  $\mathcal{S}$  be the set of solutions to the equation  $A\underline{x} = \underline{b}$  and let  $V$  be the set of solutions to the homogeneous equation  $A\underline{x} = 0$ . Then  $\mathcal{S}$  is a translate of  $V$ : if  $\underline{z}$  is any solution to  $A\underline{x} = \underline{b}$ , then  $\mathcal{S} = \underline{z} + V$ .*

**Proof.** Let  $\underline{x} \in \mathbb{R}^n$ . Then  $A(\underline{z} + \underline{x}) = \underline{b} + A\underline{x}$ , so  $\underline{z} + \underline{x} \in \mathcal{S}$  if and only if  $\underline{x} \in V$ . The result follows.  $\square$

Now would be a good time to re-read section 5.7: that section talks about obtaining solutions to an equation  $A\underline{x} = \underline{b}$  by translating  $\mathcal{N}(A)$  by a particular solution to the equation  $A\underline{x} = \underline{b}$ . In fact, Proposition 8.9 and Proposition 4.4 really say the same thing in slightly different languages, and the example in section 5.7 illustrates what these results mean in a concrete situation.

8.3.1. *The linear geometry of  $\mathbb{R}^3$ .* This topic should already be familiar.

Skew lines, parallel planes, intersection of line and plane, parallel lines.

For example, find an equation for the subspace of  $\mathbb{R}^3$  spanned by  $(1, 1, 1)$  and  $(1, 2, 3)$ . Since the vectors are linearly independent that subspace will be a plane with basis  $\{(1, 1, 1), (1, 2, 3)\}$ . A plane in  $\mathbb{R}^3$  is given by an equation of the form

$$ax + by + cz = d$$

and is a subspace if and only if  $d = 0$ . Thus we want to find  $a, b, c \in \mathbb{R}$  such that  $ax + by + cz = 0$  is satisfied by  $(1, 1, 1)$  and  $(1, 2, 3)$ . More concretely, we want to find  $a, b, c \in \mathbb{R}$  such that

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 3z &= 0. \end{aligned}$$

This is done in the usual way:

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right).$$

Thus  $c$  is an independent variable and  $a = c$  and  $b = -2c$ . We just want a particular solution so take  $c = 1$ ,  $a = 1$ , and  $b = -2$ . The subspace we seek is given by the equation

$$x - 2y + z = 0.$$

The fact that  $c$  can be anything is just saying that this plane can also be given by non-zero multiples of the equation  $x - 2y + z = 0$ . For example,  $2x - 4y + 2z = 0$  is another equation for the same plane. And so on.

**8.4. Linguistic difficulties: algebra vs. geometry.** One of my midterm questions asked students to find all points on the plane

$$x_1 - 2x_2 + 3x_3 - 4x_4 = 0$$

$$x_1 - x_2 - x_3 + x_4 = 0$$

in  $\mathbb{R}^4$ . I was surprised how many students found this a hard problem. But I suspect they could have answered the question *find all solutions to the system of equations*

$$x_1 - 2x_2 + 3x_3 - 4x_4 = 0$$

$$x_1 - x_2 - x_3 + x_4 = 0.$$

*The two problems are identical!* The students' problem was linguistic not mathematical. The first problem sounds geometric, the second algebraic. A first course on linear algebra contains many new words and ideas and understanding the subject requires one to understand the connections and relationship between the different words and ideas. A large part of this involves the connection between algebra and geometry. This connection is a fundamental aspect of mathematics. You were introduced to it in high school. You learned that the solutions to an equation like  $y = 3x - 2$  were the points on a line in  $\mathbb{R}^2$ ; that solutions to an equation like  $2x^2 + 3y^2 = 4$  corresponded to the points on an ellipse and so on. Translating between the two questions at the start of this section is the same kind of thing.

The reason students find the question of finding all solutions to the system of equations easier is that they have a *recipe* for finding the solutions: find [rref A](#), find the independent variables, etc. Recipes are a very important part of linear algebra—after all we *do* want to solve problems and we want to find methods to solve particular kinds of problems. But linear algebra involves *ideas*, and connections and interactions between those ideas. As you understand these you will learn how to come up with new recipes, and learn to recognize when a particular recipe can be used even if the problem doesn't have the “code words” that suggest a particular recipe can be used.

## 9. LINEAR DEPENDENCE AND INDEPENDENCE

The notion of linear independence is central to the course from here on out.

**9.1. The definition.** A set of vectors  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is linearly independent if the only solution to the equation

$$a_1\underline{v}_1 + \dots + a_n\underline{v}_n = \underline{0}$$

is  $a_1 = \dots = a_n = 0$ . We say  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is linearly dependent if it is not linearly independent.

In other words,  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is linearly dependent if there are numbers  $a_1, \dots, a_n \in \mathbb{R}$ , not all zero, such that

$$a_1\underline{v}_1 + \dots + a_n\underline{v}_n = \underline{0}.$$

9.1.1. *The standard basis for  $\mathbb{R}^n$  is linearly independent.* Because

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n = (a_1, a_2, \dots, a_n)$$

this linear combination is zero if and only if  $a_1 = a_2 = \cdots = a_n = 0$ . The  $\mathbf{e}_i$ s form the prototypical example of a linearly independent set.

9.1.2. *Example.* Any set containing the zero vector is linearly dependent because

$$1\cdot\mathbf{0} + 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

**9.2. Criteria for linear (in)dependence.** I won't ask you to prove the next theorem on the exam but you will learn a lot if you read and re-read the proof of the next theorem until you understand what makes it work.

**Theorem 9.1.** *The non-zero rows of a matrix in echelon form are linearly independent.*

**Proof.** The key point is that the left-most non-zero entries in each non-zero row of a matrix in echelon form move to the right as we move downwards from row 1 to row 2 to row 3 to ... etc. We can get an idea of the proof by looking at a single example. The matrix

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is in echelon form. Its non-zero rows are

$$\begin{aligned} \mathbf{v}_1 &= (1 \ 2 \ 3 \ 4 \ 5 \ 6), \\ \mathbf{v}_2 &= (0 \ 1 \ 2 \ 3 \ 4 \ 5), \\ \mathbf{v}_3 &= (0 \ 0 \ 0 \ 1 \ 2 \ 3). \end{aligned}$$

To show these are linearly independent we must show that the only solution to the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$$

is  $a_1 = a_2 = a_3 = 0$ . Suppose  $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ . The sum  $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$  is equal to

$$(a_1, 2a_1, 3a_1, 4a_1, 5a_1, 6a_1) + (0, a_2, 2a_2, 3a_2, 4a_2, 5a_2) + (0, 0, 0, a_3, 2a_3, 3a_3).$$

The left most entry of this sum is  $a_1$  so the fact that the sum is the zero vector  $\mathbf{0} = (0, 0, 0, 0, 0, 0)$  implies that  $a_1 = 0$ . It follows that  $a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$ . But  $a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (0, a_2, \dots)$  (we don't care about what is in the positions ...) so we must have  $a_2 = 0$ . Therefore  $a_3\mathbf{v}_3 = \mathbf{0}$ ; but  $a_3\mathbf{v}_3 = (0 \ 0 \ 0 \ a_3 \ 2a_3 \ 3a_3)$  so we conclude that  $a_3 = 0$  too. Hence  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent.  $\square$

**Theorem 9.2.** *Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly dependent if and only if some  $\mathbf{v}_i$  is a linear combination of the other  $\mathbf{v}_j$ s.*

**Proof.** ( $\Rightarrow$ ) Suppose  $\underline{v}_1, \dots, \underline{v}_n$  are linearly dependent. Then there are numbers  $a_1, \dots, a_n$ , not all zero, such that  $a_1\underline{v}_1 + \dots + a_n\underline{v}_n = 0$ . Suppose  $a_i \neq 0$ . Then

$$\underline{v}_i = \sum_{j \neq i} (-a_i^{-1}a_j)\underline{v}_j.$$

( $\Leftarrow$ ) Suppose  $\underline{v}_i$  is a linear combination of the other  $\underline{v}_j$ s. Then there are numbers  $b_j$  such that

$$\underline{v}_i = \sum_{j \neq i} b_j \underline{v}_j.$$

Therefore

$$1 \cdot \underline{v}_i - \sum_{j \neq i} b_j \underline{v}_j = 0,$$

thus showing that  $\underline{v}_1, \dots, \underline{v}_n$  are linearly dependent.  $\square$

9.2.1. *Using Theorem 9.2.* Theorem 9.2 is an excellent way to show that a set of vectors is linearly dependent. For example, it tells us that the vectors

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix},$$

are linearly dependent because

$$\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

Please understand why this shows these three vectors are linearly independent by re-writing it as

$$1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and re-reading the definition of linear dependence.

9.2.2. *Remark.* The important word in the next theorem is the word *unique*. It is this uniqueness that leads to the idea of coordinates which will be taken up in section 11.

**Theorem 9.3.** *Vectors  $\underline{v}_1, \dots, \underline{v}_n$  are linearly independent if and only if each vector in their linear span can be written as a linear combination of  $\underline{v}_1, \dots, \underline{v}_n$  in a unique way.*

**Proof.** Since  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_n)$  is, by definition, the set of linear combinations of  $\underline{v}_1, \dots, \underline{v}_n$  we must prove that the linear independence condition is equivalent to the uniqueness condition.

( $\Rightarrow$ ) Suppose  $\underline{v}_1, \dots, \underline{v}_n$  are linearly independent. Let  $\underline{w} \in \text{Sp}(\underline{v}_1, \dots, \underline{v}_n)$ . If  $\underline{w} = a_1\underline{v}_1 + \dots + a_n\underline{v}_n$  and  $\underline{w} = b_1\underline{v}_1 + \dots + b_n\underline{v}_n$ , then

$$\begin{aligned} 0 &= \underline{w} - \underline{w} \\ &= (a_1\underline{v}_1 + \dots + a_n\underline{v}_n) - (b_1\underline{v}_1 + \dots + b_n\underline{v}_n) \\ &= (a_1 - b_1)\underline{v}_1 + \dots + (a_n - b_n)\underline{v}_n. \end{aligned}$$

Because the set  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is linearly independent,

$$a_1 - b_1 = \dots = a_n - b_n = 0.$$

Hence the two representations of  $\underline{w}$  as a linear combination of  $\underline{v}_1, \dots, \underline{v}_n$  are the same, i.e., there is a unique way of writing  $\underline{w}$  as a linear combination of those vectors.

( $\Leftarrow$ ) Suppose each element in  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_n)$  can be written as a linear combination of  $\underline{v}_1, \dots, \underline{v}_n$  in a unique way. In particular, there is only one way to write the zero vector as a linear combination. But  $0 = 0\underline{v}_1 + \dots + 0\underline{v}_n$  so if  $a_1\underline{v}_1 + \dots + a_n\underline{v}_n = 0$ , then  $a_1 = 0, a_2 = 0, \dots, a_n = 0$ . In other words, the set  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is linearly independent.  $\square$

**Theorem 9.4.** *If  $n > m$ , then every set of  $n$  vectors in  $\mathbb{R}^m$  is linearly dependent.*

**Proof.** Let's label the vectors  $\underline{A}_1, \dots, \underline{A}_n$  and make them into the columns of an  $m \times n$  matrix that we call  $A$ . Then  $A\underline{x} = x_1\underline{A}_1 + \dots + x_n\underline{A}_n$  so the equation  $A\underline{x} = 0$  has a non-trivial solution (i.e., a solution with some  $x_i \neq 0$ ) if and only if  $\underline{A}_1, \dots, \underline{A}_n$  are linearly dependent. But  $A\underline{x} = 0$  has a non-trivial solution because it consists of  $m$  equations in  $n$  unknowns and  $n > m$  (see Proposition 5.6 and Corollary 5.7).  $\square$

A generalization of Theorem 9.4 will be proved in Theorem 11.2.

### 9.3. Linear (in)dependence and systems of equations.

**Theorem 9.5.** *The homogeneous system  $A\underline{x} = 0$  has a non-trivial solution if and only if the columns  $\underline{A}_1, \dots, \underline{A}_n$  of  $A$  are linearly dependent.*

**Proof.** The proof of Theorem 9.4 used the fact that  $A\underline{x} = x_1\underline{A}_1 + \dots + x_n\underline{A}_n$  to conclude that  $A\underline{x} = 0$  has a non-trivial solution (i.e., a solution with some  $x_i \neq 0$ ) if and only if  $\underline{A}_1, \dots, \underline{A}_n$  are linearly dependent.  $\square$

The next result is obtained directly from the previous one —it is the *contrapositive*. By that we mean the following general principle of logic: if a statement  $P$  is true if and only if a statement  $Q$  is true, then  $P$  is false if and only if  $Q$  is false.

We apply this principle to Theorem 9.5 by noting that “not having a non-trivial solution” is the same as “having only the trivial solution” (which is the same as having a unique solution for a homogeneous system), and “not linearly dependent” is the same as “linearly independent”.

**Theorem 9.6.** *The homogeneous system  $A\underline{x} = \underline{0}$  has a unique solution if and only if the columns  $\underline{A}_1, \dots, \underline{A}_n$  of  $A$  are linearly independent.*

The reason for restating Theorem 9.5 as Theorem 9.6 is to emphasize the parallel between Theorem 9.6 and Theorem 9.7.

**Theorem 9.7.** *The system  $A\underline{x} = \underline{b}$  has a unique solution if and only if  $\underline{b}$  is a linear combination of the columns  $\underline{A}_1, \dots, \underline{A}_n$  of  $A$  and  $\underline{A}_1, \dots, \underline{A}_n$  are linearly independent.*

**Proof.** We have already proved that  $A\underline{x} = \underline{b}$  has a solution if and only if  $\underline{b}$  is a linear combination of the columns of  $A$ ; see Corollary 7.2 and the discussion just before it.

( $\Rightarrow$ ) Suppose  $A\underline{x} = \underline{b}$  has a unique solution. Because  $A\underline{x} = \underline{b}$  has a solution, Corollary 7.2 and the discussion just before it tell us that  $\underline{b}$  is a linear combination of the columns of  $A$ .

We will now use the fact that  $A\underline{x} = \underline{b}$  has a *unique* solution to show that the columns of  $A$  are linearly independent. Suppose  $c_1\underline{A}_1 + \dots + c_n\underline{A}_n = \underline{0}$ ; in other words  $A\underline{c} = \underline{0}$  where

$$\underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

If  $\underline{u}$  is a solution to  $A\underline{x} = \underline{b}$ , so is  $\underline{u} + \underline{c}$  because

$$A(\underline{u} + \underline{c}) = A\underline{u} + A\underline{c} = \underline{b} + \underline{0} = \underline{b}.$$

But  $A\underline{x} = \underline{b}$  has a *unique* solution so  $\underline{u} = \underline{u} + \underline{c}$ . It follows that  $\underline{c} = \underline{0}$ . In other words, the only solution to the equation  $c_1\underline{A}_1 + \dots + c_n\underline{A}_n = \underline{0}$  is  $c_1 = \dots = c_n = 0$  which shows that the columns of  $A$  are linearly independent.

( $\Leftarrow$ ) Suppose that  $\underline{b}$  is a linear combination of the columns  $\underline{A}_1, \dots, \underline{A}_n$  of  $A$  and the columns are linearly independent. The first of these assumptions tells us there are numbers  $c_1, \dots, c_n \in \mathbb{R}$  such that  $\underline{b} = c_1\underline{A}_1 + \dots + c_n\underline{A}_n$ ; i.e.,  $A\underline{c} = \underline{b}$  where  $\underline{c}$  is the  $n \times 1$  matrix displayed above. If

$$\underline{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

were another solution to  $A\underline{x} = \underline{b}$ , then  $\underline{b} = A\underline{d} = d_1\underline{A}_1 + \dots + d_n\underline{A}_n$  so

$$(c_1 - d_1)\underline{A}_1 + \dots + (c_n - d_n)\underline{A}_n = A\underline{c} - A\underline{d} = \underline{b} - \underline{b} = \underline{0}.$$

However,  $\underline{A}_1, \dots, \underline{A}_n$  are linearly independent so  $c_1 = d_1, \dots, c_n = d_n$ , i.e.,  $\underline{c} = \underline{d}$  thereby showing that  $A\underline{x} = \underline{b}$  has a unique solution.  $\square$

Lemma 8.6 tells us that equivalent matrices have the same row space.

**Proposition 9.8.** *Let  $A$  be any matrix. Then  $A$  and  $\mathbf{rref} A$  have the same row space, and the non-zero rows of  $\mathbf{rref} A$  are linearly independent.*

**Proof.** Let  $E = \text{rref } A$ . The fact that  $A$  and  $E$  have the same row space is a special case of Lemma 8.6. Suppose  $\underline{r}_1, \dots, \underline{r}_t$  are the non-zero rows of  $E$ . Suppose the left-most 1 in  $\underline{r}_i$  appears in the  $j_i^{\text{th}}$  column. Then  $j_1 < j_2 < \dots < j_t$ .

Now suppose that  $a_1 \underline{r}_1 + \dots + a_t \underline{r}_t = 0$ .

The entry in column  $j_1$  of  $\underline{r}_1$  is 1 and the entry in column  $j_1$  of all other  $\underline{r}_i$ s is 0, so the entry in column  $j_1$  of  $a_1 \underline{r}_1 + \dots + a_t \underline{r}_t$  is  $a_1$ . Hence  $a_1 = 0$ .

The entry in column  $j_2$  of  $\underline{r}_2$  is 1 and the entry in column  $j_2$  of all other  $\underline{r}_i$ s is 0, so the entry in column  $j_2$  of  $a_1 \underline{r}_1 + \dots + a_t \underline{r}_t$  is  $a_2$ . Hence  $a_2 = 0$ .

Carry on in this way to deduce that each  $a_i$  is zero. We will then have shown that the only solution to the equation  $a_1 \underline{r}_1 + \dots + a_t \underline{r}_t = 0$  is  $a_1 = a_2 = \dots = a_t = 0$ . Hence  $\{\underline{r}_1, \dots, \underline{r}_t\}$  is linearly independent.  $\square$

After we have defined the word *basis* you will see that Proposition 9.8 says that the non-zero rows of  $\text{rref } A$  are a basis for the row space of  $A$ .

If one is interested in the column space of a matrix  $A$  rather than its row space, one can replace  $A$  by  $A^T$  and perform elementary row operations on  $A^T$ . The transposes of the rows of  $\text{rref } A^T$  span the column space of  $A$ . We will see later that they form a basis for the column space of  $A$ .

## 10. NON-SINGULAR MATRICES, INVERTIBLE MATRICES, AND INVERSES

The adjectives *singular*, *non-singular*, and *invertible*, apply to square matrices and square matrices only.

**10.1. Singular and non-singular matrices.** An  $n \times n$  matrix  $A$  is called non-singular if the only solution to the equation  $A\underline{x} = 0$  is  $\underline{x} = 0$ . We say  $A$  is singular if  $A\underline{x} = 0$  has a non-trivial solution.

When we say *the following are equivalent* in the next theorem what we mean is that if one of the three statements is true so are the others. It also means that if one of the three statements is false so are the others. Therefore to prove the theorem we must do the following: assume one of the statements is true and then show that assumption forces the other two to be true. In the proof below we show that (1) is true if and only if (2) is true. Then we show that (3) is true if (2) is. Finally we show (1) is true if (3) is. Putting all those implications together proves that the truth of any one of the statements implies the truth of the other two.

**Theorem 10.1.** *Let  $A$  be an  $n \times n$  matrix. The following are equivalent:*

- (1)  $A$  is non-singular;
- (2)  $\{\underline{A}_1, \dots, \underline{A}_n\}$  is linearly independent.
- (3)  $A\underline{x} = \underline{b}$  has a unique solution for all  $\underline{b} \in \mathbb{R}^n$ .

**Proof.** (1)  $\Leftrightarrow$  (2) This is exactly what Theorem 9.6 says.

(2)  $\Rightarrow$  (3) Suppose that (2) is true.

Let  $\underline{b} \in \mathbb{R}^n$ . By Theorem 9.4, the  $n + 1$  vectors  $\underline{b}, \underline{A}_1, \dots, \underline{A}_n$  in  $\mathbb{R}^n$  are linearly dependent so there are numbers  $a_1, \dots, a_n, a_{n+1}$ , not all zero, such

that

$$a_0 \underline{b} + a_1 \underline{A}_1 + \cdots + a_n \underline{A}_n = 0.$$

I claim that  $a_0$  can not be zero.

Suppose the claim is false. Then  $a_1 \underline{A}_1 + \cdots + a_n \underline{A}_n = 0$ . But  $\underline{A}_1, \dots, \underline{A}_n$  are linearly independent so  $a_1 = \cdots = a_n = 0$ . Thus if  $a_0 = 0$ , then *all* the  $a_i$ s are zero. That contradicts the fact (see the previous paragraph) that not all the  $a_i$ s are zero. Since the assumption that  $a_0 = 0$  leads to a contradiction we conclude that  $a_0$  must be non-zero.

Hence

$$\underline{b} = (-a_0^{-1} a_1) \underline{A}_1 + \cdots + (-a_0^{-1} a_n) \underline{A}_n.$$

Thus  $\underline{b}$  is a linear combination of  $\underline{A}_1, \dots, \underline{A}_n$ . By hypothesis,  $\underline{A}_1, \dots, \underline{A}_n$  linearly independent. Hence Theorem 9.7 tells us there is a unique solution to  $A\underline{x} = \underline{b}$ .

(3)  $\Rightarrow$  (1) Assume (3) is true. Then, in particular, the equation  $A\underline{x} = 0$  has a unique solution; i.e.,  $A$  is non-singular, so (1) is true.  $\square$

**10.2. Inverses.** Mankind used whole numbers, integers, before using fractions. Fractions were created because they are useful in the solution of equations. For example, the equation  $3x = 7$  has no solution if we require  $x$  to be an integer but it does have a solution if we allow  $x$  to be a fraction, that solution being  $\frac{7}{3}$ .

One could take the sophisticated (too sophisticated?) view that this solution was found by multiplying both sides of the equation by the inverse of 3. Explicitly,  $3x$  is equal to 7 if and only if  $\frac{1}{3} \times 3x$  equals  $\frac{1}{3} \times 7$ . Now use the associative law for multiplication and perform the complicated equation

$$\frac{1}{3} \times 3x = \left( \frac{1}{3} \times 3 \right) \times x = 1 \times x = x$$

so the previous sentence now reads  $3x$  is equal to 7 if and only if  $x$  equals  $\frac{1}{3} \times 7$ . Presto! we have solved the equation.

Sometimes one encounters a matrix equation in which one can perform the same trick. Suppose the equation is  $A\underline{x} = \underline{b}$  and suppose you are in the fortunate position of knowing there is a matrix  $E$  such that  $EA = I$ , the identity matrix. Then  $A\underline{x} = \underline{b}$  implies that  $E \times A\underline{x} = E \times \underline{b}$ . Thus, using the associative law for multiplication  $A\underline{x} = \underline{b}$  implies

$$E\underline{b} = E \times \underline{b} = E \times A\underline{x} = (E \times A) \times \underline{x} = I \times \underline{x} = \underline{x}.$$

Presto! We have found a solution, namely  $\underline{x} = E\underline{b}$ . We found this by the same process as that in the previous paragraph.

Let's look at an explicit example. Suppose we want to solve

$$(10-1) \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$

I know that the matrix

$$E = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

has the property that  $EA = I$ . You should check that. The analysis in the previous paragraph tells me that a solution to the equation (10-1) is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = E \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}.$$

**Lemma 10.2.** *Let  $A$  be an  $n \times n$  matrix and  $I$  the  $n \times n$  identity matrix. If there are matrices  $E$  and  $E'$  such that*

$$EA = AE' = I$$

*then  $E = E'$ .*

**Proof.** This is a single calculation:

$$E' = IE' = (EA)E' = E(AE') = EI = E.$$

All we used was the associative law.  $\square$

The inverse of an  $n \times n$  matrix  $A$  is the matrix  $A^{-1}$  with the property

$$AA^{-1} = I = A^{-1}A,$$

provided it exists. If  $A$  has an inverse we say that  $A$  is invertible.

### 10.3. Comments on the definition of inverse.

10.3.1. *Uniqueness of  $A^{-1}$ .* Our definition says that *the* inverse is *the* matrix ... The significance of Lemma 10.2 is that it tells us there is *at most one* matrix,  $E$  say, with the property that  $AE = EA = I$ . Check you understand that!

10.3.2. *Not every square matrix has an inverse.* The phrase *provided it exists* suggests that not every  $n \times n$  matrix has an inverse—that is correct. For example, the zero matrix does not have an inverse. We will also see examples of non-zero square matrices that do not have an inverse.

10.3.3. Recall that the derivative of a function  $f(x)$  at  $a \in \mathbb{R}$  is defined to be

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

*provided this limit exists.* The use of the phrase “*provided it exists*” in the definition of the inverse of a matrix plays the same role as the phrase “*provided this limit exists*” plays in the definition of the derivative.

10.3.4. *The inverse or an inverse.* You should compare our definition of *inverse* with that in the book. The book says an  $n \times n$  matrix is invertible if there is a matrix  $A^{-1}$  such that  $AA^{-1} = A^{-1}A = I$  and calls such  $A^{-1}$  an inverse of  $A$  (if it exists). Lemma 10.2 tells us there is at most one inverse so we can immediately speak of *the* inverse (provided it exists). You might want to look at the definition of the inverse in several books to gain a finer appreciation of this fine point.

Here is the definition from the second edition of *Linear Algebra* by M. O’Nan:

If  $A$  is an  $m \times m$  matrix and there is an  $m \times m$  matrix  $B$  such that  $AB = BA = I$ ,  $A$  is said to be **invertible** and  $B$  is said to be the **inverse** of  $A$ .

Having made this definition, the first natural question to ask is whether a matrix  $A$  can have two different inverses. Theorem 2 shows this is impossible.

**Theorem 2** Let  $A$  be an  $m \times m$  matrix with inverse  $B$ . If  $C$  is another matrix such that  $AC = CA = I$ , then  $C = B$ .

#### 10.4. Making use of inverses.

10.4.1. *If  $A$  has an inverse and  $AB = C$ , then  $B = A^{-1}C$ .* To see this just multiply both sides of the equality  $AB = C$  on the left by  $A^{-1}$  to obtain  $A^{-1}C = A^{-1}(AB) = (A^{-1}A)B = IB = B$ .

10.4.2. *Solving  $A\underline{x} = \underline{b}$  when  $A$  is invertible.* The only solution to this equation is  $\underline{x} = A^{-1}\underline{b}$ . This is a special case of the previous remark. In section 10.2 we put this remark into practice for an explicit example.

10.4.3. *Wicked notation.* Never, never, never write  $B/A$  to denote  $BA^{-1}$ .

10.4.4. *If  $AB = 0$  and  $B \neq 0$ , then  $A$  is not invertible.* If  $A$  is invertible and  $B$  is a non-zero matrix such that  $AB = 0$ , then  $B = IB = (A^{-1}A)B = A^{-1}(AB) = 0$  so contradicting the fact that  $B$  is non-zero. We are therefore forced to conclude that if  $AB = 0$  for some non-zero matrix  $B$ , then  $A$  is not invertible. Likewise, if  $BA = 0$  for some non-zero matrix  $B$ , then  $A$  is not invertible.

This is a useful observation because it immediately gives a large source of matrices that are not invertible. For example, the matrix

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

does not have an inverse because

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix};$$

in particular, the product can never be the identity matrix because the 22-entry is zero regardless of what  $w, x, y, z$  are.

10.4.5. *We can cancel invertible matrices.* Let  $A$  be an  $n \times n$  matrix. If  $AB = AC$ , then  $B = C$  because  $AB = AC$  implies  $A^{-1}(AB) = A^{-1}(AC)$  and the associative law then allows us to replace  $A(A^{-1}$  by  $I$  to obtain  $IB = IC$ , i.e.,  $B = C$ . I prefer the argument

$$B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}(AC) = (A^{-1}A)C = IC = C.$$

A similar argument shows that if  $A$  is invertible and  $EA = FA$ , then  $E = F$ .

Be warned though that you cannot cancel in an expression like  $AB = CA$  because matrices do not necessarily commute with each other.

10.4.6. *The inverse of a product.* If  $A$  and  $B$  are  $n \times n$  matrices have inverses, then  $AB$  is invertible and its inverse is  $B^{-1}A^{-1}$  because

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly  $(B^{-1}A^{-1})(AB) = I$ .

A popular error is to claim that the inverse of  $AB$  is  $A^{-1}B^{-1}$ . An analogy might be helpful: the inverse to the operation *put on socks then shoes* is *take off shoes then socks*. The inverse to a composition of operations is the composition of the inverses of the operations in the reverse order.

10.5. **The inverse of a  $2 \times 2$  matrix.** There is a simple arithmetic criterion for deciding if a given  $2 \times 2$  matrix has an inverse.

**Theorem 10.3.** *The matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse if and only if  $ad - bc \neq 0$ . If  $ad - bc \neq 0$ , then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

**Proof.** ( $\implies$ ) Suppose  $A$  has an inverse. Just prior to this theorem, in section 10.4.4, we observed that

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

does not have an inverse, so either  $c$  or  $d$  must non-zero. Therefore

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} d \\ -c \end{pmatrix} = A^{-1}A \begin{pmatrix} d \\ -c \end{pmatrix} = A^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix} = A^{-1} \begin{pmatrix} ad - bc \\ 0 \end{pmatrix}$$

so we conclude that  $ad - bc \neq 0$ .

( $\impliedby$ ) If  $ad - bc \neq 0$  a simple calculation shows that the claimed inverse is indeed the inverse.  $\square$

The number  $ad - bc$  is called the determinant of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We usually write  $\det A$  to denote the determinant of a matrix. Theorem 10.3 therefore says that a  $2 \times 2$  matrix  $A$  has an inverse if and only if  $\det A \neq 0$ .

In section 13 we define the determinant for  $n \times n$  matrices and prove that an  $n \times n$  matrix  $A$  has an inverse if and only if  $\det A \neq 0$ . That is a satisfying result. Although the formula for the determinant appears complicated at first it is straightforward to compute, even for large  $n$ , if one has a computer handy. It is striking that the invertibility of a matrix can be determined by a single computation even if doing the computation by hand can be long and prone to error.

**Theorem 10.4.** *Let  $A$  be an  $n \times n$  matrix. Then  $A$  has an inverse if and only if it is non-singular.*

**Proof.** ( $\Rightarrow$ ) Suppose  $A$  has an inverse. If  $A\underline{x} = \underline{0}$ , then

$$\underline{x} = I\underline{x} = (A^{-1}A)\underline{x} = A^{-1}(A\underline{x}) = A^{-1}\underline{0} = \underline{0}$$

so  $A$  is non-singular.

( $\Leftarrow$ ) Suppose  $A$  is non-singular. Then  $A\underline{x} = \underline{b}$  has a unique solution for all  $\underline{b} \in \mathbb{R}^n$  by Theorem 10.1. Let

$$\underline{e}_1 = (1, 0, \dots, 0)^T, \quad \underline{e}_2 = (0, 1, 0, \dots, 0), \dots, \underline{e}_n = (0, \dots, 0, 1)$$

and define  $\underline{u}_j \in \mathbb{R}^n$  to be the unique column vector such that  $A\underline{u}_j = \underline{e}_j$ . Let  $U = [\underline{u}_1, \dots, \underline{u}_n]$  be the  $n \times n$  matrix whose columns are  $\underline{u}_1, \dots, \underline{u}_n$ . Then

$$AU = [A\underline{u}_1, \dots, A\underline{u}_n] = [\underline{e}_1, \dots, \underline{e}_n] = I.$$

We will show that  $U$  is the inverse of  $A$  but we do not compute  $UA$  to do this!

First we note that  $U$  is non-singular because if  $U\underline{x} = \underline{0}$ , then

$$\underline{x} = I\underline{x} = (AU)\underline{x} = A(U\underline{x}) = A\underline{0} = \underline{0}.$$

Because  $U$  is non-singular we may apply the argument in the previous paragraph to deduce the existence of a matrix  $V$  such that  $UV = I$ . (The previous paragraph showed that if a matrix is non-singular one can multiply it on the right by some matrix to obtain the identity matrix.) Now

$$V = IV = (AU)V = A(UV) = AI = A$$

so  $AU = UA = I$ , proving that  $U = A^{-1}$ .  $\square$

We isolate an important part of the proof. After finding  $U$  and showing  $AU = I$  we were not allowed to say  $U$  is the inverse of  $A$  and stop the proof because the definition of the inverse (read it carefully, now) says that we need to show that  $UA = I$  too before we are allowed to say  $U$  is the inverse of  $A$ . It is not obvious that  $AU = I$  implies  $UA = I$  because there is no obvious reason for  $A$  and  $U$  to commute. That is why we need the last paragraph of the proof.

**Corollary 10.5.** *Let  $A$  be an  $n \times n$  matrix. If  $AU = I$ , then  $UA = I$ , i.e., if  $AU = I$ , then  $U$  is the inverse of  $A$ .*

**Corollary 10.6.** *Let  $A$  be an  $n \times n$  matrix. If  $VA = I$ , then  $AV = I$ , i.e., if  $VA = I$ , then  $V$  is the inverse of  $A$ .*

**Proof.** Apply Corollary 10.5 with  $V$  playing the role that  $A$  played in Corollary 10.5 and  $A$  playing the role that  $U$  played in Corollary 10.5.  $\square$

**Theorem 10.7.** *The following conditions on an  $n \times n$  matrix  $A$  are equivalent:*

- (1)  $A$  is non-singular;
- (2)  $A$  is invertible;

- (3)  $\text{rank } A = n$ ;
- (4)  $\text{rref } A = I$ , the identity matrix;
- (5)  $A\underline{x} = \underline{b}$  has a unique solution for all  $\underline{b} \in \mathbb{R}^n$ ;
- (6) the columns of  $A$  are linearly independent;

10.6. **If  $A$  is non-singular how do we find  $A^{-1}$ ?** We use the idea in the proof of Theorem 10.4:

the  $j^{\text{th}}$  column of  $A^{-1}$  is the solution to the equation  $A\underline{x} = \underline{e}_j$   
 so we may form the augmented matrix  $(A | \underline{e}_j)$  and perform  
 row operations to get  $A$  in row-reduced echelon form.

We can carry out this procedure in one great big military operation. If we want to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

we perform elementary row operations on the augmented matrix

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

so we get the identity matrix on the left hand side. The matrix on the right-hand side is then  $A^{-1}$ . Check it out!

## 11. BASES, COORDINATES, AND DIMENSION

11.1. **Definitions.** A basis for a subspace  $W$  is a linearly independent subset of  $W$  that spans it.

The dimension of a subspace is the number of elements in a basis for it.

To make this idea effective we must show every subspace has a basis and that all bases for a subspace have the same number of elements.

Before doing that, let us take note of the fact that  $\{\underline{e}_1, \dots, \underline{e}_n\}$  is a basis for  $\mathbb{R}^n$ , the standard basis. We made the trivial observation that these vectors span  $\mathbb{R}^n$  in section 7.3, and the equally trivial observation that these vectors are linearly independent in section 9. Combining these two facts we see that  $\{\underline{e}_1, \dots, \underline{e}_n\}$  is a basis for  $\mathbb{R}^n$ . Therefore

$$\dim \mathbb{R}^n = n.$$

Strictly speaking, we can't say this until we have proved that every basis for  $\mathbb{R}^n$  has exactly  $n$  elements but we will prove that in Corollary 11.3 below.

**Theorem 11.1.** *Suppose  $\underline{v}_1, \dots, \underline{v}_d$  is a basis for  $W$ . If  $\underline{w} \in W$ , then there are unique scalars  $a_1, \dots, a_d$  such that  $\underline{w} = a_1\underline{v}_1 + \dots + a_d\underline{v}_d$ .*

**Proof.** This theorem is a reformulation of Theorem 9.3. □

If  $\underline{v}_1, \dots, \underline{v}_d$  is a basis for  $W$  and  $\underline{w} = a_1\underline{v}_1 + \dots + a_d\underline{v}_d$  we call  $(a_1, \dots, a_d)$  the coordinates of  $\underline{w}$  with respect to the basis  $\underline{v}_1, \dots, \underline{v}_d$ . In this situation we can think of  $(a_1, \dots, a_d)$  as a point in  $\mathbb{R}^d$ .

11.2. **A subspace of dimension  $d$  is just like  $\mathbb{R}^d$ .** The function

$$T : \mathbb{R}^d \rightarrow W, \quad T(a_1, \dots, a_d) := a_1 \underline{v}_1 + \dots + a_d \underline{v}_d$$

is bijective (meaning injective and surjective, i.e., one-to-one and onto). It also satisfies the following properties:

- (1)  $T(\mathbf{0}) = \mathbf{0}$ ;
- (2) if  $\underline{x}, \underline{y} \in \mathbb{R}^d$ , then  $T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y})$ ;
- (3) if  $\underline{x} \in \mathbb{R}^d$  and  $\lambda \in \mathbb{R}$ , then  $T(\lambda \underline{x}) = \lambda T(\underline{x})$ .

These properties are all special cases of the following property:

$$(11-1) \quad T(\lambda_1 \underline{x}_1 + \dots + \lambda_s \underline{x}_s) = \lambda_1 T(\underline{x}_1) + \dots + \lambda_s T(\underline{x}_s)$$

for all  $\underline{x}_1, \dots, \underline{x}_s \in \mathbb{R}^d$  and all  $\lambda_1, \dots, \lambda_s \in \mathbb{R}$ .

The function  $T$  is an example of a *linear transformation*.

It is the existence of the bijective function  $T$  having these properties that justifies my repeated remark that a  $d$ -dimensional subspace looks exactly like  $\mathbb{R}^d$ . That is the way you should think of a  $d$ -dimensional subspace of  $\mathbb{R}^n$ . It is a copy of  $\mathbb{R}^d$  sitting inside  $\mathbb{R}^n$ . For example, all planes in  $\mathbb{R}^3$  are copies of  $\mathbb{R}^2$  sitting inside  $\mathbb{R}^3$ . Likewise, all lines in  $\mathbb{R}^3$  are copies of  $\mathbb{R}^1 = \mathbb{R}$  sitting inside  $\mathbb{R}^3$ .

11.3. **All bases for  $W$  have the same number of elements.** Compare the next result to theorem 9.4.

**Theorem 11.2.** . Every subset of  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$  having  $\geq d + 1$  vectors is linearly dependent.

**Proof.** Let  $r \geq d + 1$  and let  $\underline{w}_1, \dots, \underline{w}_r$  be vectors in  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$ . Then there are numbers  $a_{ij} \in \mathbb{R}$  for  $1 \leq i \leq r$  and  $1 \leq j \leq d$  such that

$$(11-2) \quad \underline{w}_i = a_{i1} \underline{v}_1 + a_{i2} \underline{v}_2 + \dots + a_{id} \underline{v}_d$$

for all  $i$ .

We will show that  $\{\underline{w}_1, \dots, \underline{w}_r\}$  is linearly dependent by showing there are  $c_1, \dots, c_r \in \mathbb{R}$ , not all zero, such that

$$(11-3) \quad c_1 \underline{w}_1 + \dots + c_r \underline{w}_r = \underline{0}.$$

We treat (11-3) as an equation in the unknowns  $c_1, \dots, c_r$  and show it has a non-trivial solution.

Making substitutions from (11-2), we have

$$\begin{aligned} c_1 \underline{w}_1 + \dots + c_r \underline{w}_r = & (c_1 a_{11} + c_2 a_{21} + \dots + c_r a_{r1}) \underline{v}_1 + \\ & (c_1 a_{12} + c_2 a_{22} + \dots + c_r a_{r2}) \underline{v}_2 + \\ & \vdots \\ & (c_1 a_{1d} + c_2 a_{2d} + \dots + c_r a_{rd}) \underline{v}_d. \end{aligned}$$

There are more unknowns,  $r$ , than equations,  $d$ , in the homogeneous system

$$\begin{aligned} a_{11}c_1 + a_{21}c_2 + \cdots + a_{r1}c_r &= 0 \\ a_{12}c_1 + a_{22}c_2 + \cdots + a_{r2}c_r &= 0 \\ &\vdots \\ a_{1d}c_1 + a_{2d}c_2 + \cdots + a_{rd}c_r &= 0 \end{aligned}$$

so it has a non-trivial solution by Corollary 5.7, i.e., a solution in which some  $c_i \neq 0$ . Hence  $\{\underline{w}_1, \dots, \underline{w}_r\}$  is linearly independent.  $\square$

**Corollary 11.3.** *All bases for a subspace have the same number of elements.*

**Proof.** Suppose that  $\{\underline{v}_1, \dots, \underline{v}_d\}$  and  $\{\underline{w}_1, \dots, \underline{w}_r\}$  are bases for  $W$ . Then

$$W = \text{Sp}(\underline{v}_1, \dots, \underline{v}_d) = \text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$$

and the sets  $\{\underline{v}_1, \dots, \underline{v}_d\}$  and  $\{\underline{w}_1, \dots, \underline{w}_r\}$  are linearly independent.

By Theorem 11.2, a linearly independent subset of  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_d)$  has  $\leq d$  elements so  $r \leq d$ . Likewise, by Theorem 11.2, a linearly independent subset of  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$  has  $\leq r$  elements so  $d \leq r$ . Hence  $d = r$ .  $\square$

**11.4. Every subspace has a basis.** Our next goal, Theorem 11.5, is to show that every subspace has a basis. First we need a lemma.

By Theorem 9.4, every subset of  $\mathbb{R}^n$  having  $\geq n+1$  elements is linearly independent. The lemma isolates the idea of building up a basis for a subspace one element at a time, i.e., by getting larger and larger linearly independent subsets of the subspace, a process that must eventually stop.

**Lemma 11.4.** *Let  $\{\underline{w}_1, \dots, \underline{w}_r\}$  be a linearly independent subset of  $\mathbb{R}^n$ . Let  $\underline{v}$  be an element in  $\mathbb{R}^n$  that does not belong to  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$ . Then  $\{\underline{v}\} \cup \{\underline{w}_1, \dots, \underline{w}_r\}$  is linearly independent.*

**Proof.** Suppose

$$\lambda \underline{v} + a_1 \underline{w}_1 + \cdots + a_n \underline{w}_n = 0$$

for some  $\lambda, a_1, \dots, a_n \in \mathbb{R}$ . If  $\lambda$  is non-zero, then

$$\underline{v} = (-\lambda^{-1}a_1)\underline{w}_1 + \cdots + (-\lambda^{-1}a_n)\underline{w}_n$$

which contradicts the hypothesis that  $\underline{v}$  is not in  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$ . Hence  $\lambda = 0$ . Now  $a_1 \underline{w}_1 + \cdots + a_n \underline{w}_n = 0$ . Because  $\{\underline{w}_1, \dots, \underline{w}_r\}$  is linearly independent  $a_1 = \cdots = a_n = 0$ . Hence  $\{\underline{v}, \underline{w}_1, \dots, \underline{w}_r\}$  is linearly independent.  $\square$

**Theorem 11.5.** *Every subspace of  $\mathbb{R}^n$  has a basis.*

**Proof.** Let  $W$  be a subspace of  $\mathbb{R}^n$ . We adopt the standard convention that the empty set is a basis for the subspace  $\{0\}$  so we may assume that  $W \neq 0$ . This ensures there is an element  $\underline{w} \in W$  such that  $\{\underline{w}\}$  is linearly independent.

Pick  $d$  as large as possible such that  $W$  contains a linearly independent subset having  $d$  elements. There is a largest such  $d$  because every subset of

$\mathbb{R}^n$  having  $\geq n + 1$  elements is linearly dependent (Theorem 9.4). Suppose  $\{\underline{w}_1, \dots, \underline{w}_d\}$  is a linearly independent subset of  $W$ . It is a basis for  $W$  if it spans  $W$ . However, if  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_d) \neq W$  there is a  $\underline{v}$  in  $W$  that is not in  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_d)$ . By Lemma 11.4,  $\{\underline{w}_1, \dots, \underline{w}_d, \underline{v}\}$  is linearly independent, thus contradicting the maximality of  $d$ . We conclude that no such  $\underline{v}$  can exist; i.e.,  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_d) = W$ . Hence  $\{\underline{w}_1, \dots, \underline{w}_d\}$  is a basis for  $W$ .  $\square$

**11.5. Properties of bases and spanning sets.** The following is a useful way to think of  $d = \dim W$ ;  $W$  has a linearly independent set having  $d$  elements and every subset of  $W$  having  $> d$  elements is linearly dependent.

**Corollary 11.6.** *Let  $W$  be a subspace of  $\mathbb{R}^n$ . Then  $\dim W$  is the largest number  $d$  such that  $W$  contains  $d$  linearly independent vectors.*

**Proof.** This is a consequence of the proof of Theorem 11.5.  $\square$

**Corollary 11.7.** *Let  $V$  and  $W$  be subspaces of  $\mathbb{R}^n$  and suppose  $V \subset W$ . Then*

- (1)  $\dim V < \dim W$  if and only if  $V \neq W$ ;
- (2)  $\dim V = \dim W$  if and only if  $V = W$ .

**Proof.** Let  $\{\underline{v}_1, \dots, \underline{v}_c\}$  be a basis for  $V$  and  $\{\underline{w}_1, \dots, \underline{w}_d\}$  a basis for  $W$ . Since  $\{\underline{v}_1, \dots, \underline{v}_c\}$  is a linearly independent subset of  $V$ , and hence of  $W$ ,  $c \leq d$ .

(1) ( $\Rightarrow$ ) Suppose  $c < d$ . Since all bases for  $W$  have  $d$  elements,  $\{\underline{v}_1, \dots, \underline{v}_c\}$  can't be a basis for  $W$ . Hence  $V \neq W$ .

( $\Leftarrow$ ) Suppose  $V \neq W$ . Then there is an element  $\underline{w} \in W$  that is not in  $V$ , i.e., not in  $\text{span}\{\underline{v}_1, \dots, \underline{v}_c\}$ . By Lemma 11.4,  $\{\underline{v}_1, \dots, \underline{v}_c, \underline{w}\}$  is linearly independent. It is also a subset of  $W$  and contains  $c+1$  elements so  $c+1 \leq d$ . In particular,  $c < d$ .

(2) This is the contrapositive of (1).  $\square$

**Corollary 11.8.** *Let  $W = \text{span}\{\underline{v}_1, \dots, \underline{v}_t\}$ . Some subset of  $\{\underline{v}_1, \dots, \underline{v}_t\}$  is a basis for  $W$ .*

**Proof.** If  $\{\underline{v}_1, \dots, \underline{v}_t\}$  is linearly independent it is a basis for  $W$  and there is nothing more to prove.

If  $\{\underline{v}_1, \dots, \underline{v}_t\}$  is linearly dependent, some  $\underline{v}_i$  is a linear combination of the other  $\underline{v}_j$ 's so we can remove that  $\underline{v}_i$  without changing the linear span and so get a set of  $t - 1$  vectors that spans  $W$ . By repeating this process we eventually get a linearly independent subset of  $\{\underline{v}_1, \dots, \underline{v}_t\}$  that spans  $W$  and is therefore a basis for  $W$ .  $\square$

**Theorem 11.9.** *Let  $W$  be a  $d$ -dimensional subspace of  $\mathbb{R}^n$  and let  $\mathcal{S} = \{\underline{w}_1, \dots, \underline{w}_p\}$  be a subset of  $W$ .*

- (1) *If  $p \geq d + 1$ , then  $\mathcal{S}$  is linearly dependent;*
- (2) *If  $p < d$ , then  $\mathcal{S}$  does not span  $W$ .*

- (3) If  $p = d$ , the following are equivalent:
- (a)  $\mathcal{S}$  is linearly independent;
  - (b)  $\mathcal{S}$  spans  $W$ ;
  - (c)  $\mathcal{S}$  is a basis for  $W$ .

**Proof.** Let  $V$  be the subspace spanned by  $\mathcal{S}$ . Because  $\mathcal{S} \subset W$ ,  $V \subset W$ . By Corollary 11.8, some subset of  $\mathcal{S}$  is a basis for  $V$ . Hence  $\dim V \leq p$ .

(1) is a restatement of Corollary 11.6.

(2) If  $p < d$ , then  $\dim V < \dim W$  so  $V \neq W$ .

(3) Certainly, (c) implies both (a) and (b).

(a)  $\Rightarrow$  (b) If  $\mathcal{S}$  is linearly independent it is a basis for  $V$  so  $\dim V = p = d = \dim W$ . Hence  $V = W$ .

(b)  $\Rightarrow$  (c) If  $\mathcal{S}$  spans  $W$ , then  $V = W$  so  $\dim V = \dim W$

□

**11.6. How to find a basis for a subspace.** When a subspace is described as the linear span of an explicit set of vectors, say  $W = \text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$ , the given vectors are typically linearly dependent so are *not* a basis for  $W$ . One can find a basis for  $W$  in the following way:

- if the  $\underline{w}_j$ s are row vectors form the matrix  $A$  with rows  $\underline{w}_1, \dots, \underline{w}_r$ ;  
–  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$  is equal to the row space of  $A$ ;
- if the  $\underline{w}_j$ s are column vectors form the matrix  $A$  with the rows  $\underline{w}_1^T, \dots, \underline{w}_r^T$ ;  
–  $\text{Sp}(\underline{w}_1^T, \dots, \underline{w}_r^T)$  is equal to the row space of  $A$ ;
- perform elementary row operations on  $A$  to obtain a matrix  $E$  in echelon form (in row-reduced echelon form if you want, though it isn't necessary for this);
- by Theorem 9.1, the non-zero rows of  $E$  are linearly independent;
- the non-zero rows of  $E$  are therefore a basis for the row space of  $E$ ;
- by Lemma 8.6,  $A$  and  $E$  have the same row space;
- the non-zero rows of  $E$  are therefore a basis for the row space of  $A$ ;  
– if the  $\underline{w}_j$ s are row vectors the non-zero rows of  $E$  are a basis for  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$ ;
- if the  $\underline{w}_j$ s are column vectors the transposes of the non-zero rows of  $E$  are a basis for  $\text{Sp}(\underline{w}_1, \dots, \underline{w}_r)$ .

**11.7. How to find a basis for the range of a matrix.** Proposition 8.5 proved that the range of a matrix is the linear span of its columns. A basis for  $\mathcal{R}(A)$  is therefore a basis for the column space of  $A$ . Take the transpose of  $A$  and compute  $\text{rref } A^T$ . The non-zero rows of  $\text{rref } A^T$  are a basis for the row space of  $A^T$ . The transposes of the non-zero rows in  $\text{rref } A^T$  are therefore a basis for the column space, and hence the range, of  $A$ .

Try an example!

11.8. **Rank + Nullity.** The nullity of a matrix  $A$  is the dimension of its null space.

As a consequence of the following important result we shall see that the dimension of the range of  $A$  is equal to the rank of  $A$  (Corollary 11.11). In many books the rank of a matrix is *defined* to be the dimension of its range. Accordingly, the formula in the next result is often stated as

$$\text{nullity } A + \text{rank } A = n.$$

**Theorem 11.10.** *Let  $A$  be an  $m \times n$  matrix. Then*

$$\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = n.$$

**Proof.** Pick bases  $\{\underline{u}_1, \dots, \underline{u}_p\}$  for  $\mathcal{N}(A)$  and  $\{\underline{w}_1, \dots, \underline{w}_q\}$  for  $\mathcal{R}(A)$ . Remember that  $\mathcal{N}(A) \subset \mathbb{R}^n$  and  $\mathcal{R}(A) \subset \mathbb{R}^m$ .

There are vectors  $\underline{v}_1, \dots, \underline{v}_q$  in  $\mathbb{R}^n$  such that  $\underline{w}_i = A\underline{v}_i$  for all  $i$ . We will show that the set

$$\mathcal{B} := \{\underline{u}_1, \dots, \underline{u}_p, \underline{v}_1, \dots, \underline{v}_q\}$$

is a basis for  $\mathbb{R}^n$ .

Let  $\underline{x} \in \mathbb{R}^n$ . Then  $A\underline{x} = a_1\underline{w}_1 + \dots + a_q\underline{w}_q$  for some  $a_1, \dots, a_q \in \mathbb{R}$ . Now

$$A(\underline{x} - a_1\underline{v}_1 - \dots - a_q\underline{v}_q) = A\underline{x} - a_1A\underline{v}_1 - \dots - a_qA\underline{v}_q = 0$$

so  $\underline{x} - a_1\underline{v}_1 - \dots - a_q\underline{v}_q$  is in the null space of  $A$  and therefore

$$\underline{x} - a_1\underline{v}_1 - \dots - a_q\underline{v}_q = b_1\underline{u}_1 + \dots + b_p\underline{u}_p$$

for some  $b_1, \dots, b_p \in \mathbb{R}$ . In particular,

$$\underline{x} = a_1\underline{v}_1 + \dots + a_q\underline{v}_q + b_1\underline{u}_1 + \dots + b_p\underline{u}_p$$

so  $\mathcal{B}$  spans  $\mathbb{R}^n$ .

We will now show that  $\mathcal{B}$  is linearly independent. If

$$c_1\underline{v}_1 + \dots + c_q\underline{v}_q + d_1\underline{u}_1 + \dots + d_p\underline{u}_p = 0,$$

then

$$\begin{aligned} 0 &= A(c_1\underline{v}_1 + \dots + c_q\underline{v}_q + d_1\underline{u}_1 + \dots + d_p\underline{u}_p) \\ &= c_1A\underline{v}_1 + \dots + c_qA\underline{v}_q + d_1A\underline{u}_1 + \dots + d_pA\underline{u}_p \\ &= c_1\underline{w}_1 + \dots + c_q\underline{w}_q + 0 + \dots + 0. \end{aligned}$$

But  $\{\underline{w}_1, \dots, \underline{w}_q\}$  is linearly independent so

$$c_1 = \dots = c_q = 0.$$

Therefore

$$d_1\underline{u}_1 + \dots + d_p\underline{u}_p = 0.$$

But  $\{\underline{u}_1, \dots, \underline{u}_p\}$  is linearly independent so

$$d_1 = \dots = d_p = 0.$$

Hence  $\mathcal{B}$  is linearly independent, and therefore a basis for  $\mathbb{R}^n$ . But  $\mathcal{B}$  has  $p + q$  elements so  $p + q = n$ , as required.  $\square$

We now give the long-promised new interpretation of rank  $A$ .

**Corollary 11.11.** *Let  $A$  be an  $m \times n$  matrix. Then*

$$\text{rank } A = \dim \mathcal{R}(A).$$

**Proof.** In section 5.7, we defined  $\text{rank } A$  to be the number of non-zero rows in  $\text{rref } A$ , the row-reduced echelon form of  $A$ . From the discussion in that section we obtain

$$\begin{aligned} n - \text{rank } A &= \text{the number of independent variables} \\ &\quad \text{for the equation } A\underline{x} = 0 \\ &= \dim\{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = 0\} \\ &= \dim \mathcal{N}(A). \end{aligned}$$

But  $\dim \mathcal{N}(A) = n - \dim \mathcal{R}(A)$  by Theorem 11.10, so  $\text{rank } A = \dim \mathcal{R}(A)$ .  
□

**Corollary 11.12.** *Let  $A$  be an  $m \times n$  matrix. Then*

$$\text{nullity } A + \text{rank } A = n.$$

**Corollary 11.13.** *The row space and column spaces of a matrix have the same dimension.*

**Proof.** Let  $A$  be an  $m \times n$  matrix. By Proposition 8.5, the column space of  $A$  is equal to  $\mathcal{R}(A)$ . The row space of  $A$  is equal to the row space of  $\text{rref } A$  so the dimension of the row space of  $A$  is  $\text{rank } A$ . But  $\text{rank } A = \dim \mathcal{R}(A)$ . The result follows. □

**Corollary 11.14.** *Let  $A$  be an  $m \times n$  matrix. Then*

$$\text{rank } A = \text{rank } A^T.$$

**Proof.** The rank of  $A^T$  is equal to the dimension of its row space which is equal to the dimension of the column space of  $A$  which is equal to the dimension of the row space of  $A$  which is equal to the rank of  $A$ . □

We will now give some examples that show how to use Theorem 11.10.

11.8.1.  $3 \times 3$  examples. There are four possibilities for the rank and nullity of a  $3 \times 3$  matrix.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

11.8.2. *Planes in  $\mathbb{R}^4$ .* Let  $V$  and  $V'$  be two dimensional subspaces of  $\mathbb{R}^4$ . The intersection of  $V$  and  $V'$  is a subspace of  $\mathbb{R}^4$  of dimension  $\leq 2$  so there are three possibilities: the intersection is 2-dimensional if and only if  $V = V'$ ; if the intersection is 1-dimensional, then  $V$  and  $V'$  have a line in common; if  $V \cap V' = \{\mathbf{0}\}$ , then  $V + V' = \mathbb{R}^4$ .

Let  $P$  and  $P'$  be planes in  $\mathbb{R}^4$ , i.e., translations of 2-dimensional subspaces. We say that  $P$  and  $P'$  are parallel if they are translates of the same 2-dimensional subspace. In that case,  $P \cap P' = \phi$  but that is not the only reason two planes can fail to meet.

11.8.3. *3-planes in  $\mathbb{R}^5$ .* We call a 3-dimensional subspace of  $\mathbb{R}^n$  a 3-plane. Let  $V$  and  $V'$  be 3-planes

11.8.4. *Three formulas.* There are many examples in mathematics where the same idea pops up in very different settings. Here is an example:

- If  $U$  and  $V$  are subspaces of  $\mathbb{R}^n$ , then

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$$

- If  $A$  and  $B$  are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

- Let  $A$  and  $B$  denote events; let  $A \cup B$  denote the event *either A or B occurs*, and let  $A \cap B$  denote the event *both A and B occur*. If  $P[\text{event}]$  denotes the probability of an event, then

$$P[A \cup B] = P[A] + P[B] - P[A \cap B].$$

**Lemma 11.15.** *Let  $W$  and  $W'$  be subspaces of  $\mathbb{R}^n$  and suppose that  $W \subset W'$ . If  $\underline{w}_1, \dots, \underline{w}_m$  is a basis for  $W$ , then there is a basis for  $W'$  that contains  $\underline{w}_1, \dots, \underline{w}_m$ .*

**Proof.** We argue by induction on  $\dim W' - \dim W$ . Suppose  $\dim W' = 1 + \dim W$ . Let  $\underline{w}' \in W' - W$ . Then  $\{\underline{w}', \underline{w}_1, \dots, \underline{w}_m\}$  is linearly independent, and hence a basis for  $W'$ . etc.  $\square$

**Proposition 11.16.** *If  $U$  and  $V$  are subspaces of  $\mathbb{R}^n$ , then*

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$$

**Proof.** Let  $\mathcal{B}$  be a basis for  $U \cap V$ . Then there is a basis  $\mathcal{B} \sqcup \mathcal{B}_U$  for  $U$  and a basis  $\mathcal{B} \sqcup \mathcal{B}_V$  for  $V$ . We will now show that  $\mathcal{B} \sqcup \mathcal{B}_U \sqcup \mathcal{B}_V$  is a basis for  $U + V$ .

Hence

$$\dim(U + V) = |\mathcal{B}| + |\mathcal{B}_U| + |\mathcal{B}_V|.$$

But  $|\mathcal{B}| = \dim U \cap V$ ,  $|\mathcal{B}| + |\mathcal{B}_U| = \dim U$ , and  $|\mathcal{B}| + |\mathcal{B}_V| = \dim V$ , so

$$|\mathcal{B}| + |\mathcal{B}_U| + |\mathcal{B}_V| = \dim U + \dim V - \dim U \cap V$$

thus proving the result.  $\square$

**11.9. How to compute the null space and range of a matrix.** Suppose you are given a matrix  $A$  and asked to compute its null space and range. What would constitute an answer to that question? One answer would be to provide a basis for its null space and a basis for its range.

By definition, the null space of  $A$  is the set of solutions to the equation  $A\underline{x} = 0$  so we may use the methods in sections 5.7 and 4.8 to find the solutions. Do elementary row operations to find  $\underline{rref} A$ . Write  $E$  for  $\underline{rref} A$ . Since the equations  $A\underline{x} = 0$  and  $E\underline{x} = 0$  have the same solutions,

$$\mathcal{N}(A) = \mathcal{N}(E).$$

It is easy to write down the null space for  $E$ . For example, suppose If

$$(11-4) \quad E = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 3 & 4 \\ 0 & 1 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

the dependent variables are  $x_1, x_2, x_4,$  and  $x_5$ . The independent variables  $x_3, x_6,$  and  $x_7,$  may take any values and then the dependent variables are determined by the equations

$$\begin{aligned} x_1 &= -2x_3 - 3x_6 - 4x_7 \\ x_2 &= -3x_3 - 4x_6 \\ x_4 &= 2x_6 + x_7 \\ x_5 &= -x_6 + 2x_7. \end{aligned}$$

Now set any one of the independent variables to 1 and set the others to zero. This gives solutions

$$\begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -4 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

These are elements of the null space and are obviously linearly independent. They span the null space.

MORE TO SAY/DO...basis for null space

**11.10. Orthogonal and orthonormal bases.** Two non-zero vectors  $\underline{u}$  and  $\underline{v}$  are said to be orthogonal if  $\underline{u}^T \underline{v} = 0$ , i.e., if their dot product is zero. Thus, the word *orthogonal* is synonymous with *perpendicular*. Notice that  $\underline{u}^T \underline{v} = \underline{v}^T \underline{u}$  for any pair of vectors  $\underline{u}$  and  $\underline{v}$ .

A set of non-zero vectors  $\{\underline{v}_1, \dots, \underline{v}_d\}$  is said to be orthogonal if  $\underline{v}_i$  is orthogonal to  $\underline{v}_j$  whenever  $i \neq j$ .

**Lemma 11.17.** *An orthogonal set of vectors is linearly independent.*

**Proof.** Let  $\{\underline{v}_1, \dots, \underline{v}_d\}$  be an orthogonal set of vectors and suppose that  $a_1\underline{v}_1 + \dots + a_d\underline{v}_d = 0$ . Then

$$0 = \underline{v}_i^T(a_1\underline{v}_1 + \dots + a_d\underline{v}_d).$$

However, when we multiply this out all but one term will vanish because  $\underline{v}_i^T \underline{v}_j = 0$  when  $j \neq i$ . The non-vanishing term is  $a_i \underline{v}_i^T$ , so  $a_i \underline{v}_i^T = 0$ . But  $\underline{v}_i \neq 0$ , so  $a_i = 0$ . Thus all the  $a_i$ s are zero and we conclude that  $\{\underline{v}_1, \dots, \underline{v}_d\}$  is linearly independent.  $\square$

An **orthogonal basis** is just that, a basis consisting of vectors that are orthogonal to one another. If in addition each vector has length one we call it an **orthonormal basis**.

The standard basis  $\{\underline{e}_1, \dots, \underline{e}_n\}$  for  $\mathbb{R}^n$  is an orthonormal basis.

Suppose one has a basis  $\{\underline{v}_1, \dots, \underline{v}_d\}$  for a subspace  $W$  of  $\mathbb{R}^n$ , and a vector  $\underline{w}$  in  $W$ . Then  $\underline{w} = a_1\underline{v}_1 + \dots + a_d\underline{v}_d$  for some numbers  $a_1, \dots, a_d$ . It is usually a tedious chore to find the  $a_i$ s; one has to solve a system of linear equations. However, if  $\{\underline{v}_1, \dots, \underline{v}_d\}$  is an orthonormal basis for  $W$  it is trivial to find the  $a_i$ s.

**Proposition 11.18.** *Suppose  $\{\underline{v}_1, \dots, \underline{v}_d\}$  is an orthonormal basis for  $W$ . If  $\underline{w} \in W$ , then*

$$\underline{w} = (\underline{w}^T \underline{v}_1)\underline{v}_1 + \dots + (\underline{w}^T \underline{v}_d)\underline{v}_d.$$

**Proof.** Notice first that each  $(\underline{w}^T \underline{v}_i)$  is a number!

We know there are numbers  $a_1, \dots, a_d$  such that  $\underline{w} = a_1\underline{v}_1 + \dots + a_d\underline{v}_d$ . It follows that

$$\begin{aligned} \underline{w}^T \underline{v}_i &= (a_1\underline{v}_1 + \dots + a_d\underline{v}_d)^T \underline{v}_i \\ &= (a_1\underline{v}_1^T + \dots + a_d\underline{v}_d^T) \underline{v}_i \\ &= a_1(\underline{v}_1^T \underline{v}_i) + \dots + a_d(\underline{v}_d^T \underline{v}_i). \end{aligned}$$

Because  $\{\underline{v}_1, \dots, \underline{v}_d\}$  is an orthonormal set of vectors  $\underline{v}_j^T \underline{v}_i = 0$  when  $j \neq i$  and  $\underline{v}_i^T \underline{v}_i = 1$ . Hence

$$\underline{w}^T \underline{v}_i = a_1(\underline{v}_1^T \underline{v}_i) + \dots + a_d(\underline{v}_d^T \underline{v}_i) = a_i.$$

The result now follows.  $\square$

*That* is the great virtue of orthonormal bases. Unfortunately, one is not always fortunate enough to be presented with an orthonormal basis. However, there is a standard mechanism, the Gram-Schmidt process, that will take any basis for  $W$  and produce from it an orthonormal basis for  $W$ . We turn to that next.

**11.11. The Gram-Schmidt process.** Suppose  $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_s\}$  is a basis for a subspace  $W$ . The Gram-Schmidt process is a method for constructing from  $\mathcal{B}$  an *orthogonal* basis  $\{\underline{u}_1, \dots, \underline{u}_s\}$  for  $W$ . The idea is relatively simple. First we take  $\underline{u}_1 = \underline{v}_1$ . The set  $\{\underline{u}_1\}$  is certainly an orthogonal set. Next we will carefully construct  $\underline{u}_2$  so that  $\text{Sp}(\underline{u}_1, \underline{u}_2) = \text{Sp}(\underline{v}_1, \underline{v}_2)$  and  $\underline{u}_2$  is orthogonal to  $\underline{u}_1$ ; this will give an orthogonal basis  $\{\underline{u}_1, \underline{u}_2\}$  for  $\text{Sp}(\underline{v}_1, \underline{v}_2)$ . We will then proceed inductively: suppose we have obtained an orthogonal basis  $\{\underline{u}_1, \dots, \underline{u}_r\}$  for  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_r)$ ; then we carefully construct  $\underline{u}_{r+1}$  so that  $\text{Sp}(\underline{u}_1, \dots, \underline{u}_{r+1}) = \text{Sp}(\underline{v}_1, \dots, \underline{v}_{r+1})$  and  $\underline{u}_{r+1}$  is orthogonal to  $\underline{u}_1, \dots, \underline{u}_r$ ; this will give an orthogonal basis  $\{\underline{u}_1, \dots, \underline{u}_{r+1}\}$  for  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_{r+1})$ .

Define

$$\begin{aligned}\underline{u}_1 &= \underline{v}_1 \\ \underline{u}_2 &= \underline{v}_2 - \frac{\underline{u}_1 \cdot \underline{v}_2}{\|\underline{u}_1\|^2} \underline{u}_1 \\ \underline{u}_3 &= \underline{v}_3 - \frac{\underline{u}_1 \cdot \underline{v}_3}{\|\underline{u}_1\|^2} \underline{u}_1 - \frac{\underline{u}_2 \cdot \underline{v}_3}{\|\underline{u}_2\|^2} \underline{u}_2\end{aligned}$$

etc.

**Theorem 11.19.** Suppose  $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_s\}$  is a basis for a subspace  $W$ . Then  $\{\underline{u}_1, \dots, \underline{u}_s\}$ , where the  $\underline{u}_i$ s are defined as above, is an orthogonal basis for  $W$ .

**Proof.** We must show that  $\underline{u}_i \cdot \underline{u}_j = 0$  when  $i \neq j$ . The first case is  $i = 2$  and  $j = 1$ . We get

$$\underline{u}_2 \cdot \underline{u}_1 = \underline{v}_2 \cdot \underline{u}_1 - \frac{\underline{u}_1 \cdot \underline{v}_2}{\|\underline{u}_1\|^2} \underline{u}_1 \cdot \underline{u}_1.$$

But  $\underline{u}_1 \cdot \underline{u}_1 = \|\underline{u}_1\|^2$  so those factors cancel in the second term to leave

$$\underline{u}_2 \cdot \underline{u}_1 = \underline{v}_2 \cdot \underline{u}_1 - \underline{u}_1 \cdot \underline{v}_2$$

which is zero because  $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ . Hence  $\underline{u}_2$  is orthogonal to  $\underline{u}_1$ .

The next case is  $i = 3$  and  $j = 1$ . We get

$$\underline{u}_3 \cdot \underline{u}_1 = \underline{v}_3 \cdot \underline{u}_1 - \frac{\underline{u}_1 \cdot \underline{v}_3}{\|\underline{u}_1\|^2} \underline{u}_1 \cdot \underline{u}_1 - \frac{\underline{u}_2 \cdot \underline{v}_3}{\|\underline{u}_2\|^2} \underline{u}_2 \cdot \underline{u}_1.$$

The last term is zero because  $\underline{u}_2 \cdot \underline{u}_1 = 0$ . The same argument as in the previous paragraph shows that the first two terms cancel so we get  $\underline{u}_3 \cdot \underline{u}_1 = 0$ .

Carry on like this to get  $\underline{u}_i \cdot \underline{u}_1$  for all  $i \geq 2$ .

The next case is to take  $i = 3$  and  $j = 2$ . We get

$$\underline{u}_3 \cdot \underline{u}_2 = \underline{v}_3 \cdot \underline{u}_2 - \frac{\underline{u}_1 \cdot \underline{v}_3}{\|\underline{u}_1\|^2} \underline{u}_1 \cdot \underline{u}_2 - \frac{\underline{u}_2 \cdot \underline{v}_3}{\|\underline{u}_2\|^2} \underline{u}_2 \cdot \underline{u}_2.$$

The second term on the right-hand-side is zero because  $\underline{u}_1 \cdot \underline{u}_2 = 0$ ; in the third term,  $\underline{u}_2 \cdot \underline{u}_2 = \|\underline{u}_2\|^2$  so those factors cancel to leave

$$\underline{u}_3 \cdot \underline{u}_2 = \underline{v}_3 \cdot \underline{u}_2 - \underline{u}_2 \cdot \underline{v}_3$$

which is zero. Hence  $\underline{u}_3$  is orthogonal to  $\underline{u}_2$ .

Now do the case  $i = 4$  and  $j = 2$ , and carry on like this to get  $\underline{u}_i \cdot \underline{u}_2 = 0$  for all  $i \geq 3$ . Then start again with  $i = 4$  and  $j = 3$ , et cetera, et cetera! Eventually, we prove that  $\underline{u}_i \cdot \underline{u}_j = 0$  for all  $i \neq j$ .

We still need to prove that  $\{\underline{u}_1, \dots, \underline{u}_s\}$  is a basis for  $W$ . Because the  $\underline{u}_i$ s are an orthogonal set they are linearly independent. Their linear span is therefore a vector space of dimension  $s$ .

We can rewrite the formula for  $\underline{v}_i$  by putting  $\underline{v}_i$  on one side and so obtain

$$\underline{v}_i = \underline{u}_i + \text{a linear combination of } \underline{u}_1, \dots, \underline{u}_{i-1}.$$

Hence  $\underline{v}_i \in \text{Sp}(\underline{u}_1, \dots, \underline{u}_s)$ . Hence  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_s) \subset \text{Sp}(\underline{u}_1, \dots, \underline{u}_s)$ ; but each of these linear spans has dimension  $s$  so they must be equal. However,  $\text{Sp}(\underline{v}_1, \dots, \underline{v}_s) = W$  so we conclude that  $\text{Sp}(\underline{u}_1, \dots, \underline{u}_s) = W$ , as required.  $\square$

Having obtained an orthogonal basis  $\{\underline{u}_1, \dots, \underline{u}_s\}$  it is a trivial matter to obtain an orthonormal basis: simply replace each basis vector by a multiple of itself that have length one:

$$\frac{\underline{u}_1}{\|\underline{u}_1\|}, \frac{\underline{u}_2}{\|\underline{u}_2\|}, \dots, \frac{\underline{u}_s}{\|\underline{u}_s\|}$$

is an orthonormal basis.

**11.12. Gazing into the distance: Fourier series.** As mentioned in section ??, many of the vector spaces one meets in mathematics are infinite dimensional. Typically, the elements of such vector spaces are functions.

**11.12.1. Polynomials.** For example, at high school one considers polynomial functions of arbitrary degree. If  $f(x)$  and  $g(x)$  are polynomials so is  $f(x) + g(x)$ ; so is  $\lambda f(x)$  for all  $\lambda \in \mathbb{R}$ ; the zero polynomial has the wonderful property that  $f(x) + 0 = 0 + f(x) = f(x)$ . Of course, other properties like associativity and commutativity of addition hold, and the distributive rules  $\lambda(f(x) + g(x)) = \lambda f(x) + \lambda g(x)$  and  $(\lambda + \mu)f(x) = \lambda f(x) + \mu f(x)$  hold. Every polynomial is a finite linear combination of the powers  $1, x, x^2, \dots$  of  $x$ . The powers of  $x$  are linearly independent because the only way  $a_0 + a_1x + \dots + a_nx^n$  can be the zero polynomial is if all the  $a_i$ s are zero. The powers of  $x$  form a basis for the space of polynomials. We usually write  $\mathbb{R}[x]$  for the set of all polynomials (with real coefficients!). It is an infinite dimensional vector space.

**11.12.2. Other spaces of functions.** One need not restrict attention to polynomials. Let  $C^1(\mathbb{R})$  denote the set of all differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . This is also a vector space: you can add 'em, multiply 'em by scalars, etc, and the usual associative, commutative and distributive rules apply. Likewise,  $C(\mathbb{R})$ , the set of all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a vector space.

MORE to say...

## 12. LINEAR TRANSFORMATIONS

**12.1. First observations.** Let  $V$  be a subspace of  $\mathbb{R}^n$  and  $W$  a subspace of  $\mathbb{R}^m$ . A linear transformation from  $V$  to  $W$  is a function  $F : V \rightarrow W$  such that

- (1)  $F(\underline{u} + \underline{v}) = F(\underline{u}) + F(\underline{v})$  for all  $\underline{u}, \underline{v} \in V$ , and
- (2)  $F(a\underline{v}) = aF(\underline{v})$  for all  $a \in \mathbb{R}$  and all  $\underline{v} \in V$ .

It is common to combine (1) and (2) into a single condition:  $F$  is a linear transformation if and only if

$$F(a\underline{u} + b\underline{v}) = aF(\underline{u}) + bF(\underline{v})$$

for all  $a, b \in \mathbb{R}$  and all  $\underline{u}, \underline{v} \in V$ .

**12.1.1. Equality of linear transformations.** Two linear transformations from  $V$  to  $W$  are equal if they take the same value at all points of  $V$ . This is the same way we define equality of functions:  $f$  and  $g$  are equal if they take the same values everywhere.

**12.1.2.  $T$  respects linear combinations.** Let  $T : V \rightarrow W$  be a linear transformation. If  $\underline{v}_1, \underline{v}_2, \underline{v}_3 \in V$  and  $a_1, a_2, a_3 \in \mathbb{R}$ , then

$$\begin{aligned} T(a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3) &= T((a_1\underline{v}_1 + a_2\underline{v}_2) + a_3\underline{v}_3) \\ &= T(a_1\underline{v}_1 + a_2\underline{v}_2) + a_3T(\underline{v}_3) \\ &= a_1T(\underline{v}_1) + a_2T(\underline{v}_2) + a_3T(\underline{v}_3) \end{aligned}$$

An induction argument shows that if  $\underline{v}_1, \dots, \underline{v}_p \in V$  and  $a_1, \dots, a_p \in \mathbb{R}$ , then

$$(12-1) \quad T(a_1\underline{v}_1 + \dots + a_p\underline{v}_p) = a_1T(\underline{v}_1) + \dots + a_pT(\underline{v}_p).$$

**12.1.3.  $T$  is determined by its action on a basis.** There are two ways to formalize the statement that a linear transformation is completely determined by its values on a basis.

Let  $T : V \rightarrow W$  be a linear transformation and suppose  $\{\underline{v}_1, \dots, \underline{v}_p\}$  is a basis for  $V$ .

- (1) If you know the values of  $T(\underline{v}_1), \dots, T(\underline{v}_p)$  you can compute its value at every point of  $V$  because if  $\underline{v} \in V$  there are unique numbers  $a_1, \dots, a_p \in \mathbb{R}$  such that  $\underline{v} = a_1\underline{v}_1 + \dots + a_p\underline{v}_p$  and it now follows from (12-1) that  $T(\underline{v}) = a_1T(\underline{v}_1) + \dots + a_pT(\underline{v}_p)$ .
- (2) If  $S : V \rightarrow W$  is another linear transformation such that  $S(\underline{v}_i) = T(\underline{v}_i)$  for  $i = 1, \dots, p$ , then  $S = T$ . To see this use (12-1) twice: once for  $S$  and once for  $T$ .

12.1.4. *How do we tell if a function  $T : V \rightarrow W$  is a linear transformation?* As with all such questions, the answer is to check whether  $T$  satisfies the definition or not.

PAUL - examples,

The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$f(x, y) = \begin{cases} (x + y, x - y) & \text{if } x \geq 0 \text{ and } y \geq 0 \\ (x - y, x + y) & \text{otherwise} \end{cases}$$

is not a linear transformation because \_\_\_\_\_

12.1.5. *Linear transformations send 0 to 0.* Suppose  $T : V \rightarrow W$  is a linear transformation. Let  $\underline{0}$  be the zero vector in  $V$ . Then

$$T(\underline{0}) = T(\underline{0} + \underline{0}) = T(\underline{0}) + T(\underline{0}).$$

Subtracting  $T(\underline{0})$  from both sides of the equation it follows that  $T(\underline{0})$  is the zero vector in  $W$ .

We don't usually bother to use different notation for the zero vectors in  $V$  and  $W$  so we write  $T(\underline{0}) = \underline{0}$  safe in the knowledge that the only way this equality can make sense is if the  $\underline{0}$  to the left of the  $=$  sign is in  $V$  and the  $\underline{0}$  to the right of the  $=$  sign is in  $W$ .

12.1.6. *The zero transformation.* The function  $V \rightarrow W$  that sends every  $\underline{x} \in V$  to the zero vector in  $W$  is a linear transformation (check!). We call this the *zero linear transformation* and denote it by  $0$  if necessary.

12.1.7. *The identity transformation.* The function  $V \rightarrow V$  that sends every  $\underline{x} \in V$  to itself is a linear transformation (check!). We call it the *identity transformation of  $V$*  and denote it by  $\text{id}_V$ .

12.1.8. *A composition of linear transformations is a linear transformation.* If  $S : V \rightarrow W$  and  $T : U \rightarrow V$  are linear transformations, their *composition*, denoted by  $ST$  or  $S \circ T$ , is defined in the same way as the composition of any two functions, namely

$$(ST)(\underline{u}) = S(T(\underline{u})) \quad \text{or} \quad (S \circ T)(\underline{u}) = S(T(\underline{u})).$$

This makes sense because  $T(\underline{u}) \in V$  and we can apply  $S$  to elements of  $V$ . Thus

$$ST : U \rightarrow W.$$

To see that  $ST$  is really a linear combination suppose that  $\underline{u}$  and  $\underline{u}'$  belong to  $U$  and that  $a, a' \in \mathbb{R}$ . Then

$$\begin{aligned} (ST)(a\underline{u} + a'\underline{u}') &= S(T(a\underline{u} + a'\underline{u}')) && \text{by definition of } ST \\ &= S(aT(\underline{u}) + a'T(\underline{u}')) && \text{because } T \text{ is a linear transformation} \\ &= aS(T(\underline{u})) + a'S(T(\underline{u}')) && \text{because } S \text{ is a linear transformation} \\ &= a(ST)(\underline{u}) + a'(ST)(\underline{u}') && \text{by definition of } ST \end{aligned}$$

so  $ST$  is a linear transformation.

12.1.9. *Linear combinations of linear transformations.* If  $S$  and  $T$  are linear transformations from  $U$  to  $V$  and  $a, b \in \mathbb{R}$ , we define the function  $aS + bT$  by

$$(aS + bT)(\underline{x}) := aS(\underline{x}) + bT(\underline{x}).$$

It is easy to check that  $aS + bT$  is a linear transformation from  $U$  to  $V$  too. We call it a linear combination of  $S$  and  $T$ . Of course the idea extends to linear combinations of larger collections of linear transformations.

12.2. **Linear transformations and matrices.** Let  $A$  be an  $m \times n$  matrix and define  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by  $T(\underline{x}) := A\underline{x}$ . Then  $T$  is a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . The next theorem says that *every* linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is of this form.

**Theorem 12.1.** *Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then there is a unique  $m \times n$  matrix  $A$  such that  $T(\underline{x}) = A\underline{x}$  for all  $\underline{x} \in \mathbb{R}^n$ .*

**Proof.** Let  $A$  be the  $m \times n$  matrix whose  $j^{\text{th}}$  column is  $\underline{A}_j := T(\underline{e}_j)$ . If  $\underline{x} = (x_1, \dots, x_n)^T$  is an arbitrary element of  $\mathbb{R}^n$ , then

$$\begin{aligned} T(\underline{x}) &= T(x_1\underline{e}_1 + \dots + x_n\underline{e}_n) \\ &= x_1T(\underline{e}_1) + \dots + x_nT(\underline{e}_n) \\ &= x_1\underline{A}_1 + \dots + x_n\underline{A}_n \\ &= A\underline{x} \end{aligned}$$

as required. □

In the context of the previous theorem, we sometimes say that the matrix  $A$  represents the linear transformation  $T$ .

Suppose  $T : \mathbb{R}^k \rightarrow \mathbb{R}^m$  and  $S : \mathbb{R}^m \rightarrow \mathbb{R}^n$  are linear transformations and  $A$  and  $B$  matrices such that  $T(\underline{x}) = A\underline{x}$  for all  $\underline{x} \in \mathbb{R}^k$  and  $S(\underline{y}) = B\underline{y}$  for all  $\underline{y} \in \mathbb{R}^m$ , then  $(S \circ T)(\underline{x}) = BA\underline{x}$  for all  $\underline{x} \in \mathbb{R}^k$ . Thus

*multiplication of matrices corresponds to  
composition of linear transformations*

or

*multiplication of matrices is defined so that the matrix  
representing the composition of two linear transformations  
is the product of the matrices representing each linear transformation.*

This is of great importance. It explains why the multiplication of matrices is defined as it is. That last sentence bears repeating. The fact that matrix multiplication has real meaning, as opposed to something that was defined by a committee that just wanted to be able to "multiply" matrices, is due to the fact that matrix multiplication corresponds to that most natural of operations, composition of functions, first do one, then the other.

Mathematics is not about inventing definitions just for the heck of it and seeing what one can deduce from them. Every definition is motivated by some problem, question, or naturally occurring object or feature. Linear

transformations and compositions of them are more fundamental than matrices and their product. Matrices are tools for doing calculations with linear transformations. They sometimes convey information that is concealed in other presentations of linear transformations.

We introduce you to matrices before linear transformations because they are more concrete and down to earth, not because they are more important than linear transformations.

**12.2.1. Isomorphisms.** Suppose that  $S : V \rightarrow U$  and  $T : U \rightarrow V$  are linear transformations. Then  $ST : U \rightarrow U$  and  $TS : V \rightarrow V$ . If  $ST = \text{id}_U$  and  $TS = \text{id}_V$  we say that  $S$  and  $T$  are inverses to one another. We also say that  $S$  and  $T$  are isomorphisms and that  $U$  and  $V$  are isomorphic vector spaces.

**Theorem 12.2.** *Let  $V$  be a subspace of  $\mathbb{R}^n$ . If  $\dim V = d$ , then  $V$  is isomorphic to  $\mathbb{R}^d$ .*

**Proof.** Let  $\underline{v}_1, \dots, \underline{v}_d$  be a basis for  $V$ . Define  $T : \mathbb{R}^d \rightarrow V$  by

$$T(a_1, \dots, a_d) := a_1 \underline{v}_1 + \dots + a_d \underline{v}_d.$$

MORE TO DO

$S : V \rightarrow \mathbb{R}^d$  by

□

### 12.3. How to find the matrix representing a linear transformation.

The solution to this problem can be seen from the proof of Theorem 12.1. There, the matrix  $A$  such that  $T(\underline{x}) = A\underline{x}$  for all  $\underline{x} \in \mathbb{R}^n$  is constructed one column at a time: the  $i^{\text{th}}$  column of  $A$  is  $T(\underline{e}_i)$ .

PAUL-explicit example.

### 12.4. Invertible matrices and invertible linear transformations.

**12.5. How to find the formula for a linear transformation.** First, what do we mean by a formula for a linear transformation? We mean the same thing as when we speak of a formula for one of the functions you met in calculus—the only difference is the appearance of the formula. For example, in calculus  $f(x) = x \sin x$  is a formula for the function  $f$ ; similarly,  $g(x) = |x|$  is a formula for the function  $g$ ; we can also give the formula for  $g$  by

$$g(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

The expression

$$T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a - b \\ a \\ a + b + c \\ 3c - 2b \end{pmatrix}$$

is a formula for a linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ .

Although a linear transformation may not be given by an explicit formula, we might want one in order to do calculations. For example, rotation

about the origin in the clockwise direction by an angle of  $45^\circ$  is a linear transformation, but that description is not a formula for it. A formula for it is given by

$$T \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a+b \\ b-a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Suppose  $\{v_1, \dots, v_n\}$  is a basis for  $\mathbb{R}^n$  and you are told the values of  $T(v_1), \dots, T(v_n)$ . There is a simple procedure to find the  $m \times n$  matrix  $A$  such that  $T(\underline{x}) = A\underline{x}$  for all  $\underline{x} \in \mathbb{R}^n$ . The idea is buried in the proof of Theorem 12.1.

MORE TO DO

**12.6. Rotations in the plane.** Among the more important linear transformations  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  are the rotations. Given an angle  $\theta$  we write  $T_\theta$  for the linear transformation that rotates a point by an angle of  $\theta$  radians in the counterclockwise direction.

Before we obtain a formula for  $T_\theta$  some things should be clear. First,  $T_\theta T_\psi = T_{\theta+\psi}$ . Second,  $T_0 = \text{id}$ . Third,  $T_{2n\pi} = \text{id}_{\mathbb{R}^2}$  and  $T_{(2n+1)\pi} = -\text{id}_{\mathbb{R}^2}$  for all  $n \in \mathbb{Z}$ . Fourth, rotation in the counterclockwise direction by an angle of  $-\theta$  is the same as rotation in the *clockwise* direction by an angle of  $\theta$  so  $T_\theta$  has an inverse, namely  $T_{-\theta}$ .

Let's write  $A_\theta$  for the unique  $2 \times 2$  matrix such that  $T_\theta \underline{x} = A_\theta \underline{x}$  for all  $\underline{x} \in \mathbb{R}^2$ . The first column of  $A_\theta$  is given by  $T_\theta(e_1)$  and the second column of  $A_\theta$  is given by  $T_\theta(e_2)$ . Elementary trigonometry (draw the diagrams and check!) then gives

$$(12-2) \quad A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

There are many interesting aspects to this:

- if you have forgotten your formulas for  $\sin(\theta + \psi)$  and  $\cos(\theta + \psi)$  you can recover them by using the fact that  $A_\theta A_\psi = A_{\theta+\psi}$  and computing the product on the left;
- the determinant of  $A_\theta$  is 1 because  $\cos^2 \theta + \sin^2 \theta = 1$ ;
- $A_\theta^{-1} = A_{-\theta}$  and using the formula for the inverse of a  $2 \times 2$  matrix in Theorem 10.3 you can recover the formulae for  $\sin(-\theta)$  and  $\cos(-\theta)$  if you have forgotten them.

**12.7. Reflections in  $\mathbb{R}^2$ .** Let  $L$  be the line in  $\mathbb{R}^2$  through the origin and a non-zero point  $\begin{pmatrix} a \\ b \end{pmatrix}$ . The reflection in  $L$  is the linear transformation that sends  $\begin{pmatrix} a \\ b \end{pmatrix}$  to itself, i.e., it fixes every point on  $L$ , and sends the point  $\begin{pmatrix} -b \\ a \end{pmatrix}$ ,

which is orthogonal to  $L$ , to  $\begin{pmatrix} b \\ -a \end{pmatrix}$ .<sup>10</sup> Let  $A$  be the matrix implementing this linear transformation.

Since

$$\underline{e}_1 = \frac{1}{a^2 + b^2} \left( a \begin{pmatrix} a \\ b \end{pmatrix} - b \begin{pmatrix} -b \\ a \end{pmatrix} \right)$$

and

$$\underline{e}_2 = \frac{1}{a^2 + b^2} \left( b \begin{pmatrix} a \\ b \end{pmatrix} + a \begin{pmatrix} -b \\ a \end{pmatrix} \right)$$

we get

$$A\underline{e}_1 = \frac{1}{a^2 + b^2} \left( a \begin{pmatrix} a \\ b \end{pmatrix} - b \begin{pmatrix} b \\ -a \end{pmatrix} \right) = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 - b^2 \\ 2ab \end{pmatrix}$$

and

$$A\underline{e}_2 = \frac{1}{a^2 + b^2} \left( b \begin{pmatrix} a \\ b \end{pmatrix} + a \begin{pmatrix} b \\ -a \end{pmatrix} \right) = \frac{1}{a^2 + b^2} \begin{pmatrix} 2ab \\ b^2 - a^2 \end{pmatrix}$$

so

$$A = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{pmatrix}$$

In summary, if  $\underline{x} \in \mathbb{R}^2$ , then  $A\underline{x}$  is the reflection of  $\underline{x}$  in the line  $ay = bx$ .

You might exercise your muscles by showing that all reflections are similar to one another. Why does it suffice to show each is similar to the reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}?$$

12.7.1. *Reflections in higher dimensions.* Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . We call  $T$  a reflection if  $T^2 = I$  and the 1-eigenspace of  $T$  has dimension  $n - 1$ .

12.8. **Orthogonal projections.** Let  $V$  be a subspace of  $\mathbb{R}^n$ . We define

$$V^\perp := \{\underline{w} \in \mathbb{R}^n \mid \underline{w}^T \underline{v} = 0 \text{ for all } \underline{v} \in V\},$$

i.e.,  $V^\perp$  consists of those vectors that are perpendicular to every vector in  $V$ . We call  $V^\perp$  the orthogonal to  $V$  and usually say “ $V$  perp” or “ $V$  orthogonal” when we read  $V^\perp$ .

It is easy to see that  $V^\perp$  is a subspace of  $\mathbb{R}^n$ . You should check that. You might also check that  $V \cap V^\perp = \{0\}$  because a non-zero vector can not be perpendicular to itself. It is slightly more challenging to show that

$$\dim V + \dim V^\perp = n.$$

We will prove this by using the rank+nullity formula and defining a linear transformation from  $\mathbb{R}^n$  to itself with range  $V$  and null space  $V^\perp$ .

12.9. **Invariant subspaces.**

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<sup>10</sup>Draw a picture to convince yourself that this does what you expect the reflection in  $L$  to do.

**12.10. The one-to-one and onto properties.** For a few moments forget linear algebra. The adjectives *one-to-one* and *onto* can be used with any kinds of functions, not just those that arise in linear algebra.

Let  $f : X \rightarrow Y$  be a function between any two sets  $X$  and  $Y$ .

- We say that  $f$  is one-to-one if  $f(x) = f(x')$  only when  $x = x'$ . Equivalently,  $f$  is one-to-one if it sends different elements of  $X$  to different elements of  $Y$ , i.e.,  $f(x) \neq f(x')$  if  $x \neq x'$ .
- The range of  $f$ , which we denote by  $\mathcal{R}(f)$ , is the set of all values it takes, i.e.,

$$\mathcal{R}(f) := \{f(x) \mid x \in X\}.$$

I like this definition because it is short but some prefer the equivalent definition

$$\mathcal{R}(f) := \{y \in Y \mid y = f(x) \text{ for some } x \in X\}.$$

- We say that  $f$  is onto if  $\mathcal{R}(f) = Y$ . Equivalently,  $f$  is onto if every element of  $Y$  is  $f(x)$  for some  $x \in X$ .

12.10.1. The sine function  $\sin : \mathbb{R} \rightarrow \mathbb{R}$  is not one-to-one because  $\sin \pi = \sin 2\pi$ . It is not onto because its range is  $[-1, 1] = \{y \mid -1 \leq y \leq 1\}$ . However, if we consider  $\sin$  as a function from  $\mathbb{R}$  to  $[-1, 1]$  it becomes onto, though it is still not one-to-one.

The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$  is not one-to-one because  $f(2) = f(-2)$  for example, and is not onto because its range is  $\mathbb{R}_{\geq 0} := \{y \in \mathbb{R} \mid y \geq 0\}$ . However, the function

$$f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, \quad f(x) = x^2,$$

is both one-to-one and onto.

12.10.2. We can and do apply these ideas in the context of linear algebra. If  $T$  is a linear transformation we can ask whether  $T$  is one-to-one or onto.

For example, the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

is both one-to-one and onto.

The linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix}$$

is one-to-one but not onto because  $(0, 0, 1)^T$  is not in the range. Of course, since the range of  $T$  is a plane in  $\mathbb{R}^3$  there are lots of other elements of  $\mathbb{R}^3$  not in the range of  $T$ .

**Proposition 12.3.** *A linear transformation is one-to-one if and only if its null space is zero.*

**Proof.** Let  $T$  be a linear transformation. By definition,  $\mathcal{N}(T) = \{\underline{x} \mid T(\underline{x}) = \underline{0}\}$ . If  $T$  is one-to-one the only  $\underline{x}$  with the property that  $T(\underline{x}) = T(\underline{0})$  is  $\underline{x} = \underline{0}$ , so  $\mathcal{N}(T) = \{\underline{0}\}$ .

Conversely, suppose that  $\mathcal{N}(T) = \{\underline{0}\}$ . If  $T(\underline{x}) = T(\underline{x}')$ , then  $T(\underline{x} - \underline{x}') = T(\underline{x}) - T(\underline{x}') = \underline{0}$  so  $\underline{x} - \underline{x}' \in \mathcal{N}(T)$ , whence  $\underline{x} = \underline{x}'$  thus showing that  $T$  is one-to-one.  $\square$

Another way of saying this is that a linear transformation is one-to-one if and only if its nullity is zero.

**12.11. Gazing into the distance: differential operators as linear transformations.** You do not *need* to read this section. It is only for those who are curious enough to raise their eyes from the road we have been traveling to pause, refresh themselves, and gaze about in order to see more of the land we are entering. Linear algebra is a *big* subject that permeates and provides a framework for almost all areas of mathematics. So far we have seen only a small piece of this vast land.

We mentioned earlier that the set  $\mathbb{R}[x]$  of all polynomials in  $x$  is a vector space with basis  $1, x, x^2, \dots$ . Differentiation is a linear transformation

$$\frac{d}{dx} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$$

because

$$\frac{d}{dx}(af(x) + bg(x)) = a\frac{df}{dx} + b\frac{dg}{dx}.$$

If we write  $d/dx$  as a matrix with respect to the basis  $1, x, x^2, \dots$ , then

$$\frac{d}{dx} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & 0 & \cdots \\ \vdots & & & & & & \vdots \end{pmatrix}$$

because  $\frac{d}{dx}(x^n) = nx^{n-1}$ .

The null space of  $\frac{d}{dx}$  is the set of constant functions, the subspace  $\mathbb{R} \cdot 1$ . The range of  $\frac{d}{dx}$  is all of  $\mathbb{R}[x]$ : every polynomial is a derivative of another polynomial.

Another linear transformation from  $\mathbb{R}[x]$  to itself is “multiplication by  $x$ ”, i.e.,  $T(f) = xf$ . The composition of two linear transformations is a linear transformation. In particular, we can compose  $x$  and  $\frac{d}{dx}$  to obtain another linear transformation. There are two compositions, one in each order. It is interesting to compute the difference of these two compositions. Let’s do that

Write  $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$  for the linear transformation  $T(f) = xf$  and  $D : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$  for the linear transformation  $D(f) = f'$ , the derivative of  $f$  with respect to  $x$ . Then, applying the product rule to compute the derivative of the product  $xf$ , we have

$$(DT - TD)(f) = D(xf) - xf' = f + xf' - xf' = f.$$

Thus  $DT - TD = I$ , the identity transformation.

MORE to say, Heisenberg's Uncertainty Principle.

12.11.1. *Higher-order matrix equations.* This course is all about solving equations in matrices. At high school you learned something about solving equations in which the solution is a number, i.e., a  $1 \times 1$  matrix. In this course you are learning about solving equations in which the solution is a column vector, an  $n \times 1$  matrix. An obvious next step is to consider equations in which the solutions are matrices.

For example, at high school solving the two equations  $x^2 + y^2 = 1$  and  $x - y = \frac{1}{2}$  amounts to finding the two points where the circle meets the line. The solutions are  $1 \times 2$  matrices. However, you could also consider the problem of finding all  $2 \times 2$  matrices  $x$  and  $y$  such that  $x^2 + y^2 = 1$  and  $x - y = \frac{1}{2}$ .

MORE to say

Find all  $n \times n$  matrices  $E, F, H$ , such that

$$\begin{aligned} HE - EH &= 2E \\ EF - FE &= H \\ HF - FH &= -2F. \end{aligned}$$

Because of the equivalence between matrices and linear transformations, which we discussed in section 12.2, this problem is equivalent to finding all linear transformations  $E, F$ , and  $H$ , from  $\mathbb{R}^n$  to itself that simultaneously satisfy the three equations.

Let  $\mathbb{R}[x, y]_n$  denote the  $(n + 1)$ -dimensional vector space consisting of all homogeneous polynomials in two variables having degree  $n$ . A basis for this vector space is the set

$$x^n, x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n.$$

Thus the elements of  $\mathbb{R}[x, y]_n$  consists of all linear combinations

$$a_n x^n + a_{n-1} x^{n-1} y + \dots + a_1 x y^{n-1} + a_0 y^n.$$

Let  $\partial_x$  and  $\partial_y$  denote the partial derivatives with respect to  $x$  and  $y$ , respectively. These are linear transformations  $\mathbb{R}[x, y]_n \rightarrow \mathbb{R}[x, y]_{n-1}$ . We will write  $f_x$  for  $\partial_x(f)$  and  $f_y$  for  $\partial_y(f)$ . Multiplication by  $x$  and multiplication by  $y$  are linear transformations  $\mathbb{R}[x, y]_n \rightarrow \mathbb{R}[x, y]_{n+1}$ . Therefore

$$E := x\partial_y, \quad H := x\partial_x - y\partial_y, \quad F := y\partial_x$$

are linear transformations  $\mathbb{R}[x, y]_n \rightarrow \mathbb{R}[x, y]_n$  for every  $n$ . The action of  $EF - FE$  on a polynomial  $f$  is

$$\begin{aligned} (EF - FE)(f) &= (x\partial_y y\partial_x - y\partial_x x\partial_y)(f) \\ &= x\partial_y(yf_x) - y\partial_x(xf_y) \\ &= x(f_x + yf_{xy}) - y(f_y + xf_{yx}) \\ &= xf_x + xyf_{xy} - yf_y - yxf_{yx} \\ &= xf_x - yf_y \\ &= (x\partial_x - y\partial_y)(f) \\ &= H(f). \end{aligned}$$

Since the transformations  $EF - FE$  and  $H$  act in the same way on every function  $f$ , they are equal; i.e.,  $EF - FE = H$ . We also have

$$\begin{aligned} (HE - EH)(f) &= \left( (x\partial_x - y\partial_y)(x\partial_y) - (x\partial_y)(x\partial_x - y\partial_y) \right)(f) \\ &= (x\partial_x - y\partial_y)(xf_y) - (x\partial_y)(xf_x - yf_y) \\ &= x(f_y + xf_{yx}) - yxf_{yy} - x^2f_{xy} + x(f_y + yf_{yy}) \\ &= 2xf_y \\ &= 2(x\partial_y)(f) \\ &= 2E(f). \end{aligned}$$

Therefore  $HE - EH = 2E$ . I leave you to check that  $HF - FH = -2F$ .

This example illuminates several topics we have discussed. The null space of  $E$  is  $\mathbb{R}x^n$ . The null space of  $F$  is  $\mathbb{R}y^n$ . Each  $x^{n-i}y^i$  is an eigenvector for  $H$  having eigenvalue  $n - 2i$ . Thus  $H$  has  $n + 1$  distinct eigenvalues  $n, n-2, n-4, \dots, 2-n, -n$  and, as we prove in greater generality in Theorem 14.2, these eigenvectors are linearly independent.

MORE to say – spectral lines....

### 13. DETERMINANTS

The determinant of an  $n \times n$  matrix  $A$ , which is denoted  $\det(A)$ , is a number computed from the entries of  $A$  having the wonderful property that:

$$A \text{ is invertible if and only if } \det(A) \neq 0.$$

We will prove this in Theorem 13.5 below.

Sometimes, for brevity, we denote the determinant of  $A$  by  $|A|$ .

**13.1. The definition.** We met the determinant of a  $2 \times 2$  matrix in Theorem 10.3:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

The formula for the determinant of an  $n \times n$  matrix is more complicated. It is defined inductively by which I mean that the formula for the determinant

of an  $n \times n$  matrix involves the formula for the determinant of an  $(n - 1) \times (n - 1)$  matrix.

13.1.1. *The  $3 \times 3$  case.* The determinant  $\det A = \det(a_{ij})$  of a  $3 \times 3$  matrix is the number

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

The coefficients  $a_{11}$ ,  $a_{12}$ , and  $a_{13}$ , on the right-hand side of the formula are the entries in the top row of  $A$  and they appear with alternating signs  $+1$  and  $-1$ ; each  $a_{1j}$  is then multiplied by the determinant of the  $2 \times 2$  matrix obtained by deleting the row and column that contain  $a_{1j}$ , i.e., deleting the top row and the  $j^{\text{th}}$  column of  $A$ .

In the second line of the expression for  $\det(A)$  has  $3 \times 2 = 6 = 3!$  terms. Each term is a product of three entries and those three entries are distributed so that each row of  $A$  contains exactly one of them and each column of  $A$  contains one of them. In other words, each term is a product  $a_{1p}a_{2q}a_{3r}$  where  $\{p, q, r\} = \{1, 2, 3\}$ .

13.1.2. *The  $4 \times 4$  case.* The determinant  $\det A = \det(a_{ij})$  of a  $4 \times 4$  matrix is the number

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \\ &\quad + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}. \end{aligned}$$

Look at the similarities with the  $3 \times 3$  case. Each entry in the top row of  $A$  appears as a coefficient with alternating signs  $+1$  and  $-1$  in the formula on the right. Each  $a_{1j}$  is then multiplied by the determinant of the  $3 \times 3$  matrix obtained by deleting the row and column that contain  $a_{1j}$ , i.e., deleting the top row and the  $j^{\text{th}}$  column of  $A$ . Multiplying out the formula on the right yields a total of  $4 \times 6 = 24 = 4!$  terms. Each term is a product of four entries and those four entries are distributed so that each row of  $A$  contains exactly one of them and each column of  $A$  contains one of them. In other words, each term is a product  $a_{1p}a_{2q}a_{3r}a_{4s}$  where  $\{p, q, r, s\} = \{1, 2, 3, 4\}$ .

13.1.3. *The  $n \times n$  case.* Let  $A$  be an  $n \times n$  matrix. For each  $i$  and  $j$  between 1 and  $n$  we define the  $(n - 1) \times (n - 1)$  matrix  $A^{ij}$  to be that obtained by deleting row  $i$  and column  $j$  from  $A$ . We then define

$$(13-1) \quad \det(A) := a_{11}|A^{11}| - a_{12}|A^{12}| + a_{13}|A^{13}| - \cdots + (-1)^{n-1}a_{1n}|A^{1n}|.$$

Each  $|A^{ij}|$  is called the  $ij^{\text{th}}$  minor of  $A$ .

The expression (13-1) is said to be obtained by expanding  $A$  along its first column. There are similar expressions for the determinant of  $A$  that are obtained by expanding  $A$  along any row or column.

There are a number of observations to make about the formula (13-1). When the right-hand side is multiplied out one gets a total of  $n!$  terms each of which looks like

$$\pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where  $\{j_1, \dots, j_n\} = \{1, \dots, n\}$ . In particular, each row and each column contains exactly one of the terms  $a_{1j_1}, a_{2j_2}, \dots, a_{nj_n}$ .

**Proposition 13.1.** *If  $A$  has a column of zeros, then  $\det A = 0$ .*

**Proof.** Suppose column  $j$  consists entirely of zeroes, i.e.,  $a_{1j} = a_{2j} = \cdots = a_{nj} = 0$ . Consider one of the  $n!$  terms in  $\det A$ , say  $a_{1j_1}, a_{2j_2}, \dots, a_{nj_n}$ . Because  $\{j_1, \dots, j_n\} = \{1, \dots, n\}$ , one of the factors  $a_{pj_p}$  in the product  $a_{1j_1}, a_{2j_2}, \dots, a_{nj_n}$  belongs to column  $j$ , so is zero; hence  $a_{1j_1}, a_{2j_2}, \dots, a_{nj_n} = 0$ . Thus, every one of the  $n!$  terms in  $\det A$  is zero. Hence  $\det A = 0$ .  $\square$

A similar argument shows that  $\det A = 0$  if one of the rows of  $A$  consists entirely of zeroes.

By Theorem ??,  $A$  is not invertible if its columns are linearly dependent. Certainly, if one of the columns of  $A$  consists entirely of zeroes its columns are linearly dependent. As just explained, this implies that  $\det A = 0$ . This is consistent with the statement (not yet proved) at the beginning of this section that  $A$  is invertible if and only if  $\det A \neq 0$ .

13.1.4. *The signs.* The signs  $\pm 1$  are important. First let's just note that the coefficient of  $a_{11}a_{22} \dots a_{nn}$  is  $+1$ . This can be proved by induction: we have

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} + \text{other terms}$$

but  $a_{11}$  does not appear in any of the other terms. The coefficient of  $a_{11}a_{22} \dots a_{nn}$  in  $\det A$  is therefore the same as the coefficient of  $a_{22} \dots a_{nn}$  in

$$\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

However, by an induction argument we can assume that the coefficient of  $a_{22} \dots a_{nn}$  in this smaller determinant is  $+1$ . Hence the coefficient of  $a_{11}a_{22} \dots a_{nn}$  in  $\det A$  is  $+1$ .

In general, the sign in front of  $a_{1j_1} a_{2j_2} \cdots a_{nj_n}$  is determined according to the following recipe: it is  $+1$  if one can change the sequence  $j_1, j_2, \dots, j_n$  to  $1, 2, \dots, n$  by an *even* number of switches of adjacent numbers, and is  $-1$

otherwise. For example, the coefficient of  $a_{12}a_{23}a_{34}a_{45}a_{51}$  in the expression for the determinant of a  $5 \times 5$  matrix is  $+1$  because we need 4 switches

$$23451 \rightarrow 23415 \rightarrow 23145 \rightarrow 21345 \rightarrow 12345.$$

On the other hand, the coefficient of  $a_{13}a_{24}a_{35}a_{42}a_{51}$  is  $-1$  because we need 7 switches

$$34521 \rightarrow 34251 \rightarrow 34215 \rightarrow 32415 \rightarrow 32145 \rightarrow 23145 \rightarrow 21345 \rightarrow 12345.$$

13.1.5. *The upper triangular case.* An  $n \times n$  matrix  $A$  is **upper triangular** if every entry below the main diagonal is zero; more precisely, if  $a_{ij} = 0$  for all  $i > j$ . The next result shows that it is easy to compute the determinant of an upper triangular matrix: it is just the product of the diagonal entries.

**Theorem 13.2.**

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22} \cdots a_{nn}.$$

**Proof.** The determinant is a sum of terms of the form

$$\pm a_{1j_1}a_{2j_2} \cdots a_{nj_n}$$

where  $\{j_1, \dots, j_n\} = \{1, \dots, n\}$ . For an upper triangular matrix this term can only be non-zero if  $j_1 \geq 1$ , and  $j_2 \geq 2$ , and  $\dots$ , and  $j_n \geq n$ . But this can only happen if  $j_1 = 1, j_2 = 2, \dots$ , and  $j_n = n$ . Hence the only non-zero term is  $a_{11}a_{22} \cdots a_{nn}$  which, as we noted earlier, has coefficient  $+1$ .  $\square$

Thus, for example,

$$\det(I) = 1.$$

There is an analogue of Theorem 13.2 for lower triangular matrices.

13.2. **Elementary row operations and determinants.** Computing the determinant of a matrix directly from the definition is tedious and prone to arithmetic error. However, given  $A$  we can perform elementary row operations to get a far simpler matrix, say  $A'$ , perhaps upper triangular, compute  $\det A'$  and, through repeated use of the next result, then determine  $\det A$ .

The row reduced echelon form of a square matrix is always upper triangular.

**Proposition 13.3.** *If  $B$  is obtained from  $A$  by the following elementary row operation the determinants are related in the following way:*

(13-2)

Elementary row operation

- *switch two rows*  $\det B = -\det A$
- *multiply a row by  $c \in \mathbb{R} - \{0\}$*   $\det B = c \det A$
- *replace row  $i$  by (row  $i +$  row  $k$ ) with  $i \neq k$*   $\det B = \det A$

In particular,  $\det A = 0$  if and only if  $\det B = 0$ .

**Proof.** The second of these statements is obvious. Suppose we multiply row  $p$  by the number  $c$ . Each term  $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$  in  $\det A$  contains exactly one factor from row  $p$  so each term in  $\det B$  is  $c$  times the corresponding term in  $\det A$ . Hence  $\det B$  is  $c$  times  $\det A$ .

We now prove the other two statements in the  $2 \times 2$  case.

Switching two rows changes the sign because

$$\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -(ad - bc) = -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Adding one row to another doesn't change the sign because

$$\det \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = (a+c)b - (b+d)c = ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

and

$$(13-3) \quad \det \begin{pmatrix} a & b \\ c+a & d+b \end{pmatrix} = a(d+b) - b(c+a) = -ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence all three statements are true for a  $2 \times 2$  matrix.

We now turn to the  $n \times n$  case and argue by induction on  $n$ , i.e., we assume the result is true for  $(n-1) \times (n-1)$  matrices and prove the result for the an  $n \times n$  matrix.

Actually, the induction argument is best understood by looking at how the truth of the result for  $2 \times 2$  matrices implies its truth for  $3 \times 3$  matrices. Consider switching rows 2 and 3 in a  $3 \times 3$  matrix. Since

$$(13-4) \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

it follows that

$$(13-5) \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = a_{11} \begin{vmatrix} a_{32} & a_{33} \\ a_{22} & a_{23} \end{vmatrix} - a_{12} \begin{vmatrix} a_{31} & a_{33} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix}.$$

But the determinants of the three  $2 \times 2$  matrices in this sum are the negatives of the three  $2 \times 2$  determinants in (13-4). Hence the determinant of the  $3 \times 3$  matrix in (13-5) is the negative of the determinant of the  $3 \times 3$  matrix in (13-4). This "proves" that  $\det B = -\det A$  if  $B$  is obtained from  $A$  by switching two rows.

Now consider replacing the third row of the  $3 \times 3$  matrix  $A$  in (13-4) by the sum of rows 2 and 3 to produce  $B$ . Then  $\det B$  is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} + a_{31} & a_{22} + a_{32} & a_{23} + a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{22} + a_{32} & a_{23} + a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{21} + a_{31} & a_{23} + a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{21} + a_{31} & a_{22} + a_{32} \end{vmatrix}$$

However, by the calculation in (13-3), this is equal to

$$a_{11} \begin{vmatrix} a_{32} & a_{33} \\ a_{22} & a_{23} \end{vmatrix} - a_{12} \begin{vmatrix} a_{31} & a_{33} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix}$$

i.e.,  $\det B = \det A$ .

PAUL ... didn't cover the case of an ERO that involves row 1.  $\square$

**Proposition 13.4.** *Let  $A$  and  $A'$  be row equivalent  $n \times n$  matrices. Then  $\det A = 0$  if and only if  $\det A' = 0$ .*

**Proof.** By hypothesis,  $A'$  is obtained from  $A$  by a sequence of elementary row operations. By Proposition 13.3, a single elementary row operation does not change whether the determinant of a matrix is zero or not. Therefore a sequence of elementary row operations does not change whether the determinant of a matrix is zero or not.  $\square$

### 13.3. The determinant and invertibility.

**Theorem 13.5.** *A square matrix is invertible if and only if its determinant is non-zero.*

**Proof.** Suppose  $A$  is an  $n \times n$  matrix and let  $E = \text{rref } A$ . Then  $A$  and  $E$  have the same rank.

By Proposition 5.4,  $E = I$  if  $\text{rank } E = n$ , and  $E$  has a row of zeroes if  $\text{rank } E < n$ . Since  $A$  is invertible if and only if  $\text{rank } A = n$ , it follows that  $\det E = 1$  if  $A$  is invertible and  $\det E = 0$  if  $A$  is not invertible. The result now follows from Proposition 13.4.  $\square$

### 13.4. Properties.

**Theorem 13.6.**  $\det(AB) = (\det A)(\det B)$ .

**Proof.** PAUL - prove it!  $\square$

**Corollary 13.7.** *If  $A$  is an invertible matrix, then  $\det(A^{-1}) = (\det A)^{-1}$ .*

**Proof.** It is clear that the determinant of the identity matrix is 1, so the result follows from Theorem 13.6 and the fact that  $AA^{-1} = I$ .  $\square$

$\det A = 0$  if two rows (or columns) of  $A$  are the same.

$\det(A^T) = \det A$ .

**13.5. Elementary column operations and determinants.** As you might imagine, one can define elementary column operations in a way that is analogous to our definition of elementary row operations and the effect on the determinant of those operations on the determinant of a matrix parallels that on the effect of the elementary row operations.

13.5.1. *Amusement.* Here is an amusing use of the observation about the effect of the elementary column operations on the determinant.

The numbers 132, 144, and 156, are all divisible by 12. Then the determinant of

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \\ 1 & 5 & 6 \end{pmatrix}$$

is divisible by 12 because it is not changed by adding 100 times the 1st column and 10 times the second column to the third column whence

$$\det \begin{pmatrix} 1 & 3 & 2 \\ 1 & 4 & 4 \\ 1 & 5 & 6 \end{pmatrix} = \det \begin{pmatrix} 1 & 3 & 132 \\ 1 & 4 & 144 \\ 1 & 5 & 156 \end{pmatrix} = 12 \times \det \begin{pmatrix} 1 & 3 & 11 \\ 1 & 4 & 12 \\ 1 & 5 & 13 \end{pmatrix}.$$

Similarly, the numbers 289, 391, and 867, are all divisible by 17 so

$$\det \begin{pmatrix} 2 & 8 & 9 \\ 3 & 9 & 1 \\ 8 & 6 & 7 \end{pmatrix} = \det \begin{pmatrix} 2 & 8 & 289 \\ 3 & 9 & 391 \\ 8 & 6 & 867 \end{pmatrix} = 17 \times \det \begin{pmatrix} 2 & 8 & 17 \\ 3 & 9 & 23 \\ 8 & 6 & 51 \end{pmatrix}.$$

is divisible by 17.

Amuse your friends at parties.

13.5.2. *A nice example.* If the matrix

$$A = \begin{pmatrix} b & c & d \\ a & -d & c \\ d & a & -b \\ c & -b & -a \end{pmatrix}$$

has rank  $\leq 2$ , then

- (1)  $\text{rref } A$  has  $\leq 2$  non-zero rows so
- (2) the row space of  $A$  has dimension  $\leq 2$  so
- (3) any three rows of  $A$  are linearly dependent
- (4) so  $\det(\text{any 3 rows of } A) = 0$ .

For example,

$$\begin{aligned} 0 &= \det \begin{pmatrix} b & c & d \\ a & -d & c \\ d & a & -b \end{pmatrix} \\ &= b(db - ac) - c(-ab - cd) + d(a^2 + d^2) \\ &= d(a^2 + b^2 + c^2 + d^2) \end{aligned}$$

Similarly, taking the determinant of the other three  $3 \times 3$  submatrices of  $A$  gives

$$a(a^2 + b^2 + c^2 + d^2) = b(a^2 + b^2 + c^2 + d^2) = c(a^2 + b^2 + c^2 + d^2) = 0.$$

If  $a, b, c, d$  are real numbers it follows that  $a = b = c = d = 0$ , i.e.,  $A$  must be the zero matrix. However, if  $a, b, c, d$  are complex numbers  $A$  need

not be zero because there are non-zero complex numbers  $a, b, c, d$  such that  $a^2 + b^2 + c^2 + d^2 = 0$ .

13.5.3. *Permutations.* You have probably met the word permutation meaning *arrangement* or *rearrangement* as in the question “*How many permutations are there of the numbers  $1, 2, \dots, n$ ?*” The answer, as you know, is  $n!$ .

We now define a permutation of the set  $\{1, 2, \dots, n\}$  as a function

$$\pi : \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$$

that is one-to-one (or, equivalently, onto). Thus each term in the expression for the determinant of an  $n \times n$  matrix  $(a_{ij})$  is of the form

$$(13-6) \quad \pm a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

The condition that  $\pi$  is one-to-one ensures that each row and each column contains exactly *one* of the terms in (13-6).

There are a total of  $n!$  permutations and the determinant of  $(a_{ij})$  is the sum of  $n!$  terms (13-6), one for each permutation. *However, there remains the issue of the  $\pm$  signs.*

Let  $\pi$  be a permutation of  $\{1, 2, \dots, n\}$ . Write  $\pi(1), \pi(2), \dots, \pi(n)$  one after the other. A transposition of this string of numbers is the string of numbers obtained by switching the positions of any two adjacent numbers. For example, if we start with 4321, then each of 3421, 4231, and 4312, is a transposition of 4321. It is clear that after some number of transpositions we can get the numbers back in their correct order. For example,

$$4321 \rightarrow 4312 \rightarrow 4132 \rightarrow 1432 \rightarrow 1423 \rightarrow 1243 \rightarrow 1234$$

or

$$4321 \rightarrow 4231 \rightarrow 2431 \rightarrow 2341 \rightarrow 2314 \rightarrow 2134 \rightarrow 1234.$$

If it takes an even number of transpositions to get the numbers back into their correct order we call the permutation even. If it takes an odd number we call the permutation odd. We define the sign of a permutation to be the number

$$\text{sgn}(\pi) := \begin{cases} +1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd} \end{cases}$$

It is not immediately clear that the sign of a permutation is well-defined. It is, but we won't give a proof.

The significance for us is that the coefficient of the term  $a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$  in the expression for the determinant is  $\text{sgn}(\pi)$ . Hence

$$\det(a_{ij}) = \sum_{\pi} \text{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$$

where the sum is taken over all permutations of  $\{1, 2, \dots, n\}$ .

Thus, the term  $a_{14} a_{23} a_{32} a_{41}$  appears in the expression for the determinant of a  $4 \times 4$  matrix with coefficient  $+1$  because, as we saw above, it took an even number of transpositions (six) to get 4321 back to the correct order.

## 14. EIGENVALUES

Consider a fixed  $n \times n$  matrix  $A$ . Viewed as a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ ,  $A$  will send a line through the origin to either another line through the origin or zero. Some lines might be sent back to themselves. That is a nice situation: the effect of  $A$  on the points on that line is simply to scale them by some factor,  $\lambda$  say. The line  $\mathbb{R}\underline{x}$  is sent by  $A$  to either zero or the line  $\mathbb{R}(A\underline{x})$ , so  $A$  sends this line to itself if and only if  $A\underline{x}$  is a multiple of  $\underline{x}$ . This motivates the next definition.

**14.1. Definitions and first steps.** Let  $A$  be an  $n \times n$  matrix. A number  $\lambda \in \mathbb{R}$  is an **eigenvalue** for  $A$  if there is a *non-zero* vector  $\underline{x}$  such that  $A\underline{x} = \lambda\underline{x}$ . If  $\lambda$  is an eigenvalue for  $A$  we define

$$E_\lambda := \{\underline{x} \mid A\underline{x} = \lambda\underline{x}\}.$$

We call  $E_\lambda$  the  **$\lambda$ -eigenspace** for  $A$  and the elements in  $E_\lambda$  are called **eigenvectors** or  **$\lambda$ -eigenvectors** for  $A$ .

We can define  $E_\lambda$  for any  $\lambda \in \mathbb{R}$  but it is non-zero if and only if  $\lambda$  is an eigenvalue.<sup>11</sup>

Since

$$A\underline{x} = \lambda\underline{x} \iff 0 = A\underline{x} - \lambda\underline{x} = A\underline{x} - \lambda I\underline{x} = (A - \lambda I)\underline{x},$$

it follows that

$$E_\lambda = \mathcal{N}(A - \lambda I).$$

We have therefore proved that  $E_\lambda$  is a subspace of  $\mathbb{R}^n$ . The following theorem also follows from the observation that  $E_\lambda = \mathcal{N}(A - \lambda I)$ .

**Theorem 14.1.** *Let  $A$  be an  $n \times n$  matrix. The following are equivalent:*

- (1)  $\lambda$  is an eigenvalue for  $A$ ;
- (2)  $A - \lambda I$  is singular;
- (3)  $\det(A - \lambda I) = 0$ .

**Proof.** Let  $\lambda \in \mathbb{R}$ . Then  $\lambda$  is an eigenvalue for  $A$  if and only if there is a non-zero vector  $\underline{v}$  such that  $(A - \lambda I)\underline{v} = 0$ ; but such a  $\underline{v}$  exists if and only if  $A - \lambda I$  is singular. This proves the equivalence of (1) and (2). Theorem 13.5 tells us that (2) and (3) are equivalent statements.  $\square$

**Theorem 14.2.** *Let  $A$  be an  $n \times n$  matrix and  $\lambda_1, \dots, \lambda_r$  distinct eigenvalues for  $A$ . If  $\underline{v}_1, \dots, \underline{v}_r$  are non-zero vectors such that  $A\underline{v}_i = \lambda_i \underline{v}_i$  for each  $i$ , then  $\{\underline{v}_1, \dots, \underline{v}_r\}$  is linearly independent.*

**Proof.** If there were a linear dependence relation  $a_1 \underline{v}_1 + \dots + a_r \underline{v}_r = 0$  in which not all the  $a_i$ s are zero there would be one having the smallest number of non-zero  $a_i$ s. Suppose we take that relation and let  $N$  be the number of

<sup>11</sup>*Eigen* is a German word that has a range of meanings but the connotation here is probably closest to “peculiar to”, or “particular to”, or “characteristic of”. The eigenvalues of a matrix are important characteristics of it.

non-zero  $a_i$ s in it. If  $N = 0$  we are done, so suppose that  $N \geq 1$  and that  $a_j \neq 0$ . Then

$$\begin{aligned} 0 &= A(a_1 \underline{v}_1 + \cdots + a_r \underline{v}_r) \\ &= a_1 A \underline{v}_1 + \cdots + a_r A \underline{v}_r \\ &= a_1 \lambda_1 \underline{v}_1 + \cdots + a_r \lambda_r \underline{v}_r \end{aligned}$$

and

$$\begin{aligned} 0 &= \lambda_j (a_1 \underline{v}_1 + \cdots + a_r \underline{v}_r) \\ &= a_1 \lambda_j \underline{v}_1 + \cdots + a_r \lambda_j \underline{v}_r. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= A(a_1 \underline{v}_1 + \cdots + a_r \underline{v}_r) - \lambda_j (a_1 \underline{v}_1 + \cdots + a_r \underline{v}_r) \\ &= a_1 (\lambda_1 - \lambda_j) \underline{v}_1 + \cdots + a_r (\lambda_r - \lambda_j) \underline{v}_r. \end{aligned}$$

If  $i \neq j$ , then  $\lambda_i \neq \lambda_j$  so the number of non-zero coefficients in this last linear combination is  $N - 1$ .

Now we have a linear dependence relation with  $< N$  non-zero coefficients. That contradicts our choice of  $N$  so we conclude there can be no such linear dependence relation.  $\square$

**14.2. Reflections in  $\mathbb{R}^2$ , revisited.** A simple example to consider is reflection in a line lying in  $\mathbb{R}^2$ , the situation we considered in section 12.7.

There we examined a pair of orthogonal lines,  $L$  spanned by  $\begin{pmatrix} a \\ b \end{pmatrix}$ , and  $L'$  spanned by  $\begin{pmatrix} b \\ -a \end{pmatrix}$ . We determined the matrix  $A$  that fixed the points on  $L$  and sent other points to their reflections in  $L$ . In particular,  $A$  sent  $\begin{pmatrix} b \\ -a \end{pmatrix}$  to its negative. We found that

$$A = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{pmatrix}.$$

The matrix  $A$  has eigenvectors  $\begin{pmatrix} a \\ b \end{pmatrix}$  and  $\begin{pmatrix} b \\ -a \end{pmatrix}$  with eigenvalues,  $+1$  and  $-1$ , and the corresponding eigenspaces are  $L$  and  $L'$ .

The point of this example is that the matrix  $A$  does not initially appear to have any special geometric meaning but by finding its eigenvalues and eigenspaces the geometric meaning of  $A$  becomes clear.

**14.3. The  $2 \times 2$  case.**

**Theorem 14.3.** *The eigenvalues of  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  are the solutions of the quadratic polynomial equation*

$$(14-1) \quad \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

**Proof.** We have already noted that  $\lambda$  is an eigenvalue for  $A$  if and only if  $A - \lambda I$  is singular, i.e., if and only if  $\det(A - \lambda I) = 0$ . But

$$\begin{aligned}\det(A - \lambda I) &= \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + ad - bc,\end{aligned}$$

so the result follows.  $\square$

A quadratic polynomial with real coefficients can have either zero, one, or two roots. It follows that a  $2 \times 2$  matrix with real entries can have either zero, one, or two eigenvalues. (Later we will see that an  $n \times n$  matrix with real entries can have anywhere from zero to  $n$  eigenvalues.)

14.3.1. The polynomial  $x^2 + 1$  has no real zeroes, so the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has no eigenvalues. You might be happy to observe that this is the matrix that rotates the plane through an angle of  $90^\circ$  in the counterclockwise direction. It is clear that such a linear transformation will not send any line to itself: rotating by  $90^\circ$  moves every line.

Of course, it is apparent that rotating by any angle that is not an integer multiple of  $\pi$  will have the same property: it will move every line, so will have no eigenvalues. You should check this by using the formula for  $A_\theta$  in (12-2) and then applying the quadratic formula to the equation (2x2.eigenvals).

14.3.2. The polynomial  $x^2 - 4x + 4$  has one zero, namely 2, because it is a square, so the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

has exactly one eigenvalue, namely 2. The 2-eigenspace is

$$E_2 = \mathcal{N}(A - 2I) = \mathcal{N} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

You should check that  $A$  times  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is  $2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . We know that  $\dim E_2 = 1$  because the rank of  $A - 2I$  is 1 and hence its nullity is 1.

14.3.3. The polynomial  $x^2 - x - 2$  has two zeroes, namely  $-1$  and  $2$ , so the matrix

$$B = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

has two eigenvalues,  $-1$  and  $2$ . The 2-eigenspace is

$$E_2 = \mathcal{N}(B - 2I) = \mathcal{N} \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The  $(-1)$ -eigenspace is

$$E_{-1} = \mathcal{N}(B + I) = \mathcal{N}\begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix} = \mathbb{R}\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

14.4. **The equation  $A^2 + I$ .** You already know there is no real number  $x$  such that  $x^2 + 1 = 0$ . Since a number is a  $1 \times 1$  matrix it is not unreasonable to ask whether there is a  $2 \times 2$  matrix  $A$  such that  $A^2 + I = 0$ . I will leave you to tackle that question. You can attack the question in a naive way by taking a  $2 \times 2$  matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

squaring it, setting the square equal to

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and asking whether the system of four quadratic equations in  $a, b, c, d$  so obtained have a solution. Not pretty, but why not try it and find out what happens.

As a cute illustration of the use of eigenvalues I will now show there is no  $3 \times 3$  matrix  $A$  such that  $A^2 + I = 0$ . I will prove this by contradiction; i.e., I will assume  $A$  is a  $3 \times 3$  matrix such that  $A^2 + I = 0$  and deduce something false from that, thereby showing there can be no such  $A$ .

So, let's get started by assuming  $A$  is a  $3 \times 3$  matrix such that  $A^2 + I = 0$ .

As a preliminary step I will show that  $A$  has no eigenvalues. If  $\lambda \in \mathbb{R}$  were an eigenvalue and  $\underline{v}$  a non-zero vector such that  $A\underline{v} = \lambda\underline{v}$ , then

$$-\underline{v} = -I\underline{v} = A^2\underline{v} = A(A\underline{v}) = A(\lambda\underline{v}) = \lambda A\underline{v} = \lambda^2\underline{v}.$$

But  $\underline{v} \neq 0$  so the equality  $-\underline{v} = \lambda^2\underline{v}$  implies  $-1 = \lambda^2$ . But this can not be true: there is no real number whose square is  $-1$  so I conclude that  $A$  can not have an eigenvector. (Notice this argument did not use anything about the size of the matrix  $A$ . If  $A$  is any square matrix with real entries such that  $A^2 + I = 0$ , then  $A$  does not have an eigenvalue. We should really say " $A$  does not have a *real* eigenvalue".

With that preliminary step out of the way let's proceed to prove the result we want. Let  $\underline{v}$  be any non-zero vector in  $\mathbb{R}^3$  and let  $V = \text{Sp}(\underline{v}, A\underline{v})$ . Then  $V$  is an  $A$ -invariant subspace by which I mean if  $\underline{w} \in V$ , then  $A\underline{w} \in V$ . More colloquially,  $A$  sends elements of  $V$  to elements of  $V$ . Moreover,  $\dim V = 2$  because the fact that  $A$  has no eigenvectors implies that  $A\underline{v}$  is *not* a multiple of  $\underline{v}$ ; i.e.,  $\{\underline{v}, A\underline{v}\}$  is linearly independent.

Since  $V$  is a 2-dimensional subspace of  $\mathbb{R}^3$  there is a non-zero vector  $\underline{w} \in \mathbb{R}^3$  that is not in  $V$ . By the same argument, the subspace  $W := \text{Sp}(\underline{w}, A\underline{w})$  is  $A$ -invariant and has dimension 2. Since  $\underline{w} \notin V$ ,  $V + W = \mathbb{R}^3$ . From the formula

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$$

we deduce that  $\dim(V \cap W) = 1$ . Since  $V$  and  $W$  are  $A$ -invariant, so is  $V \cap W$ . However,  $V \cap W = \mathbb{R}\underline{u}$  for some  $\underline{u}$  and the fact that  $\mathbb{R}\underline{u}$  is  $A$ -invariant implies that  $A\underline{u}$  is a scalar multiple of  $\underline{u}$ ; i.e.,  $\underline{u}$  is an eigenvector for  $A$ . But that is false: we have already proved  $A$  has no eigenvectors. Thus we are forced to conclude that there is no  $3 \times 3$  matrix  $A$  such that  $A^2 + I = 0$ .

14.4.1. *Why did I just prove that?* A recent midterm contained the following question: if  $A$  is a  $3 \times 3$  matrix such that  $A^3 + A = 0$ , is  $A$  invertible. The answer is “no”, but the question is a *very bad question* because the “proof” I had in mind is flawed. Let me explain my false reasoning: if  $A^3 + A = 0$ , then  $A(A^2 + I) = 0$  so  $A$  is not invertible because, as we observed in section 10.4.4, if  $AB = 0$  for some non-zero matrix  $B$ , then  $A$  is not invertible. However, I can not apply the observation in section 10.4.4 unless I know that  $A^2 + I$  is non-zero, and it is not easy to show that  $A^2 + I$  is not zero. The argument above is the simplest argument I know, and it isn’t so simple.

There is a shorter argument using determinants but at the time of the midterm I had not introduced either determinants or eigenvalues.

14.5. **The characteristic polynomial.** The characteristic polynomial of an  $n \times n$  matrix  $A$  is the degree  $n$  polynomial

$$\det(A - tI).$$

You should convince yourself that  $\det(A - tI)$  really is a polynomial in  $t$  of degree  $n$ . Let’s denote that polynomial by  $p(t)$ . If we substitute a number  $\lambda$  for  $t$ , then

$$p(\lambda) = \det(A - \lambda I).$$

The following result is an immediate consequence of this.

**Proposition 14.4.** *Let  $A$  be an  $n \times n$  matrix, and let  $\lambda \in \mathbb{R}$ . Then  $\lambda$  is an eigenvalue of  $A$  if and only if it is a zero of the characteristic polynomial of  $A$ .*

**Proof.** Theorem 14.1 says that  $\lambda$  is an eigenvalue for  $A$  if and only  $\det(A - \lambda I) = 0$ .  $\square$

A polynomial of degree  $n$  has at most  $n$  roots. The next result follows from this fact.

**Corollary 14.5.** *An  $n \times n$  matrix has at most  $n$  eigenvalues.*

The characteristic polynomial of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is

$$t^2 - (a + d)t + ad - bc.$$

**14.6. How to find eigenvalues and eigenvectors.** Finding eigenvalues and eigenvectors is often important. Here we will discuss the head-on, blunt, full frontal method. Like breaking into a bank, that is not always the most effective way, but it is certainly one thing to try. But keep in mind that a given matrix might have some special features that can be exploited to allow an easier method. Similarly, when breaking into a bank, there might be some special features that might suggest an alternative to the guns-ablazing method.

Let  $A$  be an  $n \times n$  matrix. By Proposition 14.4,  $\lambda$  is an eigenvalue if and only if  $\det(A - \lambda I) = 0$ , i.e., if and only if  $\lambda$  is a root/zero of the characteristic polynomial. So, the first step is to compute the characteristic polynomial. Don't forget to use elementary row operations before computing  $\det(A - tI)$  if it looks like that will help—remember, the more zeroes in the matrix the easier it is to compute its determinant. The second step is to find the roots of the characteristic polynomial.

Once you have found an eigenvalue,  $\lambda$  say, remember that the  $\lambda$ -eigenspace is the null space of  $A - \lambda I$ . The third step is to compute that null space, i.e., find the solutions to the equation  $(A - \lambda I)\underline{x} = 0$ , by using the methods developed earlier in these notes. For example, put  $A - \lambda I$  in row-reduced echelon form, find the independent variables and proceed from there.

14.6.1. *An example.* In order to compute the eigenvalues of the matrix

$$B := \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

we perform some elementary row operations on  $B - tI$  before computing its determinant: subtracting the bottom row of  $B - tI$  from each of the other rows does not change the determinant, so  $\det(B - tI)$  is equal to

$$\det \begin{pmatrix} 1-t & -1 & -1 & -1 \\ -1 & 1-t & -1 & -1 \\ -1 & -1 & 1-t & -1 \\ -1 & -1 & -1 & 1-t \end{pmatrix} = \det \begin{pmatrix} 2-t & 0 & 0 & t-2 \\ 0 & 2-t & 0 & t-2 \\ 0 & 0 & 2-t & t-2 \\ -1 & -1 & -1 & 1-t \end{pmatrix}.$$

If we replace a row by  $c$  times that row we change the determinant by a factor of  $c$ , so

$$\det(B - tI) = (2-t)^3 \det \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 1-t \end{pmatrix}.$$

The determinant does not change if we add each of the first three rows to the last row so

$$\det(B - tI) = (2 - t)^3 \det \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 - t \end{pmatrix} = (t - 2)^3(t + 2).$$

The eigenvalues are  $\pm 2$ . Now

$$E_2 = \mathcal{N}(B - 2I) = \mathcal{N} \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

The rank of  $B - 2I$ , i.e., the dimension of the linear span of its columns, is 1, and its nullity is therefore 3. Therefore  $\dim E_2 = 3$ . It is easy to find three linearly independent vectors  $\underline{x}$  such that  $(B - 2I)\underline{x} = 0$  and so compute

$$E_2 = \mathbb{R} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Finally,

$$E_{-2} = \mathcal{N}(B + 2I) = \mathcal{N} \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

14.6.2. *Another example.* Let

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of  $A$  is

$$x^2(x^3 - 5x^2 + 2x - 1).$$

The  $x^2$  factor implies that the 0-eigenspace, i.e., the null space of  $A$ , has dimension two. It is easy to see that

$$\mathcal{N}(A) = \mathbb{R} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We don't need to know the zeroes of the cubic factor  $f(x) = x^3 - 5x^2 + 2x - 1$  just yet. However, a computation gives

$$f(0) = -1, \quad f\left(\frac{2}{3}\right) > 0, \quad f(1) = -3, \quad f(10) > 0,$$

so the graph  $y = f(x)$  crosses the  $x$ -axis at points in the open intervals

$$\left(0, \frac{2}{3}\right), \quad \left(\frac{2}{3}, 1\right), \quad \text{and} \quad (1, 10).$$

Hence  $f(x)$  has three real zeroes, and  $A$  has three distinct non-zero eigenvalues. Because  $A$  has a 2-dimensional 0-eigenspace it follows that  $\mathbb{R}^5$  has a basis consisting of eigenvectors for  $A$ .

Claim: If  $\lambda$  is a non-zero eigenvalue for  $A$ , then

$$\begin{pmatrix} 1 - \lambda^{-1} - \lambda \\ 1 - \lambda^{-1} - \lambda \\ 4\lambda - 1 - \lambda^2 \\ -3 \\ 2\lambda^{-1} + 1 - \lambda \end{pmatrix}$$

is a  $\lambda$ -eigenvector for  $A$ .

Proof of Claim: We must show that  $A\underline{x} = \lambda\underline{x}$ . Since  $\lambda$  is an eigenvalue it is a zero of the characteristic polynomial; since  $\lambda \neq 0$ ,  $f(\lambda) = 0$ , i.e.,  $\lambda^3 - 5\lambda^2 + 2\lambda - 1 = 0$ . The calculation

$$A\underline{x} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \lambda^{-1} - \lambda \\ 1 - \lambda^{-1} - \lambda \\ 4\lambda - 1 - \lambda^2 \\ -3 \\ 2\lambda^{-1} + 1 - \lambda \end{pmatrix} = \begin{pmatrix} \lambda - 1 - \lambda^2 \\ \lambda - 1 - \lambda^2 \\ \lambda - 1 - \lambda^2 \\ -3\lambda \\ 2 + \lambda - \lambda^2 \end{pmatrix} = \lambda\underline{x}$$

shows that the Claim is true, *however* please, please, please, notice we used the fact that  $\lambda$  is a solution to the equation  $x^3 - 5x^2 + 2x - 1 = 0$  in the calculation in the following way: because  $\lambda^3 - 5\lambda^2 + 2\lambda - 1 = 0$  it follows that  $\lambda - 1 - \lambda^2 = \lambda(4\lambda - 1 - \lambda^2)$ , so the third entry in  $A\underline{x}$  is  $\lambda$  times the third entry in  $\underline{x}$ , thus justifying the final equality in the calculation.  $\diamond$

**Notice** we did not need to compute the eigenvalues in order to find the eigenspaces. If one wants more explicit information one needs to determine the roots of the cubic equation  $x^3 - 5x^2 + 2x - 1 = 0$ .

14.6.3. *How did I find the eigenvector in the previous example?* That is the real problem! If someone hands you a matrix  $A$  and a vector  $\underline{v}$  and claims that  $\underline{v}$  is an eigenvector one can test the truth of the claim by computing  $A\underline{v}$  and checking whether  $A\underline{v}$  is a multiple of  $\underline{v}$ .

To find the eigenvector I followed the procedure described in the first paragraph of section 14.6. By definition, the  $\lambda$ -eigenspace is the space of solutions to the homogeneous equation  $(A - \lambda I)\underline{x} = 0$ . To find those solutions

we must put

$$\begin{pmatrix} 1-\lambda & 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 0 & 1-\lambda & 1 \\ 1 & 1 & 1 & 0 & 1-\lambda \end{pmatrix}$$

in row-reduced echelon form. To do this first subtract the top row from each of the other rows to obtain

$$\begin{pmatrix} 1-\lambda & 1 & 1 & 1 & 1 \\ \lambda & -\lambda & 0 & 0 & 0 \\ \lambda & 0 & -\lambda & 0 & 0 \\ \lambda & 0 & -1 & -\lambda & 0 \\ \lambda & 0 & 0 & -1 & -\lambda \end{pmatrix}.$$

Now add the bottom row to the top row and subtract the bottom row from each of the middle three rows, then move the bottom row to the top to get

$$\begin{pmatrix} \lambda & 0 & 0 & -1 & -\lambda \\ 1 & 1 & 1 & 0 & 1-\lambda \\ 0 & -\lambda & 0 & 1 & \lambda \\ 0 & 0 & -\lambda & 1 & \lambda \\ 0 & 0 & -1 & 1-\lambda & \lambda \end{pmatrix}.$$

Write  $\nu = \lambda^{-1}$  (this is allowed since  $\lambda \neq 0$ ) and multiply the first, third, and fourth rows by  $\nu$  to get

$$\begin{pmatrix} 1 & 0 & 0 & -\nu & -1 \\ 1 & 1 & 1 & 0 & 1-\lambda \\ 0 & -1 & 0 & \nu & 1 \\ 0 & 0 & -1 & \nu & 1 \\ 0 & 0 & -1 & 1-\lambda & \lambda \end{pmatrix}.$$

Replace the second row by the second row minus the first row plus the third row to get

$$\begin{pmatrix} 1 & 0 & 0 & -\nu & -1 \\ 0 & 0 & 1 & 2\nu & 3-\lambda \\ 0 & -1 & 0 & \nu & 1 \\ 0 & 0 & -1 & \nu & 1 \\ 0 & 0 & -1 & 1-\lambda & \lambda \end{pmatrix}.$$

Now multiply the third row by  $-1$ , add the bottom row to the second row, and subtract the bottom row from the fourth row, then multiply the bottom row by  $-1$ , to get

$$B := \begin{pmatrix} 1 & 0 & 0 & -\nu & -1 \\ 0 & 0 & 0 & 2\nu + 1 - \lambda & 3 \\ 0 & 1 & 0 & -\nu & -1 \\ 0 & 0 & 0 & \nu - 1 + \lambda & 1 - \lambda \\ 0 & 0 & 1 & -1 + \lambda & -\lambda \end{pmatrix}.$$

There is no need to do more calculation to get  $A$  in row-reduced echelon form because it is quite easy to write down the solutions to the system  $B\underline{x} = 0$  which is equivalent to the system  $(A - \lambda I)\underline{x} = 0$ . Write  $\underline{x} = (x_1, x_2, x_3, x_4, x_5)^T$ . From the second row of  $B$  we see that we should take

$$x_4 = -3 \quad \text{and} \quad x_5 = 2\nu + 1 - \lambda.$$

From the other rows of  $B$  we see that we must have

$$\begin{aligned} x_1 &= \nu x_4 + x_5 &= -3\nu + (2\nu + 1 - \lambda) \\ x_2 &= \nu x_4 + x_5 &= -3\nu + (2\nu + 1 - \lambda) \\ x_3 &= (1 - \lambda)x_4 + \lambda x_5 &= 3(\lambda - 1) + \lambda(2\nu + 1 - \lambda). \end{aligned}$$

Thus

$$\underline{x} = \begin{pmatrix} 1 - \nu - \lambda \\ 1 - \nu - \lambda \\ 4\lambda - 1 - \lambda^2 \\ -3 \\ 2\nu + 1 - \lambda \end{pmatrix}.$$

This is the matrix in the statement of the claim we made on page 89.

14.6.4. *It ain't easy.* Before finding the eigenvector in the claim I made many miscalculations, a sign here, a sign there, and pretty soon I was doomed and had to start over. So, take heart: all you need is perseverance, care, and courage, when dealing with a  $5 \times 5$  matrix by hand. Fortunately, there are computer packages to do this for you. The main thing is to understand the principle for finding the eigenvectors. The computer programs simply implement the principle.

14.7. **The utility of eigenvectors.** Let  $A$  be an  $n \times n$  matrix, and suppose we are in the following good situation:

- $\mathbb{R}^n$  has a basis consisting of eigenvectors for  $A$ ;
- given  $\underline{v} \in \mathbb{R}^n$  we have an effective means of writing  $\underline{v}$  as a linear combination of the eigenvectors for  $A$ .

Then, as the next result says, it is easy to compute  $A\underline{v}$ .

**Proposition 14.6.** *Suppose that  $A$  is an  $n \times n$  matrix such that  $\mathbb{R}^n$  has a basis consisting of eigenvectors for  $A$ . Explicitly, suppose that  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is a basis for  $\mathbb{R}^n$  and that each  $\underline{v}_i$  is an eigenvector for  $A$  with eigenvalue  $\lambda_i$ . If  $\underline{v} = a_1\underline{v}_1 + \dots + a_n\underline{v}_n$ , then*

$$A\underline{v} = a_1\lambda_1\underline{v}_1 + \dots + a_n\lambda_n\underline{v}_n.$$

## 15. COMPLEX VECTOR SPACES AND COMPLEX EIGENVALUES

I assume you have already met the complex numbers. The set of complex numbers is denoted by  $\mathbb{C}$ .

You may recall that the complex numbers were introduced to provide roots for polynomials that might have no real roots. The simplest example

is the polynomial  $x^2 + 1$ . It has no real roots because the square of a real number is *never* equal to  $-1$ .

Complex numbers play a similar role in linear algebra. Sometimes a matrix (with real entries) has no (real) eigenvalues. But it might have complex eigenvalues.

**15.1.** The matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

representing counterclockwise rotation by  $90^\circ$  has no eigenvalues because its characteristic polynomial  $x^2 + 1$  has no zeroes in  $\mathbb{R}$ , it does have complex eigenvalues and eigenvectors because

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

As this calculation suggests, all the matrix algebra and linear algebra we have done so far extends with essentially no change from  $\mathbb{R}$  to  $\mathbb{C}$ . We simply allow matrices to have entries that are complex numbers and instead of working with  $\mathbb{R}^n$  we work with  $\mathbb{C}^n$ ,  $n \times 1$  column matrices, or vectors, whose entries are complex numbers.

**15.2. The complex numbers.** I am assuming you have encountered the complex numbers before so this interlude is intended only as a reminder.

The set of complex numbers is formed by enlarging the set of real numbers through adjoining a square root of  $-1$ , denoted by  $i$ . The set of complex numbers is denoted by  $\mathbb{C}$ . Because we want to add and multiply complex numbers, and want  $\mathbb{C}$  to contain  $\mathbb{R}$  we define

$$\mathbb{C} := \{a + ib \mid a, b \in \mathbb{R}\}.$$

Notice that  $\mathbb{C}$  contains  $\mathbb{R}$ . The real numbers are just those complex numbers  $a + ib$  for which  $b = 0$ . Of course we don't bother to write  $a + i0$ ; we just write  $a$ .

It is helpful to think of  $\mathbb{C}$  as a 2-dimensional vector space over  $\mathbb{R}$ , the linear span of  $1$  and  $i$ . Every complex number can be written in a unique way as  $a + ib$  for suitable real numbers  $a$  and  $b$ . We add by

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

and multiply by

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc).$$

If  $b = d = 0$  we obtain the usual addition and multiplication of real numbers.

One important feature common to both the real and complex numbers is that non-zero numbers have inverses. If  $a + ib \neq 0$ , then

$$\frac{a - ib}{a^2 + b^2} = \frac{1}{a + ib}.$$

You should check this by multiplying them together.

**15.2.1. Reassurance.** If you haven't met the complex numbers before don't be afraid. You have done something very similar before and survived that so you will survive this. Once upon a time your knowledge of numbers was limited to fractions, i.e., all you knew was  $\mathbb{Q}$ . Try to imagine yourself in that situation again. I now come to you saying I have a new number that I will denote by the symbol  $\otimes$  and I wish to enlarge the number system  $\mathbb{Q}$  to a new number system that I will denote by  $\mathbb{Q}(\otimes)$ . I propose that in my new number system  $\otimes^2 = 2$  and that  $\mathbb{Q}(\otimes)$  consists of all expressions

$$a + b \otimes$$

with the addition and multiplication in  $\mathbb{Q}(\otimes)$  defined by

$$(a + b \otimes) + (c + d \otimes) := (a + c) + (b + d) \otimes$$

and

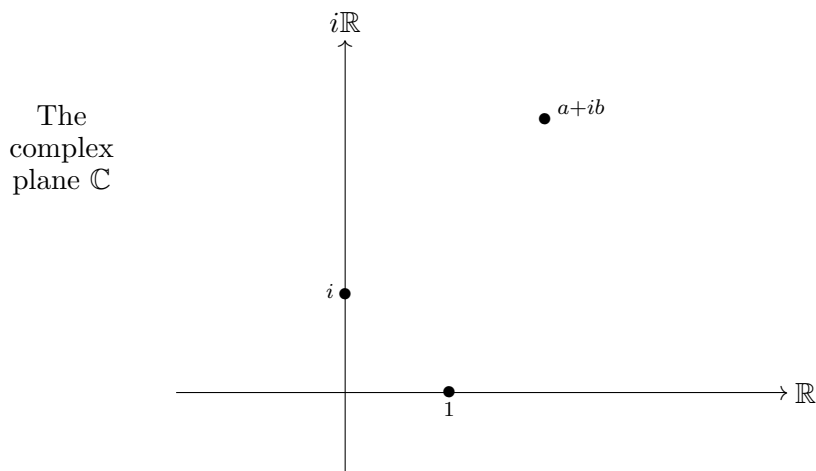
$$(a + b \otimes)(c + d \otimes) := (ac + 2bd) + (ad + bc) \otimes.$$

It might be a little tedious to check that the multiplication is associative, that the distributive law holds, and that every non-zero number in  $\mathbb{Q}(\otimes)$  has an inverse that also belongs to  $\mathbb{Q}(\otimes)$ . The number system  $\mathbb{Q}(\otimes)$  might also seem a little mysterious until I tell you that a copy of  $\mathbb{Q}(\otimes)$  can already be found inside  $\mathbb{R}$ . In fact,  $\otimes$  is just a new symbol for  $\sqrt{2}$ . The number  $a + b \otimes$  is really just  $a + b\sqrt{2}$ .

Now re-read the previous paragraph with this additional information.

**15.2.2. The complex plane.** Just as it is useful to picture the set of real numbers as the points on a line, the real line, so is it helpful to picture the set of complex numbers as the points on a plane. The horizontal axis is just the real line, sometimes called the real axis, and the vertical axis, sometimes

called the imaginary axis, is all real multiples of  $i$ . We denote it by  $i\mathbb{R}$ .



**15.2.3. Complex conjugation.** Let  $z = a + ib \in \mathbb{C}$ . The conjugate of  $z$ , denoted  $\bar{z}$ , is defined to be

$$\bar{z} := a - ib.$$

You can picture complex conjugation as reflection in the real axis. The important points are these

- $z\bar{z} = a^2 + b^2$  is a real number;
- $z\bar{z} \geq 0$ ;
- $z\bar{z} = 0$  if and only if  $z = 0$ ;
- $\sqrt{z\bar{z}}$  is the distance from zero to  $z$ ;
- $\overline{wz} = \bar{w}\bar{z}$ ;
- $\overline{w + z} = \bar{w} + \bar{z}$ ;
- $\bar{\bar{z}} = z$  if and only if  $z \in \mathbb{R}$ .

It is common to call  $\sqrt{z\bar{z}}$  the absolute value of  $z$  and denote it by  $|z|$ .

Performing the conjugate twice gets us back where we started:  $\bar{\bar{z}} = z$ . Reflecting twice in the real axis is the same as doing nothing.

**15.2.4. Roots of polynomials.** Complex numbers were invented to provide solutions to quadratic polynomials that had no real solution. The ur-example is  $x^2 + 1 = 0$  which has no solutions in  $\mathbb{R}$  but solutions  $\pm i$  in  $\mathbb{C}$ . This is the context in which you meet complex numbers at high school. You make the observation there that the solutions of  $ax^2 + bx + c = 0$  are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provided  $a \neq 0$  and the equation has no real solutions when  $b^2 - 4ac < 0$  but has two complex zeroes,

$$\lambda = -\frac{b}{2a} + i \frac{\sqrt{4ac - b^2}}{2a} \quad \text{and} \quad \bar{\lambda} = -\frac{b}{2a} - i \frac{\sqrt{4ac - b^2}}{2a}.$$

Notice that the zeroes  $\lambda$  and  $\bar{\lambda}$  are complex conjugates of each other.

It is a remarkable fact that although the complex numbers were invented with no larger purpose than providing solutions to *quadratic* polynomials, they provide all solutions to polynomials of *all* degrees.

**Theorem 15.1.** *Let  $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$  be a polynomial with coefficients  $a_1, \dots, a_n$  belonging to  $\mathbb{R}$ . Then there are complex numbers  $r_1, \dots, r_n$  such that*

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n).$$

Just as for quadratic polynomials, the zeroes of  $f$  come in conjugate pairs because if  $f(\lambda) = 0$ , then, using the properties of conjugation mentioned at the end of section 15.2.3,

$$\begin{aligned} f(\bar{\lambda}) &= \bar{\lambda}^n + a_1\bar{\lambda}^{n-1} + a_2\bar{\lambda}^{n-2} + \cdots + a_{n-1}\bar{\lambda} + a_n \\ &= \overline{\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n} \\ &= \overline{f(\lambda)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$

**15.2.5. Application to the characteristic polynomial of a real matrix.** Let  $A$  be an  $n \times n$  matrix whose entries are real numbers. Then  $\det(A - tI)$  is a polynomial with real coefficients. If  $\lambda \in \mathbb{C}$  is a zero of  $\det(A - tI)$  so is  $\bar{\lambda}$ .

Recall that the eigenvalues of  $A$  are exactly (the real numbers that are) the zeroes of  $\det(A - tI)$ . We wish to view the complex zeroes of  $\det(A - tI)$  as eigenvalues of  $A$  but to do that we have to extend our framework for linear algebra from  $\mathbb{R}^n$  to  $\mathbb{C}^n$ .

**15.3. Linear algebra over  $\mathbb{C}$ .** We now allow matrices to have entries that are complex numbers. Because  $\mathbb{R}$  is contained in  $\mathbb{C}$  linear algebra over  $\mathbb{C}$  includes linear algebra over  $\mathbb{R}$ , i.e., every matrix with real entries is a matrix with complex entries by virtue of the fact that every real number is a complex number.

Hence, left multiplication by an  $m \times n$  matrix with real entries can be viewed as a linear transformation  $\mathbb{C}^n \rightarrow \mathbb{C}^m$  in addition to being viewed as a linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ .

We extend the operation of complex conjugation to matrices. If  $A = (a_{ij})$  is a matrix with entries  $a_{ij} \in \mathbb{C}$ , we define the conjugate of  $A$  to be

$$\bar{A} := (\bar{a}_{ij}),$$

i.e., the matrix obtained by replacing every entry of  $A$  by its complex conjugate.

It is straightforward to verify that  $\overline{AB} = \bar{A}\bar{B}$  and  $\overline{A+B} = \bar{A} + \bar{B}$ .

**Proposition 15.2.** *Let  $A$  be an  $n \times n$  matrix with real entries. If  $\lambda \in \mathbb{C}$  is an eigenvalue for  $A$  so is  $\bar{\lambda}$ . Furthermore, if  $\underline{x}$  is a  $\lambda$ -eigenvector for  $A$ , then  $\bar{x}$  is a  $\bar{\lambda}$  eigenvector for  $A$ .*

**Proof.** This is a simple calculation. If  $A\underline{x} = \lambda\underline{x}$ , then

$$A\bar{x} = \overline{A\underline{x}} = \overline{\lambda\underline{x}} = \bar{\lambda}\bar{x}$$

as claimed.  $\square$

The first half of Proposition 15.2 is a consequence of the fact that if  $\lambda$  is a zero of a polynomial with real coefficients so is  $\bar{\lambda}$ .

**15.3.1. The conjugate transpose.** For complex matrices, i.e., entries having complex entries, one often combines the operations of taking the transpose and taking the conjugate. Notice that these operations commute, i.e., it doesn't matter which order one performs them: in symbols

$$(\overline{A})^{\top} = \overline{A^{\top}}.$$

We often write

$$A^* := \overline{A^{\top}}$$

and call this the conjugate transpose of  $A$ .

**15.4. The complex norm.** Let  $z \in \mathbb{C}$ . The norm of  $z$  is the number

$$\|z\| = \sqrt{z\bar{z}}.$$

If  $z \neq 0$ ,

$$z^{-1} = \bar{z}/\|z\|.$$

Check this.

We can extend the notion of norm, or “size”, from complex numbers to vectors in  $\mathbb{C}^n$ .

One important difference between complex and real linear algebra involves the definition of the norm of a vector. Actually, this change already begins when dealing with complex numbers. The distance of a real number  $x$  from zero is given by its absolute value  $|x|$ . The distance of a point  $x = (x_1, x_2) \in \mathbb{R}^2$  from zero is  $\sqrt{x_1^2 + x_2^2}$ . Picturing the complex plane as  $\mathbb{R}^2$ , the distance of  $z = a + ib$  from 0 is

$$\sqrt{a^2 + b^2} = \sqrt{z\bar{z}}.$$

If  $\mathbb{R}^n$ , the distance of a point  $\underline{x} = (x_1, \dots, x_n)^{\top}$  from the origin is given by taking the square root of its dot product with itself:

$$\|\underline{x}\| = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{\underline{x} \cdot \underline{x}} = \sqrt{\underline{x}^{\top} \underline{x}}.$$

Part of the reason this works is that a sum of squares of real numbers is always  $\geq 0$  and is equal to zero only when each of those numbers is zero. This is no longer the case for complex numbers.

The norm of  $x \in \mathbb{C}^n$  is

$$\|\underline{x}\| = \sqrt{\underline{x}^* \underline{x}}.$$

Since

$$\underline{x}^* \underline{x} = \sum_{j=1}^n \bar{x}_j x_j = \sum_{j=1}^n |x_j|^2$$

is a real number  $\geq 0$  the definition of  $\|\underline{x}\|$  makes sense—it is the non-negative square root of the real number  $\underline{x}^* \underline{x}$ .

**15.4.1.** *Real symmetric matrices have real eigenvalues.* The next result is proved by a nice application of the complex norm.

**Proposition 15.3.** *Let  $A$  be real symmetric  $n \times n$  matrix. Then all the eigenvalues of  $A$  are real.*

**Proof.** Let  $\lambda$  be an eigenvalue for  $A$ . We will show  $\lambda$  is a real number by proving that  $\lambda = \bar{\lambda}$ . Because  $\lambda$  is an eigenvalue, there is a non-zero vector  $\underline{x} \in \mathbb{C}^n$  such that  $A\underline{x} = \lambda\underline{x}$ . Then

$$\lambda \|\underline{x}\|^2 = \lambda \underline{x}^* \underline{x} = \underline{x}^* \lambda \underline{x} = \underline{x}^* A \underline{x} = (A^T \bar{\underline{x}})^T \underline{x} = (A \bar{\underline{x}})^T \underline{x}$$

where the last equality follows from the fact that  $A$  is symmetric. Continuing this string of equalities, we obtain

$$(A \bar{\underline{x}})^T \underline{x} = (\bar{\lambda} \bar{\underline{x}})^T \underline{x} = \bar{\lambda} \underline{x}^* \underline{x} = \bar{\lambda} \|\underline{x}\|^2.$$

We have shown that

$$\lambda \|\underline{x}\|^2 = \bar{\lambda} \|\underline{x}\|^2$$

from which we deduce that  $\lambda = \bar{\lambda}$  because  $\|\underline{x}\|^2 \neq 0$ . Hence  $\lambda \in \mathbb{R}$ .  $\square$

This is a clever proof. Let's see how to prove the  $2 \times 2$  case in a less clever way. Consider the symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

where  $a, b, d$  are real numbers. The characteristic polynomial of  $A$  is

$$t^2 - (a + d)t + ad - b^2 = 0$$

and your high school quadratic formula tells you that the roots of this polynomial are

$$\pm \frac{1}{2} \left( a + d \pm \sqrt{(a + d)^2 - 4(ad - b^2)} \right)$$

However,

$$(a + d)^2 - 4(ad - b^2) = (a - d)^2 + 4b^2$$

which is  $\geq 0$  because it is a sum of squares. Hence  $\sqrt{(a + d)^2 - 4(ad - b^2)}$  is a real number and we conclude that the roots of the characteristic polynomial, i.e., the eigenvalues of  $A$ , are real.

It might be fun for you to try to prove the  $3 \times 3$  case by a similar elementary argument.

## 16. SIMILAR MATRICES

**16.1.** Similar matrices are just that, similar, very similar, in a sense so alike as to be indistinguishable in their essential properties. Although a precise definition is required, and will be given shortly, it should be apparent that the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are, in the everyday sense of the word, “similar”. Multiplication by  $A$  and multiplication by  $B$  are linear transformations from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ ;  $A$  sends  $\underline{e}_1$  to itself and kills  $\underline{e}_2$ ;  $B$  sends  $\underline{e}_2$  to itself and kills  $\underline{e}_1$ . Pretty similar behavior, huh!?

### 16.2. Definition and first properties. .

Let  $A$  and  $B$  be  $n \times n$  matrices. We say that  $A$  is similar to  $B$  and write  $A \sim B$  if there is an invertible matrix  $S$  such that

$$A = SBS^{-1}.$$

Similarity of matrices is an equivalence relation meaning that

- (1)  $A \sim A$  because  $A = IAI^{-1}$ ;
- (2) if  $A \sim B$ , then  $B \sim A$  because  $A = SBS^{-1}$  implies  $S^{-1}AS = S^{-1}(SBS^{-1})S = (S^{-1}S)B(S^{-1}S) = IB I = B$ ;
- (3) if  $A \sim B$  and  $B \sim C$ , then  $A \sim C$  because  $A = SBS^{-1}$  and  $B = TCT^{-1}$  implies  $A = SBS^{-1} = S(TCT^{-1})S^{-1} = (ST)C(T^{-1}S^{-1}) = (ST)C(ST)^{-1}$ .

In particular, in this technical usage the word “similar” behaves as it does in its everyday use: it makes sense that

- (1) a thing is similar to itself;
- (2) if one thing is similar to another thing, then the other thing is similar to the first thing;
- (3) if a thing is similar to a second thing and the second thing is similar to a third thing, then the first thing is similar to the third thing.

**16.3. Example.** Let’s return to the example in section 16.1. The only difference between  $A$  and  $B$  is the choice of labeling: which order do we choose to write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or, if you prefer, which is labelled  $\underline{e}_1$  and which is labelled  $\underline{e}_2$ .

There is a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that interchanges  $\underline{e}_1$  and  $\underline{e}_2$ , namely

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad S\underline{e}_1 = \underline{e}_2 \quad \text{and} \quad S\underline{e}_2 = \underline{e}_1.$$

Since  $S^2 = I$ ,  $S^{-1} = S$  and

$$SBS^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A.$$

The calculation shows that  $A = SBS^{-1}$ ; i.e.,  $A$  is similar to  $B$  in the technical sense of the definition in section 16.2.

**16.4.** We now show that similar matrices have some of the same properties.

**Theorem 16.1.** *Similar matrices have the same*

- (1) *determinant;*
- (2) *characteristic polynomial;*
- (3) *eigenvalues.*

**Proof.** Suppose  $A$  and  $B$  are similar. Then  $A = SBS^{-1}$  for some invertible matrix  $S$ .

(1) Since  $\det(S^{-1}) = (\det S)^{-1}$ , we have

$$\det A = (\det S)(\det B)(\det S^{-1}) = (\det S)(\det B)(\det S)^{-1} = \det B.$$

We used the fact that the determinant of a matrix is a number.

(2) Since  $S(tI)S^{-1} = tI$ ,  $A - tI = S(B - tI)S^{-1}$ ; i.e.,  $A - tI$  and  $B - tI$  are similar. (2) now follows from (1) because the characteristic polynomial of  $A$  is  $\det(A - tI)$ .

(3) The eigenvalues of a matrix are the roots of its characteristic polynomial. Since  $A$  and  $B$  have the same characteristic polynomial they have the same eigenvalues.  $\square$

**16.4.1. Warning:** Although they have the same eigenvalues similar matrices do *not* usually have the same eigenvectors or eigenspaces. Nevertheless, there is a precise relationship between the eigenspaces of similar matrices. We prove that in Proposition 16.2

**16.4.2. Notation:** If  $A$  is an  $n \times n$  matrix and  $X$  a subset of  $\mathbb{R}^n$ , we use the notation

$$AX := \{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}.$$

This shorthand is similar to the notation  $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$  for the even numbers.

To check whether you understand this notation try the following problems.

- (1) Show that  $AX$  is a subspace if  $X$  is.
- (2) Show that  $A(BX) = (AB)X$  if  $A$  and  $B$  are  $n \times n$  matrices.
- (3) Show that  $IX = X$ .
- (4) If  $S$  is an invertible matrix and  $Y = SX$ , show that  $X = S^{-1}Y$ .
- (5) If  $S$  is an invertible matrix and  $X$  is a subspace of  $\mathbb{R}^n$  show that  $\dim SX = \dim X$  by proving the following claim: if  $\{\underline{v}_1, \dots, \underline{v}_d\}$  is a basis for  $X$ , then  $\{S\underline{v}_1, \dots, S\underline{v}_d\}$  is a basis for  $SX$ .

**Proposition 16.2.** *Suppose  $A = SBS^{-1}$ . Let  $E_\lambda(A)$  be the  $\lambda$ -eigenspace for  $A$  and  $E_\lambda(B)$  the  $\lambda$ -eigenspace for  $B$ . Then  $E_\lambda(B) = S^{-1}E_\lambda(A)$ , i.e.,*

$$E_\lambda(B) = \{S^{-1}\underline{x} \mid \underline{x} \in E_\lambda(A)\}.$$

*In particular, the dimensions of the  $\lambda$ -eigenspaces for  $A$  and  $B$  are the same.*

**Proof.** If  $\underline{x} \in E_\lambda(A)$ , then  $\lambda\underline{x} = A\underline{x} = SBS^{-1}\underline{x}$  so

$$B(S^{-1}\underline{x}) = S^{-1}\lambda\underline{x} = \lambda(S^{-1}\underline{x});$$

i.e.,  $S^{-1}\underline{x}$  is a  $\lambda$ -eigenvector for  $B$  or, equivalently,

$$S^{-1}E_\lambda(A) \subset E_\lambda(B).$$

Starting from the fact that  $B = S^{-1}AS$ , the same sort of argument shows that  $SE_\lambda(B) \subset E_\lambda(A)$ .

Therefore

$$E_\lambda(B) = I.E_\lambda(B) = S^{-1}S.E_\lambda(B) \subset S^{-1}.E_\lambda(A) \subset E_\lambda(B).$$

In particular,  $E_\lambda(B) \subset S^{-1}.E_\lambda(A) \subset E_\lambda(B)$  so these three sets are equal, i.e.,  $E_\lambda(B) = S^{-1}.E_\lambda(A) = \{S^{-1}\underline{x} \mid \underline{x} \in E_\lambda(A)\}$ .

That  $\dim E_\lambda(A) = \dim E_\lambda(B)$  is proved by method suggested in exercise 5 just prior to the statement of this proposition.  $\square$

If  $A$  and  $B$  are similar and  $A^r = 0$ , then  $B^r = 0$ . To see this one uses the marvelous cancellation trick,  $S^{-1}S = I$ .

**Corollary 16.3.** *If  $A$  and  $B$  are similar matrices, then  $\text{nullity}(A) = \text{nullity}(B)$  and  $\text{rank}(A) = \text{rank}(B)$ .*

**Proof.** By definition, the nullity of a matrix is the dimension of its null space. But  $\mathcal{N}(A) = E_0(A)$ , the 0-eigenspace of  $A$ . By the last sentence in the statement of Proposition 16.2,  $E_0(A)$  and  $E_0(B)$  have the same dimension. Hence  $A$  and  $B$  have the same nullity. Since  $\text{rank} + \text{nullity} = n$ ,  $A$  and  $B$  also have the same rank.  $\square$

**16.4.3. Intrinsic and extrinsic properties of matrices.** It is not unreasonable to ask whether a matrix that is similar to a symmetric matrix is symmetric. Suppose  $A$  is symmetric and  $S$  is invertible. Then  $(SAS^{-1})^T = (S^{-1})^T A^T S^T = (S^T)^{-1} A S^T$  and there is no obvious reason which this should be the same as  $SAS^{-1}$ . The explicit example

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 8 & 11 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ 3 & -2 \end{pmatrix}$$

shows that a matrix similar to a symmetric matrix need not be symmetric.

PAUL - More to say.

**16.5. Diagonalizable matrices.**

**16.5.1. Diagonal matrices.** A square matrix  $D = (d_{ij})$  is diagonal if  $d_{ij} = 0$  for all  $i \neq j$ . In other words, the non-zero entries in  $D$ , if any, lie on the main diagonal. For example, the identity matrix and the zero matrix are diagonal matrices. A diagonal matrix is one of the form

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

The  $\lambda_i$ s are the eigenvalues of the matrix. Its determinant is the product  $\lambda_1\lambda_2\cdots\lambda_n$ . Its characteristic polynomial is  $(t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n)$ .

The standard basis vectors are eigenvectors,  $D\mathbf{e}_j = \lambda_j\mathbf{e}_j$ . Thus each line/axis  $\mathbb{R}\mathbf{e}_j$  is contained in an eigenspace, but these are the only eigenspaces *if and only if* the  $\lambda_i$ s are different from one another. For example, if  $\lambda_i = \lambda_j$ , then the  $\lambda_i$ -eigenspace contains the plane  $\mathbb{R}\mathbf{e}_i + \mathbb{R}\mathbf{e}_j$ .

**16.5.2. Definition.** An  $n \times n$  matrix  $A$  is diagonalizable if it is similar to a diagonal matrix, i.e., if there is an invertible matrix  $S$  such that  $S^{-1}AS$  is diagonal.

Certainly a diagonal matrix is diagonalizable because if  $D$  is diagonal, then  $I^{-1}DI$  is!

Since similar matrices have the same characteristic polynomials, the characteristic polynomial of a diagonalizable matrix is a product of  $n$  linear terms.

**16.5.3. Example.** Let

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$S^{-1}AS = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus  $A$  is diagonalizable.

**16.5.4.** The obvious questions are *how do we determine whether  $A$  is diagonalizable or not* and *if  $A$  is diagonalizable how do we find an  $S$  such that  $S^{-1}AS$  is diagonal*. The next theorem answers both these questions. Its proof shows that  $S$  is a matrix whose columns are linearly independent eigenvectors for  $A$ . That description also shows us that there will be many such  $S$ .

**Theorem 16.4.** *Let  $A$  be an  $n \times n$  matrix. Then  $A$  is diagonalizable if and only if  $A$  has  $n$  linearly independent eigenvectors.*

**Proof.** ( $\Leftarrow$ ) Let  $\{\underline{u}_1, \dots, \underline{u}_n\}$  be a set of linearly independent eigenvectors for  $A$  and define

$$S := [\underline{u}_1, \dots, \underline{u}_n],$$

i.e.,  $S$  is the matrix whose  $j^{\text{th}}$  column is  $\underline{u}_j$ . Because the  $\underline{u}_j$ s are linearly independent  $S$  is invertible and

$$I = S^{-1}S = S^{-1}[\underline{u}_1, \dots, \underline{u}_n] = [S^{-1}\underline{u}_1, \dots, S^{-1}\underline{u}_n].$$

Hence  $S^{-1}\underline{u}_j = \underline{e}_j$ , the  $j^{\text{th}}$  standard basis vector.

Let  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  be such that  $A\underline{u}_j = \lambda_j\underline{u}_j$ . Then

$$\begin{aligned} S^{-1}AS &= S^{-1}A[\underline{u}_1, \dots, \underline{u}_n] \\ &= S^{-1}[A\underline{u}_1, \dots, A\underline{u}_n] \\ &= S^{-1}[\lambda_1\underline{u}_1, \dots, \lambda_n\underline{u}_n] \\ &= [\lambda_1S^{-1}\underline{u}_1, \dots, \lambda_nS^{-1}\underline{u}_n] \\ &= [\lambda_1\underline{e}_1, \dots, \lambda_n\underline{e}_n] \\ &= \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}. \end{aligned}$$

Thus  $A$  is diagonalizable.

( $\Rightarrow$ ) Now suppose  $A$  is diagonalizable. Then there is an invertible matrix  $C$  such that

$$C^{-1}AC = D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Now

$$\begin{aligned} CD &= C[\lambda_1\underline{e}_1, \dots, \lambda_n\underline{e}_n] \\ &= [\lambda_1C\underline{e}_1, \dots, \lambda_nC\underline{e}_n] \\ &= [\lambda_1\underline{C}_1, \dots, \lambda_n\underline{C}_n] \end{aligned}$$

where  $\underline{C}_j$  denotes the  $j^{\text{th}}$  column of  $C$ . But  $CD = AC$  so

$$\begin{aligned} [\lambda_1\underline{C}_1, \dots, \lambda_n\underline{C}_n] &= A[\underline{C}_1, \dots, \underline{C}_n] \\ &= [A\underline{C}_1, \dots, A\underline{C}_n]. \end{aligned}$$

Thus  $A\underline{C}_j = \lambda_j\underline{C}_j$  for all  $j$ . Since  $C$  is invertible its columns are linearly independent. Hence  $\{\underline{C}_1, \dots, \underline{C}_n\}$  consists of  $n$  linearly independent eigenvectors for  $A$ .  $\square$

The proof shows that the matrix  $S$  for which  $S^{-1}AS$  is diagonal is obtained by finding  $n$  linearly independent eigenvectors for  $A$  and making those the columns of  $S$ .

**Corollary 16.5.** *If an  $n \times n$  matrix has  $n$  distinct eigenvalues it is diagonalizable.*

**Proof.** Suppose  $\lambda_1, \dots, \lambda_n$  are the distinct eigenvalues for  $A$  and for each  $i$  let  $\underline{v}_i$  be a non-zero vector such that  $A\underline{v}_i = \lambda_i\underline{v}_i$ . By Theorem 14.2,  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is a linearly independent set. It now follows from Theorem 16.4 that  $A$  is diagonalizable.  $\square$

**16.5.5. Previous example revisited.** We showed above that

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

is diagonalizable. Let's suppose we did not know that and had to find out whether it was diagonalizable and, if so to find a matrix  $S$  such that  $S^{-1}AS$  is diagonal. Theorem 16.4 says we must determine whether  $A$  has two linearly independent eigenvectors.

The characteristic polynomial of  $A$  is  $t^2 - t(a + d) + ad - bc$ , i.e.,

$$t^2 - t - 2 = (t - 2)(t + 1).$$

The eigenvalues for  $A$  are 2 and  $-1$ . By Corollary 16.5,  $A$  is diagonalizable. To find an  $S$  we need to find eigenvectors for  $A$ . The 2-eigenspace is

$$E_2 = \mathcal{N}(A - 2I) = \mathcal{N} \begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The  $(-1)$ -eigenspace is

$$E_{-1} = \mathcal{N}(A + I) = \mathcal{N} \begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus  $S$  can be any matrix with one column a non-zero multiple of  $(2 \ 1)^T$  and the other a non-zero multiple of  $(1 \ 1)^T$ . For example, the matrix we used before, namely

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

works. A different  $S$  that would work is

$$S = \begin{pmatrix} 3 & -2 \\ 3 & -1 \end{pmatrix}.$$

Now

$$S^{-1}AS = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

**16.6. Orthogonal matrices.** A matrix  $A$  is orthogonal if  $A^T = A^{-1}$ .

An orthogonal linear transformation is a linear transformation implemented by an orthogonal matrix, i.e., if  $T(\underline{x}) = A\underline{x}$  and  $A$  is an orthogonal matrix we call  $T$  an orthogonal linear transformation.

Orthogonal linear transformations are of great importance because they do not change the lengths of vectors or, as a consequence of that, the angles between vectors. For this reason we often call the linear transformations given by orthogonal matrices “rigid motions”. Moving a rigid object, by which we mean some kind of physical structure, a rocket or piece of machinery, in space does not change the lengths of its pieces or the angles between them.

**16.6.1.** The identity matrix is orthogonal. A product of orthogonal matrices is an orthogonal matrix, and the inverse of an orthogonal matrix is orthogonal. The set of all  $n \times n$  orthogonal matrices is called the orthogonal group and is denoted by  $O(n)$ . It is one of the fundamental objects in mathematics.

**16.6.2. Rigid motions in  $\mathbb{R}^2$ .** We have met two kinds of rigid motions in  $\mathbb{R}^2$ , rotations and reflections. Let’s check that both are orthogonal transformations. If  $A_\theta$  denotes the matrix representing the rotation by an angle  $\theta$  in the counterclockwise direction, then

$$A_\theta(A_\theta)^T = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence  $A_\theta$  is an orthogonal matrix.

On the other hand, reflection in the line  $L$  through the origin and a non-zero point  $\begin{pmatrix} a \\ b \end{pmatrix}$  is given by the matrix

$$A = \frac{1}{a^2 + b^2} \begin{pmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{pmatrix}.$$

This is an orthogonal linear transformation because

$$A^T A = A^2 = \frac{1}{(a^2 + b^2)^2} \begin{pmatrix} a^4 + 2a^2b^2 + b^4 & 0 \\ 0 & a^4 + 2a^2b^2 + b^4 \end{pmatrix} = I.$$

Notice an important difference between rotations and reflections: the determinant of a rotation is  $+1$  but the determinant of a reflection is  $-1$ .

**Proposition 16.6.** *Every rigid motion in  $\mathbb{R}^2$  is a composition of a rotation and a reflection.*

**Proof.** Let  $A$  be an orthogonal  $2 \times 2$  matrix. □

**16.6.3.** The standard basis vectors  $\underline{e}_1, \dots, \underline{e}_n$  are orthogonal to one another. We said that an orthogonal matrix preserves the angles between vectors so, if that is correct, an orthogonal matrix  $A$  will have the property that the vectors  $A\underline{e}_j$  are orthogonal to one another. But  $A\underline{e}_j$  is the  $j^{\text{th}}$  column of  $A$ . Thus, the next result confirms what we have been saying.

**Theorem 16.7.** *An  $n \times n$  matrix is orthogonal if and only if its columns form an orthonormal basis for  $\mathbb{R}^n$ .*

**Proof.** ( $\Rightarrow$ ) Suppose  $A = (a_{ij})$  is an orthogonal matrix. Then the  $ij^{\text{th}}$  entry of  $A^T A$  is 1 if  $i = j$  and 0 otherwise. But the  $ij^{\text{th}}$  entry is

$$\sum_{k=1}^n (A^T)_{ik} A_{kj} = \sum_{k=1}^n a_{ki} a_{kj} = \underline{A}_i^T \underline{A}_j,$$

the dot product of the  $i^{\text{th}}$  column with the  $j^{\text{th}}$  column. Hence, when  $i \neq j$ , the  $i^{\text{th}}$  column is orthogonal to the  $j^{\text{th}}$  column; and, taking  $i = j$ , we see that  $\|\underline{A}_i\|^2 = 1$ . Hence the columns of  $A$  do indeed form an orthonormal basis for  $\mathbb{R}^n$ .

( $\Leftarrow$ ) Suppose  $A = (a_{ij})$  is an  $n \times n$  matrix whose columns form an orthonormal basis for  $\mathbb{R}^n$ . Then

$$\underline{A}_i^T \underline{A}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

It follows that  $A^T A = I$  and  $AA^T = I$ . Hence  $A$  is orthogonal.  $\square$

**Theorem 16.8.** *Let  $A$  be a symmetric matrix with real entries. Then there is an orthogonal matrix  $Q$  such that  $Q^{-1}AQ$  is diagonal.*

**Proof.**  $\square$

**Theorem 16.9.** *Let  $A$  be a diagonal matrix with real entries. For every orthogonal matrix  $Q$ ,  $Q^{-1}AQ$  is symmetric.*

**Proof.** A matrix is symmetric if it is equal to its transpose. We have

$$(Q^{-1}AQ)^T = Q^T A^T (Q^{-1})^T = Q^{-1} A^T (Q^T)^T = Q^{-1}AQ$$

so  $Q^{-1}AQ$  is symmetric.  $\square$

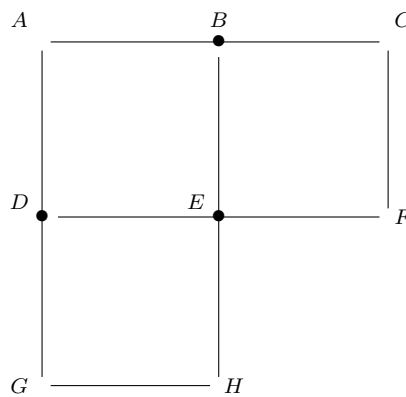
## 17. APPLICATIONS OF LINEAR ALGEBRA AND VECTOR SPACES

There are so many applications of linear algebra and vector spaces that many volumes are needed to do justice to the title of this section. Linear algebra and vector spaces are embedded deep in the heart of modern technology and the modern world.

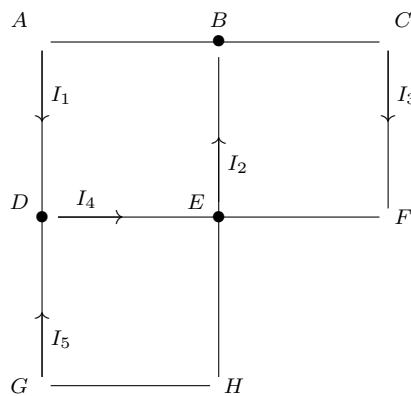
**17.1. Simple electrical circuits.** The flow of current in a simple electrical circuit consisting of batteries and resistors is governed by three laws:

- **Kirchoff's First Law** The sum of the currents flowing into a node is equal to the sum of the currents flowing out of it.
- **Kirchoff's Second Law** The sum of the voltage drops around a closed loop is equal to the total voltage in the loop.
- **Ohm's Law** The voltage drop across a resistor is the product of the current and the resistor.

To make apply these we must first draw a picture of the circuit. To simplify I will begin by leaving out the resistors and batteries. A typical circuit might look like this:



There are three loops and three nodes in this circuit. We must put arrows on each loop indicating the direction the current is flowing. It doesn't matter how we label these currents, or the direction we choose for the current flowing around each loop.



Applying Kirchoff's First Law at the nodes  $B$ ,  $D$ , and  $E$ , we deduce that

$$\begin{aligned} I_1 + I_3 &= I_2 \\ I_1 + I_5 &= I_4 \\ I_2 + I_5 &= I_3 + I_4. \end{aligned}$$

I am going to refer you to the book for more on this because I can't draw the diagrams!

**17.2. Least-squares solutions to inconsistent systems.** Recall that a system of linear equations is **consistent** if it has a solution and **inconsistent** if it doesn't. Although we have pretty much ignored inconsistent systems for what might seem sensible reasons, it *is* important to consider inconsistent systems because although there may be no solution to  $A\underline{x} = \underline{b}$  we might still want to find an  $\underline{x}$  such that  $A\underline{x}$  is as close as possible to  $\underline{b}$ . For example, a solution to the linear system  $A\underline{x} = \underline{b}$  might represent an over-optimistic goal, making a cost zero, for example, but we still want to do the best we can.

**17.2.1. A problem: find the line that best fits the data.** Given points

$$(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)$$

in  $\mathbb{R}^2$  find the line  $y = dx + c$  that best approximates those points. In general there will not be a line passing through all the points but we still want "the best line". This sort of problem is typical when gathering data to understand the relationship between two variables  $x$  and  $y$ . The data obtained might suggest a linear relationship between  $x$  and  $y$  but, especially in the social sciences, it is rare that the data gathered spells out an exact linear relationship.

We restate the question of finding the line  $y = dx + c$  as a linear algebra problem. The problem of finding the line that fits the data  $(\alpha_i, \beta_i)$ ,  $1 \leq i \leq m$  is equivalent to finding  $d$  and  $c$  such that

$$\begin{aligned} \beta_1 &= d\alpha_1 + c \\ \beta_2 &= d\alpha_2 + c \\ \beta_3 &= d\alpha_3 + c \\ &\vdots \\ \beta_m &= d\alpha_m + c \end{aligned}$$

which is, in turn, Equivalent to finding  $d$  and  $c$  such that

$$\begin{pmatrix} \alpha_1 & 1 \\ \alpha_2 & 1 \\ \alpha_3 & 1 \\ \vdots & \vdots \\ \alpha_m & 1 \end{pmatrix} \begin{pmatrix} d \\ c \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix} \quad \text{or} \quad A \begin{pmatrix} d \\ c \end{pmatrix} = \underline{b}$$

where  $A$  is an  $m \times 2$  matrix and  $\underline{b} \in \mathbb{R}^m$ .

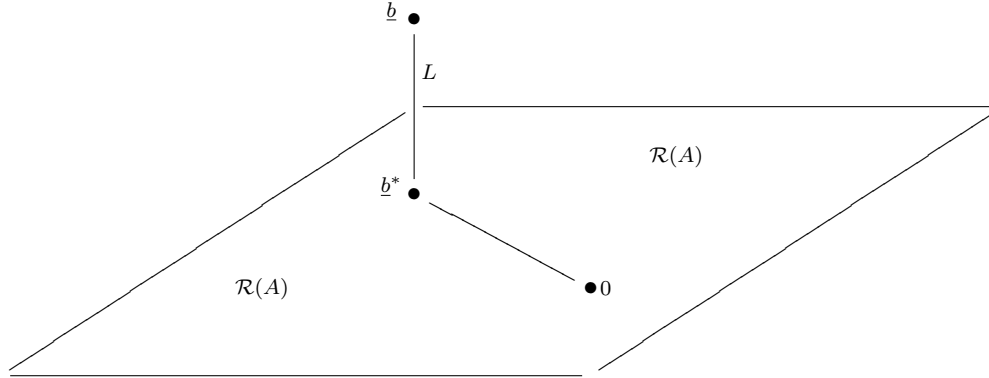
As we said, there is usually no solution  $\begin{pmatrix} d \\ c \end{pmatrix}$  to this system of linear equations but we still want to find a line that is as close as possible to the given points. The next definition makes the idea "as close as possible" precise.

**Definition 17.1.** Let  $A$  be a given  $m \times n$  matrix and  $\underline{b}$  a given vector in  $\mathbb{R}^m$ . We call  $\underline{x}^*$  a least-squares solution to  $A\underline{x} = \underline{b}$  if

$$\|A\underline{x}^* - \underline{b}\| \leq \|A\underline{x} - \underline{b}\| \quad \text{for all } \underline{x} \in \mathbb{R}^n.$$

We will now explain how to find  $\underline{x}^*$

Because  $A\underline{x}$  is in the range of  $A$  we want  $A\underline{x}$  to be the (unique!) point in  $\mathcal{R}(A)$  that is closest to  $\underline{b}$ . Let's write  $\underline{b}^*$  for the point in  $\mathcal{R}(A)$  that is closest to  $\underline{b}$ . A picture helps:



**Lemma 17.2.** Given  $\underline{b}$  not in  $\mathcal{R}(A)$ , there is a unique line  $L$  in  $\mathbb{R}^m$  that is perpendicular to  $\mathcal{R}(A)$  and passes through  $\underline{b}$ . Let  $\underline{b}^*$  be the point where  $L$  meets  $\mathcal{R}(A)$ . Then

- (1)  $\underline{b}^*$  is the point in  $\mathcal{R}(A)$  that is closest to  $\underline{b}$ ;
- (2)  $\underline{b}^*$  is the unique point in  $\mathcal{R}(A)$  such that  $\underline{b} - \underline{b}^*$  is perpendicular to  $\mathcal{R}(A)$ .
- (3)  $\underline{b}^*$  is the unique point in  $\mathcal{R}(A)$  such that  $A^T(\underline{b} - \underline{b}^*) = 0$ .

**Proof.** Parts (1) and (2) are obvious.

(3) Since  $\mathcal{R}(A)$  is spanned by the columns of  $A$ ,  $\underline{b} - \underline{b}^*$  is perpendicular to  $\mathcal{R}(A)$  if and only if it is perpendicular to every column of  $A$ . Let  $\underline{A}_j$  denote the  $j^{\text{th}}$  column of  $A$ . Now  $\underline{b} - \underline{b}^*$  is perpendicular to  $\underline{A}_j$  if and only if  $\underline{A}_j^T(\underline{b} - \underline{b}^*) = 0$ .<sup>12</sup> Hence  $\underline{b} - \underline{b}^*$  is perpendicular to  $\mathcal{R}(A)$  if and only if  $\underline{A}_j^T(\underline{b} - \underline{b}^*) = 0$  for all  $j = 1, \dots, n$ . But  $\underline{A}_j^T(\underline{b} - \underline{b}^*) = 0$  for  $j = 1, \dots, n$  if and only if  $\underline{A}^T(\underline{b} - \underline{b}^*) = 0$ .  $\square$

The point in  $\mathcal{R}(A)$  closest to  $\underline{b}$  is the unique point  $\underline{b}^*$  in  $\mathcal{R}(A)$  such that  $A^T\underline{b} = A^T\underline{b}^*$ .

Since  $\underline{b}^*$  is in  $\mathcal{R}(A)$ , there is a vector  $\underline{x}^*$  such that  $A\underline{x}^* = \underline{b}^*$ . It follows that

$$A^T A\underline{x}^* = A^T \underline{b}^* = A^T \underline{b}.$$

<sup>12</sup>The multiplication  $\underline{A}_j^T(\underline{b} - \underline{b}^*)$  is allowed because  $A^T$  is an  $n \times m$  matrix and  $\underline{b}$  is an  $m \times 1$  matrix.

The preceding argument has proved the following result.

**Proposition 17.3.** *There is a point  $\underline{x}^*$  satisfying the following equivalent properties:*

- (1)  $A^T A \underline{x}^* = A^T \underline{b}$ ;
- (2)  $A \underline{x}^*$  is the point in  $\mathcal{R}(A)$  that is closest to  $\underline{b}$ ;
- (3)  $\underline{x}^*$  is a least-squares solution to  $A \underline{x} = \underline{b}$ .

Moreover,  $\underline{x}^*$  is unique if and only if  $\mathcal{N}(A) = 0$ .

**Theorem 17.4.** *Let  $A \underline{x} = \underline{b}$  be an  $m \times n$  system of linear equations.*

- (1) *The system  $A^T A \underline{x} = A^T \underline{b}$  is consistent.*
- (2) *The solutions to the system  $A^T A \underline{x} = A^T \underline{b}$  are least-squares solutions to  $A \underline{x} = \underline{b}$ .*
- (3) *There is a unique least squares solution if and only if  $\text{rank } A = n$ .*

**17.2.2.** *Why call it least-squares?* We are finding an  $\underline{x}$  that minimizes  $\|A \underline{x} - \underline{b}\|$  and hence  $\|A \underline{x} - \underline{b}\|^2$ . But

$$\|A \underline{x} - \underline{b}\|^2 = (A \underline{x} - \underline{b})^T (A \underline{x} - \underline{b})$$

is a sum of squares!

**17.3. Approximating data by polynomial curves.** Given points

$$(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)$$

in  $\mathbb{R}^2$  we might want to find a degree 2 polynomial  $f(x)$  such that the curve  $y = f(x)$  best approximate those points. In general there will not be a degree 2 curve that passes through all the points but we still want “the best curve”. We could also ask for the best cubic curve, etc. All these problems are solved by using the general method in section 17.2. For example, if we are looking for a degree two curve we are looking for  $a, b, c$  such that  $y = ax^2 + bx + c$  best fits the data. That problem reduces to solving the system

$$\begin{pmatrix} \alpha_1^2 & \alpha_1 & 1 \\ \alpha_2^2 & \alpha_2 & 1 \\ \alpha_3^2 & \alpha_3 & 1 \\ \vdots & \vdots & \vdots \\ \alpha_m^2 & \alpha_m & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix} \quad \text{or} \quad A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{b}$$

where  $A$  is an  $m \times 3$  matrix and  $\underline{b} \in \mathbb{R}^m$ .

So, using the ideas in section 17.2 we must solve the equation

$$A^T A \underline{x}^* = A^T \underline{b}$$

to get an  $a^*$ ,  $b^*$ , and  $c^*$  that gives a parabola  $y = a^*x^2 + b^*x + c^*$  that best fits the data.

**17.4. Magic squares.** You have probably seen the following example of a magic square:

4	3	8
9	5	1
2	7	6

The numbers in each row, column, and diagonal add up to 15. Let's look at this from the point of view of linear algebra and learn a little about this magic square and then try to construct some larger ones.

Let's call any  $n \times n$  matrix with the property that the numbers in each row, column, and diagonal add up to the same number,  $\sigma$  say, a  $\sigma$ -valued magic square. We will call  $\sigma$  the magic value. We will write

$$MS_n(\sigma)$$

for the set of  $\sigma$ -valued magic squares. If the numbers in the matrix are  $1, 2, \dots, n^2$  we will call it a classical magic square.

The sum of all integers from 1 to  $n^2$  is  $\frac{1}{2}n^2(n^2 + 1)$ , so in a classical  $n \times n$  magic square the sum of the numbers in a row is  $\frac{1}{2}n(n^2 + 1)$ . For example when  $n = 3$ , the magic value is 15. When  $n = 4$  it is 34.

The  $n \times n$  matrix with every entry equal to 1 is an  $n$ -valued magic square. We will write  $E_n$  for this matrix or just  $E$  if the  $n$  is clear. Although this will not strike you as a particularly interesting  $n$ -valued magic square it will play a useful role in showing that it an appropriate system of linear equations is consistent – it will be a solution to them—and in reducing the problem from to a system of *homogenous* linear equations.

The set of all  $n \times n$  matrices forms a vector space: we can add them and multiply by scalars and the properties in Proposition ?? hold. We usually write  $M_n(\mathbb{R})$  for the set of all  $n \times n$  matrices. The subset of 0-valued magic squares  $MS_n(0)$  is a subspace of  $M_n(\mathbb{R})$ . The next lemma says that

$$MS_n(\sigma) = \frac{\sigma}{n}E + MS_n(0);$$

i.e., the set of  $\sigma$ -valued magic squares is a translation of the subspace  $MS_n(0)$ .

**Lemma 17.5.** *Let  $A$  be an  $n \times n$  matrix. Then  $A$  is a  $\sigma$ -valued magic square if and only if  $A - \frac{\sigma}{n}E$  is a 0-valued magic square.*

For example, when  $n = 3$ ,

$$\begin{pmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{pmatrix} - 5E = \begin{pmatrix} -1 & -2 & 3 \\ 4 & 0 & -4 \\ -3 & 2 & 1 \end{pmatrix}$$

is a 0-valued magic square.

We now consider the problem of finding all 0-valued  $3 \times 3$  magic squares. We will write  $x_{ij}$  for the unknown value of the  $ij^{\text{th}}$  entry. For example, the fact that sum of the top row of the matrix must be zero gives us the equation  $x_{11} + x_{12} + x_{13} = 0$ . And so on. Doing the obvious thing we get a total

of 8 homogeneous equations in 9 unknowns—3 equations from the rows, 3 from the columns, and 2 from the diagonals. By Proposition ?? a system of homogeneous equations always has a non-trivial solution when the number of rows is strictly smaller than the number of unknowns. Hence there is a non-trivial  $3 \times 3$  0-valued magic square.

Before we can write down the coefficient matrix for the system of equations we must agree on an ordering for the unknowns. We will just use the naive ordering and write them in the order

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}.$$

The coefficient matrix is

$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	main diagonal column 2 column 3 column 1 row 2 row 1 anti-diagonal row 3
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I don't want to present all the row operations in gory detail, but we can progressively obtain the equivalent matrices

$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	MD C2 C3 C1-MD AD-C3 R2+R1+R3-C1-C2-C3 R1-MD-C2-C3 row 3
--	---

then

$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 2 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	MD C2 C3 C1-MD AD-C3 R2+R1+R3-C1-C2-C3 R1-MD-C2-3C3+2AD row 3
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We can already see from this that  $x_{32}$  and  $x_{33}$  will be the independent variables, so let's take

$$x_{33} = 1, \quad x_{32} = 2.$$

Evaluating the dependent variables gives

$$x_{31} = -3, \quad x_{23} = -4, \quad x_{22} = 0, \quad x_{21} = 4, \quad x_{13} = 3, \quad x_{12} = -2, \quad x_{11} = -1,$$

which is the 0-valued magic square we obtained before.

The rank of the matrix is  $8-1=7$ , so its nullity is 2. That is equivalent to the existence of two independent variables, so let's find another solution that together with the first will give us a basis. It is obvious though that a  $3 \times 3$  matrix has various symmetries that we can exploit.

For the  $n \times n$  case we get a homogeneous system of  $2n + 2$  equations in  $n^2$  variables. If  $n > 2$  this system has a non-trivial solution.

## 18. LAST RITES

**18.1. A summary of notation.** Sets are usually denoted by upper case letters.

$f : X \rightarrow Y$  denotes a function  $f$  from a set  $X$  to a set  $Y$ . If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are functions, then  $g \circ f$  or  $gf$  denotes the composition *first do  $f$  then do  $g$* . Thus  $gf : X \rightarrow Z$  and is defined by

$$(gf)(x) = g(f(x)).$$

This makes sense because  $f(x)$  is an element of  $Y$  so  $g$  can be applied to it.

The identity function  $\text{id}_X : X \rightarrow X$  is defined by

$$\text{id}_X(x) = x \quad \text{for all } x \in X.$$

If  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are such that  $gf = \text{id}_X$  and  $fg = \text{id}_Y$  we call  $g$  the inverse of  $f$  and denote it by  $f^{-1}$ . Of course, if  $g = f^{-1}$ , then  $f = g^{-1}$ .

We denote the range of a function  $f : X \rightarrow Y$  by  $\mathcal{R}(f)$ . Thus,  $\mathcal{R}(f) = \{f(x) \mid x \in X\}$ . It is a subset of  $Y$ .

If  $A$  is an  $m \times n$  matrix we often write  $\underline{A}_1, \dots, \underline{A}_n$  for the columns of  $A$  and

$$A = [\underline{A}_1, \dots, \underline{A}_n]$$

to indicate this. This notation usually appears when we want to make use of the fact that

$$BA = B[\underline{A}_1, \dots, \underline{A}_n] = [B\underline{A}_1, \dots, B\underline{A}_n],$$

i.e., the  $j^{\text{th}}$  column of  $BA$  is  $B$  times the  $j^{\text{th}}$  column of  $A$ .

We write  $\underline{e}_j$  for the element in  $\mathbb{R}^n$  having zeroes in all positions except the  $j^{\text{th}}$  which is 1. Thus, the identity matrix is  $I = [\underline{e}_1, \dots, \underline{e}_n]$ . More generally, if  $A$  is any  $m \times n$  matrix, then  $A\underline{e}_j = \underline{A}_j$ , the  $j^{\text{th}}$  column of  $A$ . You can see this either by computation or using the fact that  $AI = A$ .

If  $\underline{x}_1, \dots, \underline{x}_n$  are vectors, then  $\mathbb{R}\underline{x}_1 + \dots + \mathbb{R}\underline{x}_n$  is synonymous with  $\text{Sp}(\underline{x}_1, \dots, \underline{x}_n)$ . It denotes the set of all linear combinations of the vectors  $\underline{x}_1, \dots, \underline{x}_n$ .

The  $\lambda$ -eigenspace is denoted by  $E_\lambda$ . If it is necessary to name the matrix for which it is the eigenspace we write  $E_\lambda(A)$ .

**18.2. A sermon.** Learning a new part of mathematics is like learning other things: cooking, riding a bike, playing chess, listening to music, playing music, tennis, tasting wine, archery, seeing different kinds of art, etc. At some point one does, or does not, make the transition to a deeper involvement where one internalizes and makes automatic the steps that are first learned by rote. At some point one does, or does not, integrate the separate discrete aspects of the activity into a whole.

You will know whether you are undergoing this transition or not. It may be happening slowly, more slowly than you want, but it is either happening or not. Be sensitive to whether it is happening. Most importantly, ask yourself whether you want it to happen and, if you do, how much work you want to do to make it happen. It will not happen without your active involvement.

Can you make fluent sentences about linear algebra? Can you formulate the questions you need to ask in order to increase your understanding? Without this internalization and integration you will feel more and more like a juggler with too many balls in the air. Each new definition, concept, idea, is one more ball to keep in the air. It soon becomes impossible unless one sees the landscape of the subject as a single entity into which the separate pieces fit together neatly.

Linear algebra will remain a great struggle if this transition is not happening. A sense of mastery will elude you. Its opposite, a sense that the subject is beyond you, will take root—paralysis and fear will set in. That is the dark side of all areas of life that involve some degree of competence, public performance, evaluation by others, and consequences that you care about.

Very few people, perhaps none, get an A in 300- and higher-level math courses unless they can integrate the separate parts into a whole in which the individual pieces not only make sense but seem inevitable and natural.

It is you, and only you, who will determine whether you learn and understand linear algebra. It is not easy.

Good luck!

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