**Remarks on Definitions**

## 1. Bad Definitions

Definitions are the foundation of mathematics. Linear algebra bulges with definitions and one of the biggest challenge for students is to master them. It is difficult. Too many students address that difficulty by complaining that questions on exams asking for precise definitions are "not mathematics" or "more like the questions in an English course than a math course". Other say they "understand the concepts" but don't know the definitions.

The assertion that definitions, or asking students to know them, is "not mathematics" seems to rest on the fact that prior to linear algebra one's experience of mathematics involves a lot of calculation and the performance of operations that one is trained to do successfully. Mathematics is developed to *avoid* or *simplify* calculation. The theorems are powerful tools that allow one to finesse and minimize calculations. The definitions encapsulate the concepts.

Enough preaching.

Some examples of inadequate definitions appear below. All examples are verbatim answers of real students. I have not changed their wording. The questions had the following format: I wrote the first part of the definition and the students then had to complete the sentence thereby giving the definition. So, below I will give the first part of the sentence that was on the exam, and then I'll follow it by a correct answer, and then by answers given by students, and a brief explanation of what is wrong with each answer.

One thing you will often see is the student's response is not the definition as given in the book, but a theorem that says some other condition is either equivalent to the definition or a consequence of it. One must learn to distinguish these three things.

You will see examples where a student inserts unnecessary words into the definition. You will see examples where a student has proposed a definition that includes parts of the definitions of other words.

(1) (This definition is not confined to the context of linear algebra.) Let $X$ and $Y$ be arbitrary sets. A function $f : X \to Y$ is <u>one-to-one</u> if ______________

<u>Correct Answer:</u>
$f(x) \neq f(x')$ whenever $x \neq x'$ <u>OR</u>
$f(x) = f(x')$ only when $x = x'$.

<u>Inadequate Answers:</u>

- for every value $x$ there is one value $y$.
  - I don't know what this sentence means. The definition should always be complete. The reader should not have to insert information into the proposed definition to complete it.
- $f(x) = f(x')$ when $x = x'$.
  - Every function has this property.
- $f(x) = f(x')$ and $x = x'$ <u>or</u> $f(x) \neq f(x')$ and $x \neq x'$.
  - It is difficult to parse this sentence. A definition should be open to only one interpretation. If the definition consists of a single sentence that sentence should has a simple grammatical structure.
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(2) (This definition is not confined to the context of linear algebra.) Let $X$ and $Y$ be arbitrary sets. The <u>range</u> of a function $f : X \to Y$ is

$$\mathcal{R}(f) := \{\_\_ \mid \_\_\_\_\_\_\_\_\}.$$

<u>Correct Answer:</u>
$\mathcal{R}(f) := \{f(x) \mid x \in X\}$ <u>OR</u>
$\mathcal{R}(f) := \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$.

<u>Inadequate Answers:</u>

- $\mathcal{R}(f) := \{f(x) \mid x \in \mathbb{R}\}$.
  - The question has nothing to do with $\mathbb{R}$. The word "range" applies to *all* functions. That is why we allow $X$ and $Y$ to be arbitrary sets. By restricting $X$ to $\mathbb{R}$ the author is destroying the generality of the definition.
- $\mathcal{R}(f) = \{\underline{x} \in X \mid f(x) \in Y\}$.
  - In this definition the symbols $\underline{x}$ and $x$ are used—different symbols stand for different things. Is this author wanting $\underline{x}$ and $x$ to stand for the same thing. Probably, but when we must guess an author's intent the proposed definition is inadequate. Even if we assume that $\underline{x}$ and $x$ stand for the same thing we can see that the author has no real understanding of what the range is because he has put $f(x)$ after "such that". Also, by starting the definition $\{\underline{x} \in X \mid \cdots\}$ the author is saying the set being defined is a subset of $X$ whereas the range is a subset of $Y$. This suggests the writer has tried to remember the definition as a sequence of words/symbols without paying attention to their *meaning*. There is no picture in the author's head of $f$ sending

elements of $X$ to elements of $Y$ and the range being the values $f$ takes.

- $\mathcal{R}(f) := f(Y)$.

(3) (This definition is not confined to the context of linear algebra.) Let $X$ and $Y$ be arbitrary sets. A function $f : X \to Y$ is onto if _______________.

Correct Answer:
$\mathcal{R}(f) = Y$; OR
if $y \in Y$, there is an $x \in X$ such that $y = f(x)$; OR
every $y \in Y$ is equal to $f(x)$ for some $x \in X$. OR
$f(X) = Y$.

Inadequate Answers:

- $f(x) = Y$. The symbol $x$ is not the same as $X$. Presumably, $x$ denotes an element in $X$. The notation $f(X)$ denotes the set of all $f(x)$ as $x$ ranges over all elements of $X$. More sucinctly, $f(X) = \{f(x) \mid x \in X\}$.
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(4) We call $\{\underline{v}_1, \dots, \underline{v}_d\}$ an orthonormal basis for a subspace $W$ if _________.

Correct Answer:
it is a basis for $W$ and is an orthogonal set and $||\underline{v}_i|| = 1$ for all $i$.

Inadequate Answers:

- $\{\underline{v}_1, \dots, \underline{v}_d\}$ are linearly independent.
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(5) Let $A$ be an $n \times n$ matrix. We call $\lambda \in \mathbb{R}$ an eigenvalue of $A$ if _______________

Correct Answer:
there is a non-zero vector $\underline{x}$ such that $A\underline{x} = \lambda\underline{x}$.

Inadequate Answers:

- $A\underline{x} = \lambda\underline{x}$ for some $\underline{x} \in \mathbb{R}$.
  - In the definition it is essential that $\underline{x}$ be non-zero. If the definition fails to exclude the possibility that $\underline{x}$ is zero, then *every* $\lambda \in \mathbb{R}$ would be an eigenvector because $A\underline{0} = \lambda\underline{0}$ for all $\lambda \in \mathbb{R}$. The next two proposed answers suffer from the same defect.
- $A\underline{x} = \lambda\underline{x}$ as long as $\lambda \in \mathbb{R}$.
- it satisfies the equation $A\underline{x} = \lambda\underline{x}$ for any $\underline{x} \in \mathbb{R}^n$.
- $\det(A - \lambda I) = 0$.

(6) Let $A$ be an $n \times n$ matrix. We call $\underline{x} \in \mathbb{R}^n$ an eigenvector for $A$ if _______________

Correct Answer:
$A\underline{x} = \lambda\underline{x}$ for some $\lambda \in \mathbb{R}$.

Inadequate Answers:

- there is a non-trivial solution to $A\underline{x} = \lambda\underline{x}$.
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(7) Let $\lambda$ be an eigenvalue for the $m \times n$ matrix $A$. The <u>λ-eigenspace</u> for $A$ is the set

$$E_\lambda := \{\_\_ \mid \_\_\_\_\_\_\_\_\}.$$

<u>Correct Answer:</u>
$E_\lambda := \{\underline{x} \mid A\underline{x} = \lambda\underline{x}\}$.

<u>Inadequate Answers:</u>
- $E_\lambda := \{A\underline{x} = \lambda\underline{x} \mid \lambda \in \mathbb{R}\}$.
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(8) Let $V$ and $W$ be subspaces. A function $T : V \to W$ is a <u>linear transformation</u> if ________

<u>Correct Answer:</u>
$T(a\underline{u} + b\underline{v}) = aT(\underline{u}) + bT(\underline{v})$ for all $\underline{u}, \underline{v} \in V$ and all $a, b \in \mathbb{R}$. <u>OR</u>
$T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ for all $\underline{u}, \underline{v} \in V$ and $T(\underline{u}) = aT(\underline{u})$for all $\underline{u} \in V$ and all $a \in \mathbb{R}$.
<u>Inadequate Answers:</u>
- $T(\underline{u}+\underline{v}) = T(\underline{u})+T(\underline{v})$ for all $\underline{u}, \underline{v} \in V$ and $T(\underline{u}) = aT(\underline{u})$ for all $\underline{u} \in V$ and some $a \in \mathbb{R}$.
- $T(\underline{u}+\underline{v}) = T(\underline{u})+T(\underline{v})$ for all $\underline{u}, \underline{v} \in V$ and $T(\underline{u}) = aT(\underline{u})$ for all $\underline{u} \in V$ and some $a \in \mathbb{R}$.
- $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ for $\underline{u}, \underline{v} \in V$ and $T(\underline{u}) = aT(\underline{u})$ for $\underline{u} \in V$.
  - The word "all" in the correct answer is an essential part the definition. To be a linear transformation $T$ must satisfy a certain property with respect to <u>all possible</u> $\underline{u}$, $\underline{v}$, $a$, and $b$.
- $T(v) \in W$.
  - The author does not understand that $T$ must be a special kind of function to be a linear transformation.
- $T(a\underline{u} + b\underline{v}) = aT(\underline{u}) + bT(\underline{v})$ where $\underline{u}, \underline{v} \in V$ and $a, b \in \mathbb{R}$.
  - The word "where" is vague, not even appropriate in this sentence. It does not convey the meaning of "for all" or "whenever". The proposed answer could be read as a statement about a single $\underline{u}$, $\underline{v}$, $a$, and $b$.

(9) The <u>range</u> of a linear transformation $T : V \to W$ is

$$\mathcal{R}(T) := \{\_\_ \mid \_\_\_\_\_\_\_\_\}.$$

<u>Correct Answer:</u>
$\{T(\underline{x}) \mid \underline{x} \in V\}$

<u>Inadequate Answers:</u>
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(10) The <u>null space</u> of a linear transformation $T : V \to W$ is

$$\mathcal{N}(T) := \{\_\_ \mid \_\_\_\_\_\_\_\_\_\}.$$

<u>Correct Answer:</u>
$\{\underline{x} \in V \mid T(\underline{x}) = 0\}$

<u>Inadequate Answers:</u>

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(11) An $n \times n$ matrix $A$ is <u>non-singular</u> if the only solution to _______________.

<u>Correct Answer:</u>
solution to $A\underline{x} = 0$ is $\underline{x} = 0$.

**Comments:**

<u>Inadequate Answers:</u>

- it is linearly independent.
- $A\underline{x} = \underline{b}$ is $A\underline{b}^{-1}$
- $A_1x_1 + \cdots A_nx_n$ is $A_1 = \cdots = A_n = 0$.
- the characteristic polynomial is zero.

(12) A vector $\underline{w}$ is a <u>linear combination</u> of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if _________

<u>Correct Answer:</u>
$\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some $a_1, \dots, a_n \in \mathbb{R}$.

**Comments:**

<u>Inadequate Answers:</u>

- $\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$.
- its components are $\{\underline{v}_i + \underline{v}_j\}$.
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(13) The <u>linear span</u> of $\{\underline{v}_1, \dots, \underline{v}_n\}$ consists of _________

<u>Correct Answer:</u>
all linear combinations of $\underline{v}_1, \dots, \underline{v}_n$.

**Comments:**

<u>Inadequate Answers:</u>

- the linear combination of $\{\underline{v}_1, \dots, \underline{v}_n\}$.
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(14) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is <u>linearly independent</u> if the only solution to_______________

<u>Correct Answer:</u>
the equation $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ is $a_1 = \cdots = a_n = 0$.

**Comments:**

<u>Inadequate Answers:</u>

- $a_1\underline{v}_1, \cdots, a_n\underline{v}_n = 0$ and $a_1 = \cdots = a_n = 0$.
  - First, this is not a sentence. Second, there is no equation; well, maybe $a_n\underline{v}_n = 0$ is an equation. Third, there is no information about the solution to any equation: that is probably because the author used the word "and" when he/she should have used "is".
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- no $\underline{v}_i$ is a multiple of $\underline{v}_j$

(15) A subset $W$ of $\mathbb{R}^n$ is a subspace if it satisfies the following three conditions: _______________.

Correct Answer:
$0 \in W$; $\underline{u} + \underline{v} \in W$ whenever $\underline{u} \in W$ and $\underline{v} \in W$; $a\underline{u} \in W$ for all $a \in \mathbb{R}$ and all $\underline{u} \in W$.

Inadequate Answers:
- must contain the zero space; must be open when adding, and open when multiplying.
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(16) Let $W$ be a subspace of $\mathbb{R}^n$. A subset of $W$ is a basis for $W$ if _______________.

Correct Answer:
it is linearly independent and spans $W$.

Inadequate Answers:
- it is linearly independant (sic) and it is a span of $W$
- the subset's columns are linearly independent and span $W$.
- the linear combination of the subset spans $W$.
- it is an orthogonal set that spans $W$.
- it is a linearly independent set of $n$ vectors.

(17) The dimension of a subspace $W \subset \mathbb{R}^n$ is _______________.

Correct Answer:
the number of elements in a basis for it.

Inadequate Answers:
- the number of elements in $W$;
- $n$
- the number of elements in the basis for $W$;
- the space spanned by $W$.

(18) Two $n \times n$ matrices $A$ and $B$ are similar if _______________

there is an invertible matrix $S$ such that $B = S^{-1}AS$.

(19) An $n \times n$ matrix $A$ is diagonalizable if _______________

Correct Answer:
it is similar to a diagonal matrix OR
there is an invertible matrix $S$ such that $S^{-1}AS$ is diagonal.

Inadequate Answers:

- they have the same row reduced echelon form.
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(20) An $n \times n$ matrix $Q$ is orthogonal if ______________

Correct Answer:
$Q^T = Q^{-1}$

Inadequate Answers:

- its columns form a basis for $\mathbb{R}^n$
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