- We will write $\underline{A}_1, \dots, \underline{A}_n$ for the columns of an $m \times n$ matrix A.
- Several questions involve an unknown vector $\underline{x} \in \mathbb{R}^n$. We will write $x_1, \dots, x_n$ for the entries of $\underline{x}$; thus $\underline{x} = (x_1, \dots, x_n)^T$.
- The null space and range of a matrix A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively.
- The linear span of a set of vectors is denoted by $\mathrm{Sp}(\underline{v}_1, \dots, \underline{v}_n)$.
- We will write $\underline{e}_1, \dots, \underline{e}_n$ for the standard basis for $\mathbb{R}^n$. Thus $\underline{e}_i$ has a 1 in the i^{th} position and zeroes elsewhere.
- In order to save space I will often write elements of $\mathbb{R}^n$ as row vectors, particularly in questions about linear transformations. For example, I will write $T(x, y) = (x + y, x - y)$ rather than

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}.$$

Scoring. On the True/False section you get $+1$ for each correct answer, -1 for each incorrect answer and 0 if you choose not to answer the question.

Part A.

Complete the definitions.
You do not need to write the part I have already written.
Just complete the sentence.

(1) (This definition is not confined to the context of linear algebra.) Let X and Y be arbitrary sets. A function $f : X \to Y$ is <u>one-to-one</u> if ______________

$f(x) \neq f(x')$ whenever $x \neq x'$
OR
$f(x) = f(x')$ only when $x = x'$.

(2) (This definition is not confined to the context of linear algebra.) Let X and Y be arbitrary sets. The <u>range</u> of a function $f : X \to Y$ is

$$\mathcal{R}(f) := \{\_\_ \mid \_\_\_\_\_\_\_\_\}.$$

$\mathcal{R}(f) := \{f(x) \mid x \in X\}$.
OR
$\mathcal{R}(f) := \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$.

(3) (This definition is not confined to the context of linear algebra.) Let X and Y be arbitrary sets. A function $f : X \to Y$ is <u>onto</u> if ______________.

$\mathcal{R}(f) = Y$;
OR
if $y \in Y$, there is an $x \in X$ such that $y = f(x)$;
OR
every $y \in Y$ is equal to $f(x)$ for some $x \in X$.

(4) (This definition is not confined to the context of linear algebra.) The composition of functions $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f :? \to ?$ defined by _______________.

$(g \circ f) : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

(5) The symbol i is used to denote a complex number that is _________.

a square root of -1.
Comments: We do not say i is *the* square root of -1 because there are *two* square roots of -1. To say i is "*the square root of* -1" is incorrect.

(6) The set of complex numbers $\mathbb{C}$ is

$$\mathbb{C} := \{\_\_ \mid \_\_\_\_\_\_\_\_\_\}.$$

$$\mathbb{C} := \{a + ib \mid a, b \in \mathbb{R}\}.$$

(7) The product of the complex numbers $a+ib$ and $c+id$ is equal to _________

$$(ac - bd) + i(ad + bc)$$

(8) The conjugate of the complex number $a + ib$ is equal to _________

$$a - ib$$

(9) The norm of the complex number $a + ib$ is equal to _________

$$\sqrt{a^2 + b^2}$$

(10) The norm of a vector $\underline{x} \in \mathbb{R}^n$ is denoted by _____ and is defined by _________.

$||\underline{x}||$ and is defined by $||\underline{x}|| := \sqrt{\underline{x}^T \underline{x}}$.

(11) The norm of a vector $\underline{x} \in \mathbb{C}^n$ is denoted by _________ and is defined by _________.

$||\underline{x}||$ and is defined by $||\underline{x}|| := \sqrt{\overline{\underline{x}}^T \underline{x}}$ where $\overline{\underline{x}}$ is the conjugate of $\underline{x}$.

(12) Two non-zero vectors $\underline{u}$ and $\underline{v}$ are orthogonal if _________.

$\underline{u}^T \underline{v} = 0$, or $\underline{v}^T \underline{u} = 0$.

(13) We call $\{\underline{v}_1, \ldots, \underline{v}_d\}$ an orthogonal set of vectors if _________.

$\underline{v}_i^T \underline{v}_j = 0$ for all $i \neq j$.

(14) We call $\{\underline{v}_1, \dots, \underline{v}_d\}$ an orthogonal basis for a subspace W if _________.

it is a basis for W and is an orthogonal set.

(15) We call $\{\underline{v}_1, \dots, \underline{v}_d\}$ an orthonormal basis for a subspace W if _________.

it is a basis for W and is an orthogonal set and $||\underline{v}_i|| = 1$ for all i.

(16) An $n \times n$ matrix A is orthogonal if _________.

$A^T = A^{-1}$.

(17) Let $\mathcal{B} = \{\underline{w}_1, \dots, \underline{w}_d\}$ be a basis for a subspace W of $\mathbb{R}^n$. Let $\underline{x} \in W$. We call $(a_1, \dots, a_d)$ the coordinates of x with respect to $\mathcal{B}$ if _________

$\underline{v} = a_1\underline{v}_1 + \cdots + a_d\underline{v}_d$.

(18) Let V and W be subspaces. A linear transformation from V to W is ________

a function $T : V \to W$ such that $T(a\underline{u}+b\underline{v}) = aT(\underline{u})+bT(\underline{v})$ for all $\underline{u}, \underline{v} \in V$.

(19) Let A be an $n\times n$ matrix. We call $\lambda \in \mathbb{R}$ an eigenvalue of A if _______________

there is a non-zero vector $\underline{x}$ such that $A\underline{x} = \lambda\underline{x}$.

(20) Let A be an $n \times n$ matrix. We call $\underline{x} \in \mathbb{R}^n$ an eigenvector for A if _______________

$A\underline{x} = \lambda\underline{x}$ for some $\lambda \in \mathbb{R}$.

(21) Let λ be an eigenvalue for the $m \times n$ matrix A. The λ-eigenspace for A is the set

$$E_\lambda := \{\_\_ \mid \_\_\_\_\_\_\_\_\_\}.$$

$$E_\lambda := \{\underline{x} \mid A\underline{x} = \lambda\underline{x}\}.$$

(22) Let A be an $n\times n$ matrix. The characteristic polynomial of A is _______________

$\det(A - tI)$.

(23) Write down the determinant

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

(24) Write down the determinant

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

(25) Two $n \times n$ matrices A and B are similar if _______________

there is an invertible matrix S such that $B = S^{-1}AS$.

(26) An $n \times n$ matrix A is diagonalizable if _______________

it is similar to a diagonal matrix
OR
there is an invertible matrix S such that $S^{-1}AS$ is diagonal.

(27) An $n \times n$ matrix Q is orthogonal if _______________

$Q^T = Q^{-1}$

Part B.

Complete the statements of the following results.
You do not need to write the part I have already written.
Just complete the sentence.

(28) The eigenvalues of a matrix A are the zeroes of _______________.

its characteristic polynomial.

(29) The zeroes of the characteristic polynomial of a matrix A are _______________.

its eigenvalues.

(30) Every orthogonal set of vectors is _________.

linearly independent.

(31) Let $\lambda_1, \dots, \lambda_r$ be distinct (i.e., all different) eigenvalues for a matrix A. If $\underline{v}_1, \dots, \underline{v}_r$ are non-zero vectors such that $\underline{v}_i$ is an eigenvector for A with eigenvalue λ_i, then $\{\underline{v}_1, \dots, \underline{v}_r\}$ is _________.

linearly independent.

(32) Let $a_1, \dots, a_n \in \mathbb{R}$ and let $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. Then there are complex numbers $r_1, \dots, r_n$ such that _________.

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n).$$

(33) Furthermore, in the context of the previous question if $f(\lambda) = 0$, then _________.

$f(\bar{\lambda}) = 0$.

(34) The λ-eigenspace of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$ because it is equal to _________.

the null space of $A - \lambda I$.

(35) Consequently, the λ-eigenspace of A is non-zero if and only if _________ singular.

the matrix $A - \lambda I$ is singular.

(36) If $\det A = 0$, then A is _________.

singular.

(37) A square matrix has an inverse if and only if its determinant is _________.

non-zero.

(38) For the purposes of this and the two questions we will say that an $m \times n$ matrix A <u>represents</u> the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ if $T(\underline{x}) = A\underline{x}$ for all $\underline{x} \in \mathbb{R}^n$.

How do you compute A from T?

The j^{th} column of A is $T(\underline{e}_j)$.

(39) If A represents $T : \mathbb{R}^n \to \mathbb{R}^m$ and B represents a linear transformation $S : \mathbb{R}^m \to \mathbb{R}^k$, what matrix represents ST?

BA.

(40) If A represents $T : \mathbb{R}^n \to \mathbb{R}^m$ and B represents $S : \mathbb{R}^n \to \mathbb{R}^m$, what matrix represents $aS + bT$?

$aA + bB$.

(41) If $\{\underline{v}_1, \ldots, \underline{v}_d\}$ is an orthogonal basis for W, then $\{$_________ $\}$ is an orthonormal basis for W.

$$\left\{ \frac{\underline{v}_1}{||\underline{v}_1||}, \frac{\underline{v}_2}{||\underline{v}_2||}, \ldots, \frac{\underline{v}_d}{||\underline{v}_d||} \right\}.$$

(42) Suppose $\mathcal{B} = \{\underline{w}_1, \ldots, \underline{w}_d\}$ is an orthogonal basis for W and let $\underline{v} \in W$. The simplest way to compute the coordinates $(a_1, \ldots, a_d)$ of $\underline{v}$ is to compute

$$a_i = \_\_\_\_\_\_\_\_\_.$$

$$a_i = \frac{\underline{v}^T \underline{w}_i}{||\underline{w}_i||^2} = \frac{\underline{v}^T \underline{w}_i}{\underline{w}_i^T \underline{w}_i}$$

(43) Let $\mathcal{B} = \{\underline{w}_1, \dots, \underline{w}_d\}$ be a basis for W. Applying the Gram-Schmidt process to $\mathcal{B}$ produces _________ for W.

an orthogonal basis.

(44) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is and V is a subspace of $\mathbb{R}^n$, then $T(V)$ is a _________

subspace of $\mathbb{R}^m$.

(45) A square matrix is singular if and only if its characteristic polynomial _________

has zero as a root.

(46) If $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations such that $S(\underline{e}_i) = T(\underline{e}_i)$ for all $i = 1, \dots, n$, then _________

$S = T$.

(47) The $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that is rotation in the counterclockwise direction by an angle θ. Then the matrix A_θ such that $T_\theta(\underline{x}) = A_\theta \underline{x}$ for all $\underline{x} \in \mathbb{R}^2$ is

$$A_\theta =$$

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(48) An $n \times n$ matrix is orthogonal if and only if its columns _________

form an orthonormal basis for $\mathbb{R}^n$.

(49) The set consisting of $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and _____ is an orthonormal basis for $\mathbb{R}^2$.

$\pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(50) The function $\sin : \mathbb{R} \to \mathbb{R}$ is not a linear transformation because _________

$$\sin(x+y) \neq \sin x + \sin y.$$

(51) Suppose $\{\underline{v}_1, \dots, \underline{v}_n\}$ is a basis for $\mathbb{R}^n$. Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations such that $S(\underline{v}_i) = T(\underline{v}_i)$ for all i. Then _________ because _________

$S = T$ because if $\underline{w} \in \mathbb{R}^n$, then $\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$ and

$$\begin{aligned} S(\underline{w}) &= S(a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_n\underline{v}_n) \\ &= a_1 S(\underline{v}_1) + a_2 S(\underline{v}_2) + \cdots + a_n S(\underline{v}_n) \\ &= a_1 T(\underline{v}_1) + a_2 T(\underline{v}_2) + \cdots + a_n T(\underline{v}_n) \\ &= T(a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_n\underline{v}_n) \\ &= T(\underline{w}). \end{aligned}$$

(52) If it exists, what is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} =?$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(53) Let A be an $n \times n$ matrix. If B is obtained from A by
(a) swapping two *adjacent* rows of A, then $\det B =?$
(b) multiplying a row in A by $c \in \mathbb{R}$, then $\det B =?$
(c) replacing row i by row i + a multiple of row $k \neq i$, then $\det B =?$

(a) $\det B = -\det A$
(b) $\det B = c \det A$
(c) $\det B = \det A$

(54) The linear span $\mathrm{Sp}\{\underline{u}_1, \ldots, \underline{u}_r)$ is the same as the linear span $\mathrm{Sp}(\underline{v}_1, \ldots, \underline{v}_s)$ if and only if every $\underline{u}_i$ is a linear _________ and every $\underline{v}_j$ is _________

every $\underline{u}_i$ is a linear combination of $\underline{v}_1, \ldots, \underline{v}_s$ and every $\underline{v}_j$ is in $\mathrm{Sp}\{\underline{u}_1, \ldots, \underline{u}_r)$.
OR
every $\underline{u}_i$ is a linear combination of $\underline{v}_1, \ldots, \underline{v}_s$ and every $\underline{v}_j$ is a linear combination of $\underline{u}_1, \ldots, \underline{u}_r$.

(55) Let Q be an orthogonal $n \times n$ matrix. Then $\det Q =$ _________

± 1

(56) Let Q be an orthogonal $n \times n$ matrix. Then $||Q\underline{x}|| =$ _________

$||\underline{x}||$ for all $\underline{x}$ in $\mathbb{R}^n$.

(57) Let Q be an orthogonal $n \times n$ matrix. Then $(Q\underline{x})^T(Q\underline{v}) =$ _________

$\underline{x}^T\underline{v}$ for all $\underline{x}$ and $\underline{v}$ in $\mathbb{R}^n$.

(58) Let A be a matrix having only real eigenvalues. Then there is an orthogonal matrix Q such that _________

$Q^{-1}AQ$ is upper triangular.

(59) Let A be a symmetric matrix. Then there is an orthogonal matrix Q such that _________

$Q^{-1}AQ$ is diagonal.

(60) Let D be a diagonal matrix and Q an orthogonal matrix. Then $Q^{-1}DQ$ is _________

a symmetric matrix.

Part C.

These questions involve some short calculations or arguments but I will simply grade them right or wrong so you don't need to show your work.

(61) The matrix that represents the linear transformation $T(x, y) = (x+2y, x-y)$ is _________

$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$.

(62) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (y, x)$ and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (-x, y)$. Then $ST(x, y)$= _________ and $TS(x, y)$= _________

$ST(x, y) = (-y, x)$ and $TS(x, y) = (y, -x)$.

(63) If $a + ib$ is a non-zero complex number its inverse is _________.

$$\frac{a - ib}{a^2 + b^2}.$$

(64) The eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

are the solutions to the equation _________.

$\lambda^2 - (a+d)\lambda + ad - bc = 0.$

(65) The 2×2 matrix _________ has no real eigenvalues.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(66) The 2×2 matrix _________ has exactly one real eigenvalues.

I

(67) The 2×2 matrix _________ has two distinct real eigenvalues.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(68) Let A and B be invertible $n \times n$ matrices. Simplify the following expression as much as possible:

$$(B^{-1}A^T)^{-1}B^{-1}(ABA)^T\left((AB)^T\right)^{-1}$$

(69) I claim that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix

$$A = \frac{1}{a^2+b^2}\begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{pmatrix}$$

is reflection in the line $ay = bx$. Verify this by
(a) showing that A has eigenvalues $+1$ and -1;
(b) showing that the 1-eigenspace is the line $ay = bx$;
(c) showing that the (-1)-eigenspace is the line perpendicular to $ay = bx$;

Part D.

True or False—just write T or F

(1) If the characteristic polynomial of A is $(t-1)^3(t+1)^2$, then A is singular.

F

(2) If the characteristic polynomial of A is $t^3(t+1)^2$, then A is singular.

T

(3) If a square matrix has a row of zeroes its is singular.

T

Comments: There are several different ways to see this. If the i^{th} row of A is zero, then the i^{th} row of the product AB is zero for every matrix B, so AB can never be the identity matrix. Or, A^T has a column consisting entirely of zeroes so it its columns are linearly dependent (every set of vectors that contains the zero vector is linearly dependent), and it is therefore singular because a square matrix is non-singular if and only if its columns are linearly dependent.

(4) There is an invertible matrix A such that $AA^T = 0$.

F
Comments: If A had a inverse and $AA^T = 0$, then $0 = AA^T = A^{-1}(AA^T) = (A^{-1}A)A^T = IA^T = A^T$. But then $A = (A^T)^T = 0$ and that is absurd—the zero matrix does not have an inverse.

(5) (I make no assumptions about the matrix A in this question.) A solution to the equation $A\underline{x} = \underline{b}$ is given by $\underline{x} = A^{-1}\underline{b}$.

F
Comments: Because no assumptions about A are made, A need not have an inverse. It may not even be a square matrix.

(6) Consider a consistent system of linear equations in 5 unknowns. If there are 4 left-most 1s in its reduced echelon form, there are 4 dependent variables.

T
Comments:

(7) The matrix AA^T is always symmetric.

T
Comments:

(8) The matrix AA^T is always square.

T
Comments:

(9) If $BD = EB = I$, then $D = E$.

T
Comments:

(10) Every set of five vectors in $\mathbb{R}^4$ is linearly dependent.

T
Comments: If a subspace W has dimension p, then every set of $p + 1$ elements in W is linearly dependent. That is a theorem in section 3.5(?) the book. It is very important that you not only know the answer to this and the next question but you know *why* the answers are what they are.

(11) If a subset of $\mathbb{R}^4$ spans $\mathbb{R}^4$ it is linearly independent.

F

Comments: If a subspace W has dimension p, a set that spans W is linearly independent if and only if it has exactly p elements. That is a theorem in section 3.5(?) the book.

(12) A homogeneous linear system of 15 equations in 16 unknowns always has a non-zero solution.

T
Comments: See the section in the book about homogeneous linear systems.

(13) If S is a linearly dependent subset of $\mathbb{R}^n$ so is every subset of $\mathbb{R}^n$ that contains S.

T
Comments: It is very important that you know the answer to this and the next question.

(14) Every subset of a linearly independent set is linearly independent.

T
Comments:

(15) A square matrix is singular if and only if its transpose is singular.

T
Comments: First, a square matrix is either singular or non-singular. A matrix is non-singular if and only if it has an inverse. But if A has an inverse then $(A^{-1})^T$ is the inverse of A^T. Thus a matrix is non-singular if and only if its transpose is. Hence a matrix is singular if and only if its transpose is.

(16) If A is singular and similar to B, then B is singular.

T

(17) If A has no real eigenvalues and is similar to B, then B has no real eigenvalues.

T

(18) The set $W = \{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = 0\}$ is a subspace of $\mathbb{R}^5$.

T
Comments: It is very important that you not only know the answer to this and the next question but you know *why* the answers are what they are. The W in this question is a subspace because the set of solutions to *any* set of homogeneous equations is a subspace.

(19) The set $W = \{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a subspace of $\mathbb{R}^4$.

F
Comments: $0 \notin W$.

(20) The solutions to a system of homogeneous linear equations always form a subspace.

T
Comments: Look at the proof in the book – it is important to understand why this is true.

(21) The solutions to a system of linear equations always form a subspace.

F
Comments: Question 25 gives a counterexample. Do you know what I mean by *a counterexample*.

(22) If $\mathrm{Sp}(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$, then $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$.

T
Comments: $\underline{u}$ is certainly in $\mathrm{Sp}(\underline{u}, \underline{v}, \underline{w})$ because it is equal to $1.\underline{u} + 0.\underline{v} + 0\underline{w}$. The hypothesis that $\mathrm{Sp}(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$ therefore implies that $\underline{u} \in Sp(\underline{v}, \underline{w})$. But $Sp(\underline{v}, \underline{w})$ is, by definition, all linear combinations of $\underline{v}$ and $\underline{w}$, so $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$

An important step towards mastering the material in this course is to be able to answer this and the next question *instantly*, and to know why the answers are what they are. In fact, I would go so far as to say that if you can't answer this and the next question *instantly* you are struggling.

(23) If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $Sp(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$.

T
Comments:

(24) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, then $\{\underline{v}_1 - 2\underline{v}_2, 2\underline{v}_2 - 3\underline{v}_3, 3\underline{v}_3 - \underline{v}_1\}$ is linearly dependent.

T
Comments: $1.(\underline{v}_1 - 2\underline{v}_2) + 1.(2\underline{v}_2 - 3\underline{v}_3) + 1.(3\underline{v}_3 - \underline{v}_1) = 0.$

(25) The null space of an $m \times n$ matrix is contained in $\mathbb{R}^m$.

F
Comments:

(26) The range of an $m \times n$ matrix is contained in $\mathbb{R}^n$.

F
Comments:

(27) If $\underline{a}$ and $\underline{b}$ belong to $\mathbb{R}^n$, then the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 0\}$ is a subspace of $\mathbb{R}^n$.

T

Comments: See comment about question 15.

(28) If $\underline{a}$ and $\underline{b}$ belong to $\mathbb{R}^n$, then the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 1\}$ is a subspace of $\mathbb{R}^n$.

F
Comments: $0 \notin W$

(29) If $\underline{a}$ and $\underline{b}$ belong to $\mathbb{R}^n$, then the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 1\} \cup \{\underline{0}\}$ a subspace of $\mathbb{R}^n$?

F
Comments: If $\underline{x} \in W$ and $\underline{x} \neq 0$, then $2\underline{x} \notin W$.

(30) If A and B are non-singular $n \times n$ matrices, so is AB.

T
Comments: If $AB\underline{x} = 0$, then $B\underline{x} = 0$ because A is non-singular, but $B\underline{x} = 0$ implies $\underline{x} = 0$ because B is non-singular.

(31) If A and B are non-singular $n \times n$ matrices, so is $A + B$.

F
Comments: For example, I and $-I$ are non-singular but there sum is the zero matrix which is singular.

(32) Let A be an $n \times n$ matrix. If the rows of A are linearly dependent, then A is singular.

T
Comments:

(33) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

T
Comments: It is easy to check, and the book does show you this, that if A' is obtained from A by a *single* elementary row operation, then A obtained from A' by a single elementary row operation. Now just string together a sequence of elementary row operations, and reverse each one of them to get back from B to A.

(34) There is a matrix whose inverse is $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$.

F

Comments: The inverse of a matrix A has an inverse, namely A. The given matrix is obviously not invertible because its columns (and rows) are not linearly dependent. So it cannot be the inverse of a matrix.

(35) If $A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

F
Comments: A is a 2×2 matrix and E is a 2×3 matrix. The product BA only exists when B is an $m \times 2$ matrix for some m and in that case BA is an $m \times 2$ matrix, so cannot equal a 2×3 matrix.

(36) Any linearly independent set of five vectors in $\mathbb{R}^5$ is a basis for $\mathbb{R}^5$.

T
Comments:

(37) The row space of the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

T
Comments: The rows are clearly linearly independent (if that is not clear to you then you do not understand linear (in)dependence) but any three linearly independent vectors form a basis for $\mathbb{R}^3$.

Perhaps it is better to answer this directly: if (a, b, c) is any vector in $\mathbb{R}^3$, then

$$(a, b, c) = \tfrac{1}{3}(3, 0, 0) + \tfrac{1}{2}(0, 2, 0) + \tfrac{1}{3}(0, 0, 3)$$

and it is clear that (a, b, c) can't be written as a linear combination of the rows in any other way.

(38) The subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is the same as the subspace spanned by $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

T
Comments: Each is a multiple of the other: the linear span of a single non-zero vector is the line through that vector and 0, i.e., consists of all multiples of the given vector.

(39) The subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}6\\4\\2\end{pmatrix}$ is the same as the subspace spanned by $\begin{pmatrix}3\\2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\\6\end{pmatrix}$.

T
Comments:

(40) For all matrices A and B, $\mathcal{N}(A) \subset \mathcal{N}(AB)$.

F
Comments:

(41) For all matrices A and B, $\mathcal{N}(B) \subset \mathcal{N}(AB)$.

T
Comments:

(42) For all matrices A and B, $\mathcal{N}(AB) \subset \mathcal{N}(A)$.

F
Comments:

(43) $\{24x \mid x \in \mathbb{Z}\} \subset \{8x \mid x \in \mathbb{Z}\}$

T
Comments: Every integer multiple of 24 is an integer multiple of 8.

(44) For all matrices A and B, $\mathcal{R}(AB) \subset \mathcal{R}(A)$.

T
Comments: This is no more complicated than the fact that every integer multiple of 6 is an even number. Let's prove that. If n is an integer multiple of 6, then $n = 6m$ for some m. But $6 = 2 \times 3$, so $n = 2 \times 3m$. We have just shown that

$$\{6x \mid x \in \mathbb{Z}\} \subset \{2x \mid x \in \mathbb{Z}\}.$$

Of course, we usually call $\{2x \mid x \in \mathbb{Z}\}$ the set of even numbers and call $\{6x \mid x \in \mathbb{Z}\}$ the set of integer multiples of 6.

If $\underline{x} \in \mathcal{R}(AB)$, then $\underline{x} = AB\underline{u}$ for some $\underline{u}$, so $\underline{x} = A(B\underline{u})$. But the last equation says that $\underline{x}$ is a multiple of A so $\underline{x} \in \mathcal{R}(A)$.

(45) For all matrices A and B, $\mathcal{R}(AB) = \mathcal{R}(BA)$.

F
Comments:

(46) For all matrices A and B, $\mathcal{R}(A) \subset \mathcal{R}(AB)$.

F

Comments:

(47) If $\underline{u}$ and $\underline{v}$ are $n \times 1$ column vectors then $\underline{u}^T\underline{v} = \underline{v}^T\underline{u}$.

T

(48) If $A^2 = I$ and $B^2 = I$, then $(AB)^{-1} = BA$.

(49) If A is invertible every eigenvalue for A is an eigenvalue for A^{-1}.

(50) If A is invertible every eigenvector for A is an eigenvector for A^{-1}.

(51) If A is a non-singular 5×5 matrix and $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is a linearly independent subset of $\mathbb{R}^5$, then $\{A\underline{u}_1, A\underline{u}_2, A\underline{u}_3\}$ is also a linearly independent subset of $\mathbb{R}^5$.

(52) If A is a non-singular 5×5 matrix and $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is a linearly dependent subset of $\mathbb{R}^5$, then $\{A\underline{u}_1, A\underline{u}_2, A\underline{u}_3\}$ is also a linearly dependent subset of $\mathbb{R}^5$.

(53) If W is a subspace of $\mathbb{R}^n$ having dimension d, then W contains exactly d vectors.

(54) If W is a subspace of $\mathbb{R}^n$ that contains $\underline{u} + \underline{v}$, then W contains $\underline{u}$ and $\underline{v}$.

(55) If a is a non-zero number and W is a subspace of $\mathbb{R}^n$ containing $a\underline{u}$, then W contains $\underline{u}$.

(56) If W is a subspace of $\mathbb{R}^n$ having dimension d, then W contains exactly d vectors.

(57) If W is a subspace of $\mathbb{R}^n$ having dimension d, then W contains exactly d vectors.

Part E. Multiple Choice

+1 for each correct answer, -1 for each incorrect answer.

For each of these questions there is only one correct answer. You should mark at most bubble.

These questions are taken from http://scherk.pbworks.com/Quiz

(101) Let A be a 3×4 matrix, and let B be a 4×3 matrix. Which of AB, BA, $A+B$, $A-B$ make sense?
(a) All make sense.
(b) Only AB and BA make sense.
(c) Only AB makes sense.
(d) Only BA makes sense.
(e) Only $A+B$ and $A-B$ make sense.
(f) None make sense.

(102) Let A be a 3×4 matrix, and let B be a 4×5 matrix. Which of AB, BA, $A+B$, $A-B$ make sense?
(a) All make sense.
(b) Only AB and BA make sense.
(c) Only AB makes sense.
(d) Only BA makes sense.
(e) Only $A+B$ and $A-B$ make sense.
(f) None make sense.

(103) Let A be a 3×4 matrix, and let B be a 3×4 matrix. Which of AB, BA, $A+B$, $A-B$ make sense?
(a) All make sense.
(b) Only AB and BA make sense.
(c) Only AB makes sense.
(d) Only BA makes sense.
(e) Only $A+B$ and $A-B$ make sense.
(f) None make sense.

(104) Let A be a 3×3 matrix, and let B be a 3×3 matrix. Which of AB, BA, $A+B$, $A-B$ make sense?
(a) All make sense.
(b) Only AB and BA make sense.
(c) Only AB makes sense.
(d) Only BA makes sense.
(e) Only $A+B$ and $A-B$ make sense.
(f) None make sense.

(105) If X is a row vector and Y a column vector, then YX is
(a) A row vector, if both vectors have the same number of entries.
(b) A column vector, in all cases.
(c) A column vector, if both vectors have the same number of entries.
(d) Nothing; this operation cannot be defined in general.
(e) A number.
(f) A row vector, in all cases.
(g) A matrix.

(106) If X is a row vector and Y a column vector, then XY is
(a) A matrix.

(b) A row vector.
(c) A number, if the two vectors have the same number of entries, and nothing (undefined) otherwise.
(d) Nothing; this operation cannot be defined in general.
(e) A column vector.
(f) A number.

(107) If one adds a row vector to a column vector, one gets
(a) An L-shaped vector.
(b) Nothing; this operation cannot be defined in general.
(c) A row vector.
(d) A column vector.
(e) A number, if the two vectors have the same number of entries, and nothing (undefined) otherwise.
(f) A number.
(g) A matrix.

(108) If A is a matrix and X a column vector, then AX is
(a) A column vector, if the number of rows of the matrix equals the number of rows of the vector.
(b) A column vector, if the number of columns of the matrix equals the number of rows of the vector.
(c) A number.
(d) A matrix.
(e) Nothing; this operation cannot be defined in general.
(f) A column vector, if the number of rows of the matrix equals the number of columns of the vector.

(109) If X is a column vector and A a matrix, XA is:
(a) A column vector, if the number of rows of the matrix matches the number of columns of the vector.
(b) A column vector, if the number of columns of the matrix matches the number of rows of the vector.
(c) Nothing; this operation cannot be defined in general.
(d) A column vector, if the number of rows of the matrix matches the number of rows of the vector.
(e) A matrix.
(f) A number.
(g) A row vector.

(110) Let A be a matrix. Under what conditions will A^2 make sense?
(a) A must be a square matrix.
(b) A must have at least as many rows as columns.
(c) A must be a column vector.
(d) A^2 makes sense for any matrix .
(e) A must be a row vector.
(f) A must have at least as many columns as rows.
(g) A must be in reduced row-echelon form.

(111) If A is a 3×5 matrix, then the determinant of A is
(a) A matrix.
(b) Undefined.
(c) A subspace of $\mathbb{R}^3$.

(d) A number (possibly non-zero).
(e) A matrix.
(f) A subspace of $\mathbb{R}^5$.
(g) Zero.

(112) If A is a 3×5 matrix, then the rank of A is
(a) A subspace of $\mathbb{R}^3$.
(b) A subspace of $\mathbb{R}^5$.
(c) A 5×3 matrix.
(d) Undefined.
(e) A 3×5 matrix.
(f) Zero.
(g) A number (possibly non-zero).

(113) If A is a 3×5 matrix, then the transpose of A is
(a) A number (possibly non-zero).
(b) A subspace of $\mathbb{R}^5$.
(c) Zero.
(d) Undefined.
(e) A subspace of $\mathbb{R}^3$.
(f) A 5×3 matrix.
(g) A 3×5 matrix.

(114) If A is a 3×5 matrix, then the inverse of A is
(a) A 5×3 matrix.
(b) A subspace of $\mathbb{R}^3$.
(c) Undefined.
(d) A number (possibly non-zero).
(e) A 3×5 matrix.
(f) A subspace of $\mathbb{R}^5$.
(g) Zero.

(115) If A is a 3×5 matrix, then the range of A is
(a) A 5×3 matrix.
(b) A subspace of $\mathbb{R}^3$.
(c) Undefined.
(d) A number (possibly non-zero).
(e) A 3×5 matrix.
(f) A subspace of $\mathbb{R}^5$.
(g) Zero.

(116) If A is a 3×5 matrix, then the null space of A is
(a) A 5×3 matrix.
(b) A subspace of $\mathbb{R}^3$.
(c) Undefined.
(d) A number (possibly non-zero).
(e) A 3×5 matrix.
(f) A subspace of $\mathbb{R}^5$.
(g) Zero.

(117) If A is a 3×5 matrix, then the row-reduced echelon form of A is
(a) Zero.
(b) A number (possibly non-zero).
(c) Undefined.

(d) A subspace of $\mathbb{R}^5$.
(e) A 3×5 matrix.
(f) A subspace of $\mathbb{R}^3$.
(g) A 5×3 matrix.

(118) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$. The null space of T is
(a) $\{(0, 0, 0, x_4) \mid \text{where } x_4 \text{ is a real number}\}$
(b) $\{(0, x_1, x_2, x_3) \mid \text{where } x_1, x_2, x_3 \text{ are real numbers}\}$
(c) $\{(x_4, 0, 0, 0) \mid \text{where } x_4 \text{ is a real number}\}$
(d) $\{(x_1, x_2, x_3, 0) \mid \text{where } x_1, x_2, x_3 \text{ are real numbers}\}$
(e) $\{(0, 0, 0, 1)\}$
(f) $\{(1, 0, 0, 0)\}$
(g) $\{(x_1, 0, 0, 0)) \mid \text{where } x_1 \text{ is a real number}\}$

(119) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation $T(x_1, x_2) = (x_1, 0)$. The null space of T is
(a) $\{(0, t) \mid t \text{ is a real number}\}$
(b) $\{(0, 1)\}$
(c) $\{(1, 0)\}$
(d) $\{(x_1, 0) \mid x_1 \text{ is a real number}\}$
(e) 1
(f) $\{(x_1, 0)\}$
(g) $\{(0, x_2)\}$

(120) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation $T(x_1, x_2, x_3, x_4) = (x_2, x_3, 0, 0)$. The null space of T consists of all vectors of the form
(a) $(x_1, 0, 0, x_4) \mid \text{where } x_1 \text{ and } x_4 \text{ are real numbers}\}$
(b) $(x_2, x_3, 0, 0) \mid \text{where } x_2 \text{ and } x_3 \text{ are real numbers}\}$
(c) $(0, 0, 0, 1)$ and $(1, 0, 0, 0)$
(d) $(0, 0, 0, 1)$ and $(0, 0, 1, 0)$
(e) $(0, x_2, x_3, 0) \mid \text{where } x_2 \text{ and } x_3 \text{ are real numbers}\}$
(f) $(0, 0, x_1, x_4) \mid \text{where } x_1 \text{ and } x_4 \text{ are real numbers}\}$
(g) $(0, 0, x_2, x_3) \mid \text{where } x_2 \text{ and } x_3 \text{ are real numbers}\}$

(121) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the transformation $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_3)$. The range of T has many bases; one of them is the set of vectors
(a) $(1, 0, 0)$ and $(0, 1, 0)$
(b) $(x_1, 0, 0, 0)$, $(0, 0, x_2, 0)$, and $(0, 0, 0, x_2)$
(c) $(1, 0, 0, 0)$, $(0, 0, 0, 1)$, and $(0, 0, 1, 1)$
(d) $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$
(e) $(x_1, 0, 0)$ and $(0, x_2, 0)$
(f) $(x_1, 0, 0)$, $(0, x_2, 0)$, and $(0, 0, x_3)$

(122) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the transformation $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_2)$. The nullspace of T has many bases; one of them is the set of vectors
(a) $(0, 0, 1)$
(b) $(0, 0, x_3)$
(c) $(1, 0, 0, 0)$ and $(0, 0, 1, 1)$
(d) $(1, 0, 0)$ and $(0, 1, 0)$
(e) $(0, 1, 0, 0)$
(f) $(x_1, x_2, 0)$

(123) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$. The range of T consists of all vectors of the form

(a) $(0, 0, 0, x_4) \mid$ where x_4 is a real number}
(b) $(0, x_1, x_2, x_3) \mid$ where x_1, x_2, x_3 are real numbers}
(c) $(x_1, x_2, x_3, 0) \mid$ where x_1, x_2, x_3 are real numbers}
(d) $(x_4, 0, 0, 0) \mid$ where x_4 is a real number}
(e) $(x_1, x_2, x_3, 0) \mid$ where x_1, x_2, x_3 are real numbers}
(f) $(x_1, x_2, x_3, x_4) \mid$ where x_1, x_2, x_3, x_4 are real numbers}
(g) $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$

(124) If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ is one-to-one, then

(a) Its rank is five and its nullity is two.
(b) Its rank and nullity can be any pair of non-negative numbers that add up to five.
(c) Its rank is three and its nullity is two.
(d) Its rank is two and its nullity is three.
(e) Its rank is three and its nullity is zero.
(f) Its rank and nullity can be any pair of non-negative numbers that add up to three.
(g) The situation is impossible.

(125) If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ is onto, then

(a) Its rank is five and its nullity is two.
(b) Its rank is two and its nullity is three.
(c) Its rank is three and its nullity is zero.
(d) Its rank and nullity can be any pair of non-negative numbers that add up to three.
(e) Its rank is three and its nullity is two.
(f) Its rank and nullity can be any pair of non-negative numbers that add up to five.
(g) The situation is impossible.

(126) If a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ is onto, then

(a) Its rank is three and its nullity is two.
(b) Its rank is two and its nullity is three.
(c) Its rank is five and its nullity is two.
(d) Its rank and nullity can be any pair of non-negative numbers that add up to five.
(e) Its rank is three and its nullity is zero.
(f) Its rank and nullity can be any pair of non-negative numbers that add up to three.
(g) The situation is impossible.

(127) If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ is onto, then

(a) Its rank is two and its nullity is three.
(b) The situation is impossible.
(c) Its rank is three and its nullity is zero.
(d) Its rank is three and its nullity is two.
(e) Its rank is five and its nullity is two.
(f) Its rank and nullity can be any pair of non-negative numbers that add up to three.

(g) Its rank and nullity can be any pair of non-negative numbers that add up to five.

(128) Let $T : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear transformation. Then
(a) T is one-to-one if and only if its rank is three; T is never onto.
(b) T is onto if and only if its rank is two; T is never one-to-one.
(c) T is onto if and only if its rank is three; T is never one-to-one.
(d) T is one-to-one if and only if its rank is five; T is never onto.
(e) T is one-to-one if and only if its rank is two; T is never onto.
(f) T is onto if and only if its rank is five; T is never one-to-one.

(129) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation. Then
(a) T is onto if and only if its rank is two; T is never one-to-one.
(b) T is one-to-one if and only if its rank is two; T is never onto.
(c) T is one-to-one if and only if its rank is five; T is never onto.
(d) T is onto if and only if its rank is five; T is never one-to-one.
(e) T is invertible if and only if its rank is five.
(f) T is onto if and only if its rank is three; T is never one-to-one.
(g) T is one-to-one if and only if its rank is three; T is never onto.

(130) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation. Then
(a) T is onto if and only if its nullity is two; T is never one-to-one.
(b) T is invertible if and only if its nullity is zero.
(c) T is one-to-one if and only if its nullity is zero; T is never onto.
(d) T is onto if and only if its nullity is three; T is never one-to-one.
(e) T is one-to-one if and only if its nullity is two; T is never onto.
(f) T is one-to-one if and only if its nullity is three; T is never onto.
(g) T is onto if and only if its nullity is zero; T is never one-to-one.

(131) Let $T : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear transformation. Then
(a) T is onto if and only if its nullity is zero; T is never one-to-one.
(b) T is one-to-one if and only if its nullity is two; T is never onto.
(c) T is onto if and only if its nullity is two; T is never one-to-one.
(d) T is invertible if and only if its nullity is zero.
(e) T is one-to-one if and only if its nullity is zero; T is never onto.
(f) T is onto if and only if its nullity is three; T is never one-to-one.
(g) T is one-to-one if and only if its nullity is three; T is never onto.

Part F. More Multiple Choice

+1 for each correct answer, -1 for each incorrect answer.

There may be more than one correct answer to each of these questions. You may mark as many bubbles as you want. You will get +1 for each marked bubble that is correct and -1 for each that is incorrect.

(132) Let A and B be $n \times n$ matrices.
(a) If AB is singular, so is A.
(b) If AB is singular, so is BA.
(c) If AB is non-singular, so is BA.
(d) If A is singular, so is BA.
(e) If AB is non-singular, so are A and B.
(f) If AB is singular, so are A and B.
(g) If A and B are non-singular, so is AB.

(h) If A and B are non-singular, so is $A + B$.

(133) If A and B are similar matrices, then
(a) they have the same range;
(b) they have the same rank;
(c) they have the same null space;
(d) they have the same nullity;
(e) they have the same eigenvalues;
(f) they have the same eigenvectors;
(g) they have the same characteristic polynomial;
(h) all the above are true;
(i) none of the above is true;

(134) Let A an $n \times n$ matrix. The determinant of A is
(a) the same as that of A^T;
(b) the same as that of the matrix obtained by switching the first and third rows of A;
(c) the same as that of the matrix obtained by replacing its **third** row by 5 times the first row plus the third row;
(d) the same as that of the matrix obtained by replacing its **first** row by 5 times the first row plus the third row;
(e) the same as that of A^{-1};
(f) 5 times that of the matrix obtained by replacing its third row by 5 times the first row plus the third row;
(g) $\frac{1}{2}$ times that of the matrix obtained by replacing the third row by 2 times the third row.

(135) Let W be a subspace of $\mathbb{R}^m$. There is always a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ whose range is W if
(a) $n \geq m$.
(b) $n \geq \dim W$.
(c) n is any integer.

(136) Let A be a square matrix having eigenvalue 3. Then
(a) 28 is an eigenvalue of $A^3 + I$.
(b) 28 is an eigenvalue of $9A + I$.
(c) 28 is an eigenvalue of $A + 27I$.
(d) $\frac{1}{3}$ is an eigenvalue of A^{-1}.
(e) $\frac{1}{3}$ is an eigenvalue of A^T.
(f) 3 is an eigenvalue of A^T.

(137) A system of n homogeneous linear equations in 16 unknowns always
(a) has a non-zero solution if $n = 15$.
(b) has a non-zero solution if $n < 15$.
(c) has a non-zero solution if $n = 16$.
(d) has a non-zero solution for all n.
(e) has a unique non-zero solution if $n = 16$.

(138) Which of the following formulas define linear transformations:
(a) $T(a, b) = (a + b, a - b)$
(b) $T(x_2, x_1) = (x_2 + x_1, x_2 - x_1)$
(c) $T(a, b, c) = a + b + c$
(d) $T(a, b, c) = abc$
(e) $T(a, b, c) = 0$

(f) $T(a, b, c) = 1$
(g) $T(a, b, c) = (a, b, c)$.

(139) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(\underline{x}) = A\underline{x}$ is orthogonal if

(a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(e) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(f) $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

(140) Which of the following are subspaces:
(a) The solutions to a system of homogeneous linear equations;
(b) The solutions to a system of linear equations;
(c) The set $W = \{\underline{x} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 \mid x_1^2 = x_2^2\}$.
(d) The set of all eigenvectors for a matrix A.
(e) The set of all eigenvalues for a matrix A.
(f) The set of eigenvectors having eigenvalue zero.
(g) The set of eigenvectors having eigenvalue three.
(h) The vectors $(x, y, z)^T$ in $\mathbb{R}^3$ such that $2x + z = 0$.
(i) The vectors $(x, y, z)^T$ in $\mathbb{R}^3$ such that $x + y + z = 1$.

(141) Which of the following matrices are symmetric:
(a) 0
(b) I
(c) $A + A^T$
(d) AA^T
(e)

(142) Let U and V be subspaces of $\mathbb{R}^n$. Which of the following are subspaces:
(a) $U + V$
(b) $U - V$
(c) UV and VU
(d) $U \cap V$
(e) $U \cup V$.
(f) U^{-1} and V^{-1}.

(143) Let A and B be similar matrices. Which of the following are true:
(a) if $A = I$, then $B = I$.
(b) if $A = 0$, then $B = 0$.
(c) if A is diagonal, so is B.
(d) if A is diagonalizable, so is B.
(e) if A is orthogonal, so is B.

(144) Let A and B be similar matrices. Which of the following are true:

(a) if λ is an eigenvalue of A it is also an eigenvalue of B.
(b) if $\underline{v}$ is an eigenvector for A it is also an eigenvector for B.
(c) $\mathcal{N}(A) = \mathcal{N}(B)$;
(d) $\operatorname{rank} A = \operatorname{rank} B$.
(e) If $A\underline{x} = \underline{b}$, then $B\underline{x} = \underline{b}$.

(145) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then T is a linear transformation is linear if and only if
(a) The graph of T takes the form $y = mx + c$.
(b) There exists a matrix A such that $T\underline{x} = A\underline{x}$ for all $\underline{x} \in \mathbb{R}^n$.
(c) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ and $T(c\underline{v}) = cT(\underline{v})$ for all vectors $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^n$ and all scalars $c \in \mathbb{R}$.
(d) T is one-to-one and onto.
(e) No condition required (all transformations are linear).
(f) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ for all vectors $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^n$.
(g) The image of T is either $\{0\}$, a line through the origin, or a plane through the origin.