Math 308 Midterm Answers and Comments November 6, 2009

- We will write $\underline{A}_1, \dots, \underline{A}_n$ for the columns of an $m \times n$ matrix A.
- Several questions involve an unknown vector $\underline{x} \in \mathbb{R}^n$. We will write $x_1, \dots, x_n$ for the entries of $\underline{x}$; thus $\underline{x} = (x_1, \dots, x_n)^T$.
- The null space and range of a matrix A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively.
- The linear span of a set of vectors is denoted by $\mathrm{Sp}(\underline{v}_1, \dots, \underline{v}_n)$.
- We will write $\underline{e}_1, \dots, \underline{e}_n$ for the standard basis for $\mathbb{R}^n$. Thus $\underline{e}_i$ has a 1 in the i^{th} position and zeroes elsewhere.

Scoring. On the True/False section you get +2 for each correct answer, −2 for each incorrect answer and 0 if you choose not to answer the question.

General comments on your answers: People did well on Part B, the definitions. That's good. Part C was also quite well done. A large number of people fell apart on the True/False questions. That was worth 60 points, but very few people scored above 40. Part A gave you the most problems. I was surprised by how difficult it proved to be. It was worth 27 points. If you got under 18 on it you have some serious weaknesses that you must address as soon as possible,

I suggest you read my answers and comments on the following pages very carefully and ASK a question in class or office hours if there is anything in it you do not fully understand. Please do that—be absolutely rigorous about reading it and understanding it. Once you understand <u>all</u> that is on the following pages you will have a much better understanding of linear algebra.

The Google group math308autumn2009 at

http://groups.google.com/group/math308autumn2009?hl=en

could be a really wonderful resource for you. Send your questions back and forth to one another and get some discussions going. If you all wait for someone else to do it first, nothing will happen.

Your Performance: The average score was 70 (out of a possible 131) and the median was 72. The number of people in the various ranges was:

≤ 29	$30-39$	$40-49$	$50-59$	$60-69$	$70-79$	$80-89$	$90-99$	≥ 100
5	7	9	10	29	26	25	12	8

The scores correspond to the following grades and the number of people in each range is

≤ 49	$50-64$	$65-84$	≥ 85
D	C	B	A
21	25	52	33

By making the True/False questions worth ±2 I over-emphasized that section. Next time I will make it ±1. To get a better sense of how you are doing I suggest dividing your True/False score by 2 and then recomputing your total (now out of 101), and assign yourself a grade according to the following scheme:

≤ 40	$41-52$	$53-69$	≥ 70
D	C	B	A

Final remark: To those of you who did not present your work clearly, legibly, neatly, and according to the usual rules of grammar: I will assign points on the final exam for overall presentation of your work.

Part A.

Short answer questions

Scoring: 3 points per question. No partial credit.

(1) Let A be a 3×4 matrix and $\underline{b} \in \mathbb{R}^4$. Suppose that the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 3 & 0 & 0 & \mid & 2 \\ 0 & 0 & 1 & 2 & \mid & 2 \\ 0 & 0 & 0 & 0 & \mid & 0 \end{pmatrix}$$

Say which the independent and dependent variables are and write down all solutions to the equation $A\underline{x} = \underline{b}$.

The independent variables are x_2 and x_4; they can take any values, and then one must have $x_1 = 2 - 3x_2$ and $x_3 = 2 - 2x_4$.

Comments: Almost everyone got this right. That's good. It tells me that once you have put the original augmented matrix $(A \mid \underline{b})$ to row reduced echelon form you understand how to read off all solutions to the original system of equations.

(2) Find three linearly dependent vectors $\underline{u}, \underline{v}, \underline{w}$ in the linear span of the vectors

$$(1,1,1,1),\ (1,2,3,4),\ (1,-1,1,-1)$$

such that $\{\underline{u}, \underline{v}\}$, $\{\underline{u}, \underline{w}\}$, and $\{\underline{w}, \underline{v}\}$, are linearly independent.

There are infinitely many solutions, but the simplest is to choose two linearly independent vectors in the linear span for $\underline{u}$ and $\underline{v}$, and then take $\underline{w} = \underline{u} + \underline{v}$. This is what most people did—they chose $\underline{u}$ and $\underline{v}$ to be two of the three given vectors and then took $\underline{w}$ to be their sum. For example, $\underline{u} = (1,1,1,1)$, $\underline{v} = (1,2,3,4)$, and $\underline{w} = \underline{u} + \underline{v} = (2,3,4,5)$ was a popular solution. I liked the solution

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

Cute! And

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

had a nice symmetry to it.

Comments: Some people gave three vectors $\underline{u}, \underline{v}, \underline{w}$ such that each of the sets $\{\underline{u}, \underline{v}\}$, $\{\underline{u}, \underline{w}\}$, and $\{\underline{w}, \underline{v}\}$, is linearly independent, BUT not all of $\underline{u}$, $\underline{v}$, and $\underline{w}$ belonged to the linear span of the three given vectors. If you lost points for this question and want to check whether the vectors you gave are in the linear span it might be good to take note of the fact that the vectors

$$(1,0,1,0),\ (0,1,0,1),\ (0,0,1,1)$$

are a basis for the subspace spanned by $(1,1,1,1)$, $(1,2,3,4)$, and $(1,-1,1,-1)$.

Some people gave the "answer"

$$\underline{u} = (1,1,1,1),\ \underline{v} = (1,2,3,4),\ \underline{w} = (1,-1,1,-1)$$

but these three vectors are linearly independent and the question asked for three "linearly dependent vectors" $\underline{u}, \underline{v}, \underline{w}$.

Several people gave the incorrect answer

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

but the question asked for "*vectors* $\underline{u}, \underline{v}, \underline{w}$ *in the linear span of the vectors*

$$(1,1,1,1),\ (1,2,3,4),\ (1,-1,1,-1)$$

and none of the vectors $\underline{e}_i$ is in the linear span of the three given vectors.

(3) Find integers a and b between -6 and $+6$ so that $\begin{pmatrix}1 & 2 & 3\\ 2 & a & 1\\ -1 & b & 1\end{pmatrix}$ is singular.

Almost everyone knew that a square matrix is singular if and only if its columns (or rows) are linearly dependent, and then tried to find a and b so that the columns are linearly dependent. However, at this stage not everyone remembered that a set of vectors, in this case the columns, is linearly dependent if and only if one of them is a linear combination of the others. That is a pity, and it made the problem more difficult for those that did not think of using that fact.

There are many values of a and b that make the matrix singular, but all I asked for was *one* solution. I had hoped that you would all be able to find a solution just by "looking" at the matrix. For example, the columns are linearly dependent if the sum of the first two columns is the third—thus $(a,b) = (-1,2)$ is a solution. Similarly, the columns are linearly dependent if one of them is a multiple of another, so $(a,b) = (4,-2)$ is a solution.

Although I did not ask you this, it is not hard to show that (a,b) is a solution if and only if

$$4a + 5b = 6.$$

Comments: Quite a large number of people took the following approach: the columns are linearly dependent if there are numbers r, s, and t, not all zero, such that

$$r\begin{pmatrix}1\\2\\-1\end{pmatrix} + s\begin{pmatrix}2\\a\\b\end{pmatrix} + t\begin{pmatrix}3\\1\\1\end{pmatrix} = 0. \tag{1}$$

That is true, the matrix is singular if this equation has a solution with r, s, and t, not all zero. But when one writes this out one gets the three equations

$$\begin{aligned} r + 2s + 3t &= 0\\ 2r + as + t &= 0\\ -r + bs + t &= 0 \end{aligned}$$

and that looks complicated! Not just to me, but also to those who took this approach.

However, displaying the equation (1) and looking at the top entries in each column might make you think "oh, 1+2=3" and then notice one can obviously choose a and b such that

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

That is how I found the solution $(a, b) = (-1, 2)$.

Perhaps the message to be drawn from this is that it is a good idea to remember that a set of vectors, in this case the columns, is linearly dependent if and only if one of them is a linear combination of the others.

(4) Find an equation for the plane in $\mathbb{R}^3$ containing $(1, 0, 1)$ and $(0, 2, 3)$.

You need to find numbers a, b, c such that the two given points are solutions to the equation $ax+by+cz = 0$. For example, the plane is given by the equation

$$2x + 3y - 2z = 0$$

and by any non-zero multiples of this.

Comments: Most people got this correct and most of those who got it wrong could have seen they had it wrong by checking whether the given points really did lie on the line they gave. For example, if you wrote $x - 3y + 2z = 0$ and plugged in $(1, 0, 1)$ you would get $3 = 0$, obviously an error.

(5) This problem takes place in $\mathbb{R}^4$. Find two linearly independent vectors belonging to the plane that consists of the solutions to the system of equations

$$\begin{aligned} 4x_1 - 3x_2 + 2x_3 - x_4 &= 0 \\ x_1 - x_2 - x_3 + x_4 &= 0. \end{aligned}$$

There are infinitely many solutions to the problem. One way to find such points is to start by taking $x_1 = x_2 = 1$, and then you need $2x_3 - x_4 = -1$ and $-x_3 + x_4 = 0$; the second of these gives $x_3 = x_4$, and then the first gives $x_3 = -1$; so $(1, 1, -1, -1)$ is one point. Now look for a solution with $x_1 = 0$ and $x_2 = 1$; you then need $2x_3 - x_4 = 3$ and $-x_3 + x_4 = 1$; the second of these gives $x_4 = 1 + x_3$; substituting that into $2x_3 - x_4 = 3$ gives $x_3 - 1 = 3$, so $x_3 = 4$ and $x_4 = 5$, so $(0, 1, 4, 5)$ is another point.

The justification, which I did not ask for, for this approach is that two equations are independent so the system is rank two and will have two independent variables. In the previous paragraph I am taking x_1 and x_2 for the independent variables, and then express x_3 and x_4, the dependent variables, as linear combinations of the independent variables.

Other points on the plane that turned up in the answers of various students were

$$(1, 2, 3, 4),\ (3, 4, 1, 2),\ (4, 5, 0, 1),\ (0, 1, 4, 5),\ (5, 6, -1, 0),\ (-1, 0, 5, 6),$$
$$(2, 3, 2, 3),\ (1, 1, -1, -1)$$

Comments: A systematic approach to the problem is to write down the augmented matrix

$$\begin{pmatrix} 4 & -3 & 2 & -1 & | & 0 \\ 1 & -1 & -1 & 1 & | & 0 \end{pmatrix}$$

and then do row operations to get

$$\begin{pmatrix} 1 & -1 & -1 & 1 & | & 0 \\ 4 & -3 & 2 & -1 & | & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & -1 & 1 & | & 0 \\ 0 & 1 & 6 & -5 & | & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 5 & -4 & | & 0 \\ 0 & 1 & 6 & -5 & | & 0 \end{pmatrix}.$$

The independent variables are x_3 and x_4. Taking $(x_3, x_4) = (0, 1)$ gives $(x_1, x_2) = (4, 5)$. Taking $(x_3, x_4) = (1, 0)$ gives $(x_1, x_2) = (-5, -6)$. Hence $(4, 5, 0, 1)$ and $(-5, -6, 1, 0)$ lie on the plane and are obviously linearly independent.

Several people made the following error. They formed the augmented matrix above and did the row operations correctly to obtain the row reduced echelon matrix that I got in the previous paragraph. But then they went wrong by saying that the rows of the row reduced echelon matrix give the solutions, i.e., they claimed that $(1, 0, 5, -4)$ and $(0, 1, 6, -5)$ are points lying on the plane. *But that is not the correct way to use the row reduced echelon matrix to get the solutions.* Instead proceed as I did in the last three sentences of the previous paragraph. (Many people who made this error were able to answer question 1 correctly, so if you got this question wrong perhaps it would be a good idea to look at what you did in Question 1.)

Some people looked ahead to question 7 and checked that $(1, 2, 3, 4)$ is on the plane (that's fine) and then gave $(2, 4, 6, 8)$ as the second point on the plane. But those two vectors are linearly *dependent*

$$(2, 4, 6, 8) - 2(1, 2, 3, 4) = (0, 0, 0, 0)$$

as are any two vectors when one is a multiple of another. Remember that a set of vectors is linearly dependent if and only if one is a linear combination of the others.

Some students did not realize that finding points on the plane is *exactly the same problem* as finding solutions to the system consisting of the two given linear equations. That is an important thing to know.

With a question like this you should always check your answer–for example, one person gave the answer $\{(1, 1, 0, 1), (0, 1, 0, 1)\}$ but the first of those points does not satisfy the equation $x_1 - x_2 - x_3 + x_4 = 0$ and the second does not satisfy the equation $4x_1 - 3x_2 + 2x_3 - x_4 = 0$. The point $(1, 1, 0, 1)$ lies on the subspace given by $4x_1 - 3x_2 + 2x_3 - x_4 = 0$ and the point $(0, 1, 0, 1)$ lies on the subspace given by $x_1 - x_2 - x_3 + x_4 = 0$, but neither belongs to the intersection of those 3-dimensional subspaces, and that was what the question asked for.

Some people gave two vectors, one satisfying the equation $4x_1 - 3x_2 + 2x_3 - x_4 = 0$ and the other satisfying the equation $x_1 - x_2 - x_3 + x_4 = 0$, but neither point satisfying both equations. You needed to find tow points each of which satisfied both equations.

I'm wondering if some people had a fundamental misunderstanding of the question from the beginning. In $\mathbb{R}^4$, the solutions to a single homogeneous linear equation form a 3-dimensional subspace of $\mathbb{R}^4$. Conversely, every 3-dimensional subspace of $\mathbb{R}^4$ is the set of solutions to a single homogeneous linear equation. A plane in $\mathbb{R}^4$ is the intersection of two 3-dimensional subspaces so is the set of solutions to a system of two homogeneous linear equations. In other words, every 2-dimensional subspace of $\mathbb{R}^4$ is the set of solutions to a a system of two homogeneous linear equation. Conversely, the set of solutions to a a system of two homogeneous linear equations is a 2-dimensional subspace of $\mathbb{R}^4$ (unless one the equations is a multiple of the other). Similar considerations apply in higher dimensions, and to larger systems of linear equations.

(6) Find a basis for the plane in the previous question.

Your answer to the previous question is an answer to this question! Most people realized that (good!).

Comments:

(7) Is $(1, 2, 3, 4)$ a linear combination of the two vectors in your answer to the previous question? Explain why.

YES (provided your answer to the previous question is correct), because it lies on the given plane [and every point on the plane is a linear combination of the vectors in a basis for the plane]. I did not expect you to give the part in [...]—I just assumed you knew that once you had said "because it lies on the plane."

Comments: Most people got this and the previous two questions correct. That is encouraging. But it was also discouraging to see too many people saying that $(1, 2, 3, 4)$ does not lie on the plane—it does, and it is an elementary matter to check this by plugging $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$, into the two equations and doing the elementary arithmetic to see that both equations are then true. [If you were one of those people who said $(1, 2, 3, 4)$ does not lie on the plane ask yourself whether you plugged it in and checked, and if so try doing it again to see if you can do the calculation accurately.]

Some people who had gotten an incorrect answer to the previous question said NO, and that was correct, but then they had difficulty explaining why it was not a linear combination.

Some people who had gotten an incorrect answer to the previous question said "YES, because the point lies on the plane." However, although it *does* lie on the plane it was not usually a linear combination of the two vectors they gave in the previous answer, so I deducted points for that.

The following answer is *almost* correct: *"Yes, because* $(1, 2, 3, 4)$ *is in the plane and the plane contains all linear combinations of the two vectors."* We need to look at it closely to see the hole—it is true that the plane contains all linear combinations of the two vectors, but that in itself is not a reason why $(1, 2, 3, 4)$ is a linear combination of the two vectors. The student is saying only that the set of linear combinations of the two vectors is a subset of the plane, which is true, but why should $(1, 2, 3, 4)$ belong to that subset. One could correct the answer as follows: *"Yes, because* $(1, 2, 3, 4)$ *is in the plane* *and every point in the plane is a linear combination of the two vectors."* The student is not using or stating the fact that the plane is the linear span of the two vectors.

One student said *"YES, because* $(1, 2, 3, 4) \in \mathbb{R}^4$ *and the basis can form any vector."* First the phrase "can form any vector" is not the right way to say that a basis spans a certain subspace, or that one vector is a linear combination of the others. Second, the basis for the plane only spans the plane, not all of $\mathbb{R}^4$ so the premise that $(1, 2, 3, 4) \in \mathbb{R}^4$ can't lead to the conclusion the student wanted; the premise should be that $(1, 2, 3, 4)$ lies on the given plane.

(8) Find a basis for the line $x_1 + 2x_2 = x_1 - 2x_3 = x_2 - x_4 = 0$ in $\mathbb{R}^4$.

Any non-zero point on the line is a basis for it—that is true for any line, i.e., for any 1-dimensional subspace. So $(2, -1, 1, -1)$ is a basis for example.

Comments: A surprisingly large number of people gave me a "basis" for the line consisting of two or more vectors. That shows a serious misunderstanding. Every line through the origin in every $\mathbb{R}^n$ has a basis consisting of *one* element—any point that lies on the line, other than 0, is a basis for the line. A line is a 1-dimensional thing and the definition of dimension is designed to accord with our everyday usage and intuition. Likewise a basis for a plane consists of exactly two (linearly independent) vectors.

In this case it was easy to find a basis because each of the three equations defining the line, i.e., the equations

$$x_1 + 2x_2 = 0, \quad x_1 - 2x_3 = 0, \quad x_2 - x_4 = 0$$

involves just two unknowns, so once you know the value of one of those unknowns the value of the other is completely determined. For example, if we take $x_4 = 1$, the third equation tells us that $x_2 = 1$, then the first equation tells us $x_1 = -2$, and then the middle equation tells us that $x_3 = -1$, so $(-2, 1, -1, 1)$ is a solution to the system of the three homogeneous equations that are given, and therefore lies on the line that consists of all solutions to that system of equations.

(9) Let A be an $m \times n$ matrix. Express $A\underline{x}$ as a linear combination of the columns of A.

$A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$. Notice that at the top of the first page of the exam I said that

> *We will write* $\underline{A}_1, \ldots, \underline{A}_n$ *for the columns of an* $m \times n$ *matrix* A. *Several questions involve an unknown vector* $\underline{x} \in \mathbb{R}^n$. *We will write* $x_1, \ldots, x_n$ *for the entries of* $\underline{x}$*; thus* $\underline{x} = (x_1, \ldots, x_n)^T$.

Comments: Here are some incorrect answers:

- $A\underline{x} = a_1x_1 + \cdots + a_nx_n$. That is no good because you do not say what $a_1, \ldots, a_n$ are. We have never used the notation $a_1, \ldots, a_n$ for the columns of the matrix. In fact, my guess is that the notation a_1 and a_n has been used *exclusively for integers.*
- $A\underline{x} = \underline{A}_1\underline{x}_1 + \cdots + \underline{A}_n\underline{x}_n$. That is no good because you do not say what $\underline{x}_1, \ldots, \underline{x}_n$ are. In fact, I have always reserved the underline notation $\underline{x}$ for vectors.

- $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ where $\underline{x} \in \mathbb{R}$. But $\underline{x}$ is not in $\mathbb{R}$

All these errors are a matter of precision. I think those students more or less know what the answer is, perhaps even know exactly what the answer is, but you must write down the correct answer to get the points. Unfortunately, you do not get the points for knowing the correct answer but writing down an incorrect answer.

Part B.

Complete the following definitions. **You do not need to write the part I have already written.** Just complete the sentence. Make sure you give the definition rather than a theorem which is a consequence of the definition.

Scoring: 2 points per question. No partial credit.

Comments: Most people did very well on this part of the exam. I was pleased to see that. It is a tautology that you can't do linear algebra if you don't know what the words mean, i.e., if you don't know the definitions. The only problem I saw from those who did know the definitions was that they sometimes gave a condition that is *equivalent* to, or a consequence of, the definition. For example, the definition of the range of an $m \times n$ matrix A is $\{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}$ whereas the fact that the range is equal to the linear span of the columns of A is a *consequence* of the definition.

This distinction between definitions and equivalent conditions may not strike you as an important difference. It is certainly subtle, but it is important. Let me try to explain this in the case of the word *range*. As I said in class, we also use the word *range* when talking about functions from $\mathbb{R}$ to $\mathbb{R}$. There, if f is a function defined on all of $\mathbb{R}$ we define the range of f to be $\{f(x) \mid x \in \mathbb{R}\}$. For example, the range of the sine function is the closed interval $[-1, 1] = \{y \mid -1 \le y \le 1\}$. Likewise, if f is the function from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by $f(x, y, z) = (x^2 + y^2, \sqrt{|z|})$, the range of f is the northeast quadrant $\{(u, v) \mid u \ge 0, v \ge 0\}$.

We often think of an $m \times n$ matrix A as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that sends $\underline{x} \in \mathbb{R}^n$ to $A\underline{x} \in \mathbb{R}^m$. The range of this function is $\{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}$ and we call it the *range of A*.

Experience has taught us that the values a function takes are important and we therefore want a word to describe that set of values. The word we have chosen is *range*. You will find the word *range* appearing in every area of mathematics, not just in linear algebra, or in the context of functions between $\mathbb{R}^m$ and $\mathbb{R}^n$. For different kinds of functions we often ask "what is the range of that function" or "how do we find the range of that particular function"? Then one needs to prove a theorem that will give a method for determining the range of the function. The method will vary depending on the kind of function under consideration. For the function $f(\underline{x}) = A\underline{x}$ the theorem says that the range is equal to the linear span of the columns of A. That is a good result because it gives a straightforward description of the range that we can read off immediately from the data, i.e., the matrix A, that gives the function, i.e., from the definition of the function.

(1) Two systems of linear equations are equivalent if ______________.

they have the same solutions.
Comments: The answer "*they have the same row reduced echelon form*" is not correct. There is a Theorem that says Two systems of linear equations are equivalent if and only if they have the same row reduced echelon form, but that is a consequence of the definition. The importance of that theorem is that it provides a method to decide if two systems of equations have the same set of solutions.

It is obviously important to know when two systems of linear equations have the same set of solutions so we introduce a single word for two systems having that property, namely "equivalent". It is easier to say two systems are equivalent than two systems have the same solutions.

(2) The range of an $m \times n$ matrix A is $\{...\}$.

$\{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}$
Comments:

(3) An $n \times n$ matrix A is non-singular if the only ______________.

solution to $A\underline{x} = 0$ is $\underline{x} = 0$.
Comments:

(4) A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if _________

$\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some $a_1, \dots, a_n \in \mathbb{R}$.
Comments:

(5) The linear span of $\{\underline{v}_1, \dots, \underline{v}_n\}$ consists of _________

all linear combinations of $\underline{v}_1, \dots, \underline{v}_n$.
Comments:

(6) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent if the only solution to______________

the equation $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ fis $a_1 = \cdots = a_n = 0$.
Comments:

(7) A subset W of $\mathbb{R}^n$ is a subspace if it satisfies the following three conditions: ______________.

$0 \in W$; $\underline{u} + \underline{v} \in W$ whenever $\underline{u} \in W$ and $\underline{v} \in W$; $a\underline{u} \in W$ for all $a \in \mathbb{R}$ and all $\underline{u} \in W$.

Comments: Some people got the symbols $\in$ and $\subset$ mixed up:

$\in$ means "is an element of" or "belongs to."

$\subset$ means "is a subset of."

Thus $\in$ is applied to *elements* whereas $\subset$ is applied to sets. Remember that a set is a collection of things, and those things are called the elements of the set. Consider the set of integers $\mathbb{Z}$ and the set of even numbers E.The elements of $\mathbb{Z}$ are the integers, the whole numbers. The elements of E are the even numbers. Thus $2 \in \mathbb{Z}$ but $E \subset \mathbb{Z}$. It is also correct to say that $\{2\} \subset \mathbb{Z}$, i.e., the set consisting of the single element 2 is a subset of $\mathbb{Z}$ because every element in $\{2\}$ is an element of $\mathbb{Z}$.

We distinguish between the number 2 and the set $\{2\}$. First, $2 \neq \{2\}$. Second, $2 \in \{2\}$.

(8) Let W be a subspace of $\mathbb{R}^n$. A subset of W is a basis for W if ______________.

Comments:

(9) The dimension of a subspace $W \subset \mathbb{R}^n$ is ______________.

the number of elements in a basis for it.

Comments: Several people said "*the number of elements in the basis for W*". That is incorrect because it suggests that there is only one basis for W. That is not the case unless $W = \{0\}$ in which case the only basis for W is the empty set ϕ. If $W \neq \{0\}$, then W has infinitely many bases. We proved that all of them have the same number of elements so the notion of dimension is well-defined.

(10) The intersection of two sets A and B is the set $A \cap B =$ ______________

$\{x \mid x \in A \text{ and } x \in B\}$.

Comments:

Part C.

Complete the following sentences. Most of the sentences are theorems or consequences of theorems or definitions. **You do not need to write the part I have already written.** Just complete the sentence.

Scoring: 2 points per question. No partial credit.

(1) A homogeneous system of m linear equations in n unknowns has a non-zero solution if ______________. (Answer involves m and n.)

$m < n$.

Comments: The colloquial way of saying this is that there is a non-trivial solution to a system of homogeneous linear equations if there are fewer equations than variables.

(2) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a ________ of the columns of A.

linear combination
Comments:

(3) A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.

a linear combination
Comments:

(4) If U and V are subspaces of $\mathbb{R}^n$ so are _________ and _________.

$U + V$ and $U \cap V$.
Comments: I had expected to see $U \cup V$ and sure enough there were a few of those, but what I had not expected was UV, VU, or V^{-1}. I find it hard to imagine what those people had in mind. Are they thinking that because we use upper case letters to denote both matrices and subspaces that they are the same things or that the same operations can be applied to them? I don't know. Perhaps they thought U and V were matrices although the question says they are subspaces. And V is a subspace, not a matrix, so V^{-1} has no meaning.

(5) Let A be an $m \times n$ matrix. Then $\dim \mathcal{N}(A) + \dim \mathcal{R}(A) =$ ________

n
Comments:

(6) If A is a 2×4 matrix whose null space is a plane its range is ________.

$\mathbb{R}^2$.
Comments: Use the formula in Question 5 to deduce that the rank of the matrix is two, so the range is a plane. However, that plane is contained in $\mathbb{R}^2$ so must be all of $\mathbb{R}^2$. I deducted a point if someone said a plane, or a plane in $\mathbb{R}^2$ because such an answer suggests that they don't know that the range is $\mathbb{R}^2$. It is always useful to keep in mind the idea that an $m \times n$ matrix gives a function (a linear transformation) from $\mathbb{R}^n$ to $\mathbb{R}^m$ so the range is in $\mathbb{R}^m$.

(7) If A is a 4×2 matrix whose null space is a plane its range is ________.

$\{0\}$, the zero subspace.
Comments: Use the formula in Question 5. You are told that $\dim \mathcal{N}(A) = 2$, so $\dim \mathcal{R}(A) = 0$. The only subspace whose basis has zero elements is the zero subspace, $\{0\}$. The universal convention is that the empty set, denoted ϕ, is the only basis for the zero subspace. The empty set contains no elements at all. It has zero elements.

Some people wrote ϕ for the answer—that is incorrect because the empty set is not the same as the set $\{0\}$. The set $\{0\}$ contains one element, namely 0.

(8) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ belongs to the linear span of ________

the columns of A.
Comments:

(9) Let A be an $m \times n$ matrix and $\underline{b} \in \mathbb{R}^m$. The equation $A\underline{x} = \underline{b}$ has a unique solution if the rank of A is ___________

(10) The matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible if and only if ___________.

$wz - xy \neq 0$.
Comments: Many people gave answers along the lines "*if there is a matrix B such that*

$$B\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but I was *not* asking for the definition of what it means for a general matrix to be invertible. I was asking a question about a particular 2×2 matrix, and there is a very simple way to decide if it is invertible: compute $wz - xy$. Likewise, I gave no points for the answer "*if the columns are linearly independent*".

(11) If the matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible its inverse is ___________.

$$\frac{1}{wz - xy}\begin{pmatrix} z & -x \\ -y & w \end{pmatrix}$$

Comments:

(12) If a subspace W has dimension p, then every set of $p + 1$ elements in W is ___________.

linearly dependent.
Comments: Most people got this correct. Good!

Part D.

True or False—just write T or F

Scoring: You get +2 for each correct answer and -2 for each incorrect answer, and 0 if you do not answer the question.

(1) Every set of four vectors in $\mathbb{R}^5$ is linearly independent.

F
Comments: The vectors $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(1,1,0,0,0)$, and $(1,2,0,0,0)$, are not linearly independent.

(2) If a subset of $\mathbb{R}^4$ spans $\mathbb{R}^4$ it is linearly independent.

F
Comments: The vectors $(1,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(1,1,1,1)$ span $\mathbb{R}^4$ but are not linearly independent.

(3) If a subspace W has dimension p, a set that spans W is linearly independent if and only if it has exactly p elements.

T
Comments: We proved this theorem in class.

(4) A linearly independent subset of a p-dimensional subspace W spans W if and only if it has exactly p elements.

T
Comments: We proved this theorem in class.

(5) If S is a linearly dependent set so is every set that contains S.

T
Comments:

(6) Every subset of a linearly independent set is linearly independent.

T
Comments:

(7) The smallest subspace containing subspaces V and W is $V \cup W$.

F
Comments: $V \cup W$ is rarely a subspace. For example, if V is not contained in W and W is not contained in V, then $V \cup W$ is not a subspace because $\underline{v} + \underline{w}$ is in neither V nor W when $\underline{v} \in V$, $\underline{v} \notin W$, $\underline{w} \in W$, and $\underline{w} \notin V$.

For example, if V and W are different lines through the origin in $\mathbb{R}^2$ their union is not a subspace of $\mathbb{R}^2$ because it is not closed under $+$. We showed that the only subspaces of $\mathbb{R}^2$ are the zero subspace $\{0\}$, the lines through the origin, and $\mathbb{R}^2$ itself.

The only time $V \sup W$ is a subspace is when $V \subset W$ or $W \subset V$.

(8) A square matrix having a column of zeroes is always singular.

T
Comments: Any set of vectors that contains the zero vector is linearly dependent because

$$1.\underline{0} + 0.\underline{v}_1 + \cdots + 0.\underline{v}_n = \underline{0}.$$

(9) There is an invertible 3×3 matrix A such that $A^3 = 0$.

F
Comments: If A and B are invertible so is AB. Its inverse is $B^{-1}A^{-1}$. If C is also invertible then $ABC = (AB)C$ is invertible. Thus, if A is invertible so is AAA, i.e., A^3 is invertible. But it is clear that the zero matrix does not have an inverse.

(10) Suppose $\underline{v}$ is a solution to $A\underline{x} = \underline{b}$ and $\underline{u} \in \mathcal{N}(A)$. Then $\underline{u} + \underline{v}$ is a solution to $A\underline{x} = \underline{b}$.

T
Comments: We proved this in class. If $A\underline{v} = \underline{b}$ and $A\underline{u} = 0$, then

$$A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{b} = \underline{b}.$$

(11) The solutions to a system of linear equations always form a subspace.

F
Comments: The system must be homogeneous for the space of solutions to be a subspace. A subspace contains the zero vector $\underline{0}$ so $(0, \dots, 0)$ must be a solution to the system of equations, but a linear equation that has $(0, \dots, 0)$ as a solution must be homogeneous. Check the definition of "homogeneous" if necessary.

(12) If $\mathrm{Sp}(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$, then $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$.

T
Comments: Certainly $\underline{u}$ belongs to $\mathrm{Sp}(\underline{u}, \underline{v}, \underline{w})$ (why?), so the equality $\mathrm{Sp}(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$ implies that $\underline{u}$ belongs to $\mathrm{Sp}(\underline{v}, \underline{w})$. But $\mathrm{Sp}(\underline{v}, \underline{w})$ consists of all linear combinations of $\underline{v}$ and $\underline{w}$.

(13) If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $Sp(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$.

T
Comments: It is clear that $Sp(\underline{v}, \underline{w})$ is always contained in $Sp(\underline{u}, \underline{v}, \underline{w})$—check you understand why. However, if $\underline{u} = a\underline{v} + b\underline{w}$, then an element $c\underline{u} + d\underline{v} + e\underline{w}$ in $Sp(\underline{u}, \underline{v}, \underline{w})$ is equal to

$$c(a\underline{v} + b\underline{w}) + d\underline{v} + e\underline{w} = (ca + d)\underline{v} + (cb + e)\underline{w}$$

which is an element of $Sp(\underline{v}, \underline{w})$. This shows that $Sp(\underline{u}, \underline{v}, \underline{w})$ is contained in $Sp(\underline{v}, \underline{w})$. Hence the two sets are equal.

(14) The null space of a $p \times q$ matrix is contained in $\mathbb{R}^p$.

F
Comments: For this and the next question it is helpful to keep in mind the fact that a $p \times q$ matrix A can be viewed as a function from $\mathbb{R}^q$ to $\mathbb{R}^p$. It is the function sending $\underline{x}$ to $A\underline{x}$.

(15) The range of an $p \times q$ matrix is contained in $\mathbb{R}^q$.

F

(16) Let $\underline{a}, \underline{b} \in \mathbb{R}^n$. Then the set $\{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T \underline{x} = \underline{b}^T \underline{x}\}$ is a subspace of $\mathbb{R}^n$.

T
Comments: Just check the three conditions, or observe that the set consists of all solutions to the equation $(\underline{a}^T - \underline{b}^T)\underline{x} = 0$ so is the null space of the matrix $\underline{a}^T - \underline{b}^T$ and use the theorem that the null space of a matrix is always a subspace.

(17) If A is singular and B is non-singular then AB is always singular.

T
Comments: Because A is singular there is a non-zero vector $\underline{x}$ such that $A\underline{x} = 0$. It follows that

$$(AB)(B^{-1}\underline{x}) = A(BB^{-1})\underline{x} = AI\underline{x} = A\underline{x} = 0$$

and $B^{-1}\underline{x}$ is not zero because if it were then $B(B^{-1}\underline{x})$ would also be zero; but the latter is $I\underline{x} = \underline{x}$ which is non-zero.

(18) If A is singular and B is non-singular then BA is always singular.

T
Comments: Because A is singular there is a non-zero vector $\underline{x}$ such that $A\underline{x} = 0$. It follows that $(BA)\underline{x} = 0$ so BA is singular.

(19) If A and B are non-singular so is AB.

T
Comments: Non-singularity is equivalent to invertibility so the fact that a product of invertible matrices is invertible tells us that a product ofnon-singular matrices is non-singular.

(20) If A and B are singular $n \times n$ matrices, so is $A + B$.

F
Comments: For example, if $A + (-A)$ is certainly not invertible although A invertible implies $-A$ is invertible.

(21) An $n \times n$ matrix is non-singular if and only if its columns are a basis for $\mathbb{R}^n$.

T
Comments: A matrix is non-singular if and only if its columns are linearly independent, but any n linearly independent vectors in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$.

(22) There is a matrix whose inverse is $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$.

F
Comments: The inverse of a matrix is invertible: $(A^{-1})^{-1} = A$. The third column of the given matrix is the sum of the first two columns so the given matrix is not invertible—therefore it can't be the inverse of any matrix.

(23) If $A^{-1} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

T
Comments: Take $B = EA^{-1}$.

(24) The column space of the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

T
Comments: It is "obvious" that the columns are linearly independent.

(25) The subspace of $\mathbb{R}^3$ spanned by $(1, -2, 1)$ is the same as the subspace spanned by $(-1, 2, -1)$.

T
Comments: Since $(1, -2, 1) = -(-1, 2, -1)$, every multiple of $(1, -2, 1)$ is a multiple of $(-1, 2, -1)$ and conversely, i.e., the two vectors span the same line.

(26) The subspace spanned by $(1, 0, 2)$ and $(2, 0, 1)$ is the same as the subspace spanned by $(1, 0, -1)$ and $(-1, 0, 1)$.

F
Comments: The vectors $(1, 0, 2)$ and $(2, 0, 1)$ are linearly independent so span a plane whereas $(1, 0, -1)$ and $(-1, 0, 1)$ are linearly *dependent* so span a line.

(27) The subspace spanned by $(1, 0, 2)$ and $(2, 0, 1)$ is the same as the subspace spanned by $(1, 0, 1)$ and $(-1, 0, 1)$.

T

Comments: The vectors $(1, 0, 2)$ and $(2, 0, 1)$ are linearly independent so span a plane. The vectors $(1, 0, 1)$ and $(-1, 0, 1)$ are also linearly independent so span a plane. Both planes contain $(1, 0, 0)$ and (0,0,1)*soareequaltothelinearspanof*(1,0,0)*and*

(28) For all matrices A and B, $\mathcal{N}(AB)$ contains $\mathcal{N}(B)$.

T

Comments: If $B\underline{x} = 0$, then $AB\underline{x} = 0$.

(29) For all matrices A and B, $\mathcal{R}(AB)$ is contained in $\mathcal{R}(A)$.

T

Comments: Every element in $\mathcal{R}(AB)$ is of the form $AB\underline{x}$; but $AB\underline{x} = A(B\underline{x})$ so is in the range of A.

(30) For all matrices A and B, $\mathcal{R}(AB) = \mathcal{R}(BA)$.

F

Comments: It is possible to have $AB = 0$ but $BA \neq 0$.