Math 308 **Midterm** **November 6, 2009**

We will write $\underline{A}_1, \ldots, \underline{A}_n$ for the columns of an $m \times n$ matrix A. If $\underline{x} \in \mathbb{R}^n$, we will write $\underline{x} = (x_1, \ldots, x_n)^T$.

The null space and range of a matrix A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. The linear span of a set of vectors is denoted by $\mathrm{Sp}(\underline{v}_1, \ldots, \underline{v}_n)$.

Scoring. On the True/False section you get $+2$ for each correct answer, -2 for each incorrect answer and 0 if you choose not to answer the question.

Part A.

Short answer questions

(1) A system of linear equations can have either
  (a) a _________ solution, or
  (b) _________ solutions, or
  (c) _________ solutions.

  unique; infinitely many; no.

(2) An $n \times n$ matrix is non-singular if and only if it is row equivalent _________

  to the identity matrix.

(3) A set of vectors is linearly dependent if it contains the _________

  zero vector.

(4) A set of $n+1$ vectors is $\mathbb{R}^n$ is _________

  linearly dependent.

(5) A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.

  a linear combination

(6) The only $n \times n$ row-reduced echelon matrix of rank n is the _________

  identity matrix.

(7) The meaning of the notation $A \subset B$ is that every element in A _________

  is an element of B.

(8) If $x \in A$ and $A \subset B$ then x _________

  $\in B$.

(9) Give an example of three linearly dependent vectors $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^3$, such that $\{\underline{u}, \underline{v}\}$, $\{\underline{u}, \underline{w}\}$, and $\{\underline{w}, \underline{v}\}$, are linearly independent.

There are many solutions to this. Take any two linearly independent vectors, and a non-zero vector that is a linear combination of the other two: for example,

$$\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_1 + \underset{\sim}{e}_2\} = \{(1,0,0),(0,1,0),(1,1,0)\}$$

or,

$$\{(1,2,3),(3,2,1),(1,1,1)\}.$$

Comments: It would be a good idea to understand why the second example answers the question.

(10) Choose values for a and b so that $\begin{pmatrix} a & a & 0 \\ 1 & b & 2 \\ -1 & 1 & 1 \end{pmatrix}$ is singular.

You must choose a and b so that the rows, or columns, are linearly dependent. The simplest solution is to take $a = 0$ because then the top row is the zero vector and any set containing the zero vector is linearly dependent. If you take $a \neq 0$, then b **must equal** 5.

Comments: The reason b must equal 5 if a is non-zero is that singularity of the matrix is equivalent to the existence of elements $r, s, t \in \mathbb{R}^3$, not all zero, such that

$$r\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \textit{i.e.,} \begin{pmatrix} (r+s)a \\ r+sb+2t \\ s-r+t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $(r+s)a = 0$ and $a \neq 0$, $r+s$ must equal zero. Looking at the third row, $-r+s+t$ must also be zero so $t = 2r$. Finally, for the second row to be zero we need $r - rb + 4r = 0$. This forces b to be 5.

(11) Choose values for a and b so that the columns of the matrix in question 10 form a basis for $\mathbb{R}^3$.

a can be any non-zero number, and b can be any number except 5.

Comments: There are infinitely many solutions. You must choose a and b so that the columns, are linearly independent, i.e., you simply need to avoid the values of a and b that are valid answers to question 1.

Some people still confuse *singular* and *non-singular*. Make sure you are not one of them.

(12) Choose values for a and b so that the columns of the matrix in question 10 are not a basis for $\mathbb{R}^3$.

You must pick a and b such that the columns are not linearly independent, i.e., such that the matrix is singular, so this is really the same question as question 10.

(13) What is the relation between the range and the columns of a matrix?

The range is equal to the linear span of the columns.

Comments: An answer such as *The columns of a matrix are the range* is not good enough. There is a connection between the columns and the range, but you must say *precisely* what it is. There are elements of the range that are not columns of a matrix but *linear combinations of the columns.*

Another incorrect answer was *the range spans the columns of the matrix.* That needs rearranging.

(14) If the following system of equations in the unknowns x, t, w is written as a matrix equation $A\underline{x} = \underline{b}$, what are A, $\underline{x}$, and $\underline{b}$?

$$\begin{aligned} 2x + 3w - t &= -1 \\ t + 2 &= -x - w \\ 2x + 3t &= 1 + 2t \end{aligned}$$

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x \\ w \\ t \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix},$$

Comments:

(15) Suppose A is a 3×4 matrix and $\underline{b} \in \mathbb{R}^4$. Suppose that the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 0 & | & 5 \\ 0 & 0 & 1 & 4 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Write down all solutions to the equation $A\underline{x} = \underline{b}$.

x_2 and x_4 can be anything and $x_1 = 5 - 2x_2$ and $x_3 = 1 - 4x_4$.
Comments:

(16) (You are not allowed to use the inverse of C in your answer to this question.) If $CD = EC = I$ show that $D = E$.

$D = ID = (EC)D = E(CD) = EI = E$.
Comments: Many, many people seemed to forget the definition of *inverse*: a matrix C has an inverse if there is a matrix B such that $CB = BC = I$. The point of this question was to show that the apparently weaker statement that there are matrices D and E (not assumed to be the same!) such that $CD = EC = I$ implies that $D = E$ and hence C has an inverse, namely D.

You must make it absolutely clear to the reader why each $=$ sign in your solution is justified. For example, if you begin by writing $C^{-1}CD = ECC^{-1}$ you have assumed that C has an inverse. But the definition is that C has an inverse if there is a matrix B such that $BC = CB = I$, and the hypothesis does not tell you that such a B exists. Indeed, the point of the problem is to show the existence of such a B by showing that $D = E$.

Even if you were told C has an inverse (which you weren't), it is not true that $C^{-1}M = MC^{-1}$ for all matrices M so you need to justify writing $C^{-1}CD = ECC^{-1}$ by writing $C^{-1}CD = C^{-1}I = IC^{-1} = ECC^{-1}$.

(17) If BAB has an inverse show A and B have inverses.

We will use the fact that a matrix has an inverse if and only if it is non-singular. If $B\underline{x} = 0$, then $(BAB)\underline{x} = 0$ which implies that $\underline{x} = 0$. Hence B is non-singular and therefore has an inverse. Since a product of two matrices having inverses has an inverse, $B^{-1}(BAB)B^{-1}$ has an inverse, i.e., A has an inverse.

Alternatively: By hypothesis, there is a matrix C such that $CBAB = BABC = I$. This equation says that B has an inverse.[1] Since a product of two matrices having inverses has an inverse, $B^{-1}(BAB)B^{-1}$ has an inverse, i.e., A has an inverse.

Comments: Many people simply wrote down $(BAB)^{-1} = B^{-1}A^{-1}B^{-1}$ but that answer presupposes A and B have inverses! *After* you have proved that A and B have inverses you are entitled to write $(BAB)^{-1} = B^{-1}A^{-1}B^{-1}$. But there is no need to write that down anyway because the question does not ask what the inverse of BAB is.

See also the comments to question 6.

(18) Let

$$A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad Q = A^T C.$$

Compute the **inverse** of Q **without computing** Q.

Because $Q^{-1} = C^{-1}(A^T)^{-1} = C^{-1}(A^{-1})^T$,

$$Q^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 7 & 1 \end{pmatrix}.$$

Comments:

(19) Find the reduced echelon matrix that is equivalent to the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 2 \\ 0 & 0 & 4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Comments:

(20) Give an example of subspaces $V \subset W \subset \mathbb{R}^4$ such that $\dim V = 1$ and $\dim W = 3$, and $W \cap \{\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4\} = \phi$.

There are many examples. For example, take $V = \mathbb{R}(1,1,0,0)$, all multiples of the vector $(1,1,0,0)$, and $W = \mathrm{Sp}((1,1,0,0),(1,2,3,4),(0,0,1,1))$.

(21) List all the subspaces of $\mathbb{R}^3$.

[1] Although we don't need to make this observation, by the previous problem the inverse of B is $CBA = ABC$.

The zero subspace, $\mathbb{R}^3$ itself, every line through the origin, and every plane containing the origin.

Comments: One person gave the answer

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

but this is a list of *elements* in $\mathbb{R}^3$ not *subspaces*.

(22) Let E be set of all even numbers, O the set of all odd numbers, and T the set of all integer multiples of 3. Then
(a) $E \cap O =$________ (use a single symbol for your answer)
(b) $E \cup O =$________ (use a single symbol for your answer)
(c) $E \cap T =$________ (use words for your answer)

$E \cap O = \phi$, the empty set, because no number is both even and odd;
$E \cup O = \mathbb{Z}$ because every integer is either even or odd;
$E \cap T =$ all multiples of 6 because a number is divisible by both 2 and 3 if and only if it is divisible by 6.

(23) If A is a square matrix such that $A^2 = A$, simplify $(I - A)^2$, and $(I - A)^{15}$.

Since $(I-A)^2 = I-2A+A^2 = I-2A+A = I-A$ an induction argument shows that $(I - A)^n = I - A$ for all $n \geq 0$.

Comments: Some people forgot that $IA = A$ so wrote $(I - A)^2 = I - 2IA + A^2 = I - 2IA + A$ and were unable to simplify that.

Some thought that A had to be either 0 or I, but there are lots of A such that $A^2 = A$. For example, let

$$A = \begin{pmatrix} 9 & -6 \\ 12 & -8 \end{pmatrix}.$$

(24) What is the relation between the rank and nullity of an $m \times n$ matrix?

rank+nullity $= n$.
Comments:

(25) Let A be a 3×2 matrix whose null space is a line. Complete the sentence "The range of A is a ________ in ________"

line in $\mathbb{R}^3$.

Comments: An answer such as *subspace of* $\mathbb{R}^3$ is true but less precise than it should be. Such an answer fails to use the information that the null space is a line.

You *must* know that if A is an $m \times n$ matrix, then $rank(A) + nullity(A) = n$. Questions using that fact will certainly be on the final exam.

(26) Let A be a 2×3 matrix whose null space is a line. Complete the sentence "The range of A is a ________"

2-dimensional subspace of $\mathbb{R}^2$, therefore all of $\mathbb{R}^2$.

Comments:

An answer such as *subspace of* $\mathbb{R}^2$ is true but less precise than it should be. Such an answer fails to use the information that the null space is a line. Likewise, an answer like *a 2-dimensional subspace of* $\mathbb{R}^2$, which is correct, conveys the impression that there are several two-dimensional subspaces of $\mathbb{R}^2$ and that you are unable to specify which one.

(27) Give an example of a 2×3 matrix A having rank one and nullity two.

Since rank+nullity=3 in this case, A must be non-zero and the rows must be linearly dependent. There are infinitely many solutions but one answer is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Comments:

(28) If $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly independent show that $\{\underline{u} + 2\underline{v}, \underline{v}, \underline{u} - \underline{v} + 3\underline{w}\}$ is linearly independent.

If $a(\underline{u}+2\underline{v})+b\underline{v}+c(\underline{u}-\underline{v}+3\underline{w}) = 0$, then $(a+c)\underline{u}+(2a+b-c)\underline{v}+3c\underline{w} = 0$. Because $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly independent $a + c = 2a + b - c = 3c = 0$. It follows that $c = 0$ because $3c = 0$, then$a = 0$ because $a + c = 0$, then $b = 0$ because $2a+b-c = 0$. Hence $\{\underline{u}+2\underline{v}, \underline{v}, \underline{u}-\underline{v}+3\underline{w}\}$ is linearly independent.

(29) What is the equation of the plane (i.e., the two-dimensional subspace) in $\mathbb{R}^3$ containing $(1, 1, 1)$ and $(0, 2, 3)$?

The plane is $3y - 2z - x = 0$.

I was just looking for the equation, not an explanation of why that is the correct equation. However, here is the reasoning that leads to the answer. Let P be the plane we are looking for. A plane in $\mathbb{R}^3$ is given by a single equation of the form $ax + by + cz = 0$. So we are looking for a, b, c with the property that

$$P = \{(x, y, z) \mid ax + by + cz = 0\}.$$

Because $(0, 2, 3) \in P$ we need $2b + 3c = 0$. One solution to this is given by $b = 3$ and $c = -2$. Because $(1, 1, 1) \in P$, we also need $a + b + c = 0$; because we have chosen $b = 3$ and $c = -2$ we must have $a = -1$. Hence the result.

Notice that another answer is given by $6y - 4z - 2x = 0$, or any other non-zero multiple of $ax + by + cz = 0$.

(30) Write down two linearly independent vectors in the plane $2x + 3y - z = 0$.

Infinitely many answers. One such is $(1, 0, 2)$ and $(0, 1, 3)$. I did not care whether your answer consisted of row or column vectors.

Comments: Some incorrect answers:

- Several students wrote down vectors that did not lie on the plane.
- Some students gave $(0, 0, 0)$ as one of the vectors but every set that contains the zero vector is linearly dependent (why?).

- One student gave the answer

$$\underline{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

  but those vectors belong to $\mathbb{R}^2$ and the plane is lying in $\mathbb{R}^3$ so points on it have *three* coordinates, i.e., they are either 3×1 or 1×3 vectors.

(31) Write down a basis for the plane $2x + 3y - z = 0$ in $\mathbb{R}^3$.

You can use your answer to the previous question, e.g., $\{(1, 0, 2), (0, 1, 3)\}$. **Comments:** Many people failed to realize they could use their answer from the previous question. A basis for the plane $2x + 3y - z = 0$ in $\mathbb{R}^3$ is a linearly independent subset of the plane that spans it. Since a plane has dimension two any two linearly independent vectors on the plane will be a basis for the plane.

(32) Write down a basis for the line $x + y = x - z = 0$ in $\mathbb{R}^3$.

Any non-zero multiple of $(1, -1, 1)$.
Comments: A line has dimension one so the basis will consist of one non-zero vector that lies on the line.

(33) Write down two linearly independent vectors lying on the plane

$$x_1 + 2x_2 + 2x_4 = x_1 - 2x_2 - x_3 - x_4 = 0$$

in $\mathbb{R}^4$.

There are many correct answers. One is $(-2, 0, -3, 1)$ and $(-2, 1, -4, 0)$.

(34) Is $(9, 8, 7, 6)$ a linear combination of the two vectors in your answer to question 33? Explain why?

No because $(9, 8, 7, 6)$ does not lie on the plane, and the points on the plane are precisely all linear combinations of $(-2, 0, -3, 1)$ and $(-2, 1, -4, 0)$.

(35) List all the singular 1×1 matrices.

A 1×1 matrix is a number so 0 is the only singular 1×1 matrix, i.e., 0 is the only number that does not have an inverse.

(36) Let

$$A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$$

and $\underline{b}$ any vector in $\mathbb{R}^2$. Explain why there is or is not a solution to the equation $A\underline{x} = \underline{b}$.

Part B.

Complete the definitions.

You do not need to write the part I have already written.
Just complete the sentence.

(1) A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if there are

numbers $a_1, \dots, a_n$ such that $\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$.

(2) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is

linear combination of the columns of A.

(3) The linear span of $\{\underline{v}_1, \dots, \underline{v}_n\}$ consists of

all linear combinations of $\underline{v}_1, \dots, \underline{v}_n$.

(4) Express $A\underline{x}$ as a linear combination of the columns of A.

$A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$.

(5) If A is an $m \times n$ matrix of rank n, then the equation $A\underline{x} = \underline{b}$ has a

unique solution.

(6) $\mathbb{R}^n$ denotes the set of all

$n \times 1$ matrices.

(7) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent if the only solution to_______________

the equation $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ is $a_1 = \cdots = a_n = 0$.

(8) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly dependent if the equation _______________ has a solution such that _______________

$a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ has a solution such that at least one of the a_is is non-zero.

(9) An $n \times n$ matrix A is non-singular if _______________.

the only solution to the equation $A\underline{x} = 0$ is $\underline{x} = 0$.
Comments: Some people said *if it has an inverse* and/or *its rows are linearly independent* and/or *its columns are linearly independent* and/or *it is row equivalent to an identity matrix.* These conditions on a matrix are *equivalent* to the condition that a matrix be non-singular but they are *consequences* of the definition, not the actual definition itself.

(10) A subset $S = \{\underline{v}_1, \dots, \underline{v}_d\}$ of $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$ if _______________.

it is linearly independent and spans $\mathbb{R}^n$.

Comments:

(11) The dimension of a subspace $W \subset \mathbb{R}^n$ is ______________.

the number of elements in a basis for W.

Comments: Several people said *the number of elements in W*. If $W \neq \{0\}$, then W has infinitely many elements. It is also incorrect to say *the number of elements in the basis for W* because that suggests that W has only *one* basis. If $\dim W \geq 1$, W has infinitely many bases.

Another "answer": *Number of basis vector in W*. Presumably the author expects the reader to rewrite this sentence. But that is the author's job, not the reader's. Let's see how much is wrong with this. First, it should begin with the word *the*, as in *the number of*. Number of what? Vectors. Which vectors? Those in a basis for W. But the author does not say *the number of vectors*. Indeed, no plural is used at all. It is hard to think of any grammatically correct sentence in which *number of* is followed by a singular noun.

(12) A subset W of $\mathbb{R}^n$ is a subspace if ______________.

it is contains the zero vector, and whenever $\underline{u}$ and $\underline{v}$ are in W, and $a \in \mathbb{R}$, $\underline{u} + \underline{v}$ and $a\underline{v}$ are in W.

Comments: Many garbled versions given as answers.

(13) The linear span $\mathrm{Sp}(\underline{v}_1, \dots, \underline{v}_d)$ of a subset $\{\underline{v}_1, \dots, \underline{v}_d\} \subset \mathbb{R}^n$ is ______________.

the set of vectors $a_1\underline{v}_1 + \cdots + a_d\underline{v}_d$ as $a_1, \dots, a_d$ range over all of $\mathbb{R}$.

Comments: One person said *the set of all vectors $a_1\underline{v}_1 + \cdots + a_d\underline{v}_d$, a takes on all possible values in $\mathbb{R}^N$*. What is a? What does $\mathbb{R}^N$ have to do with the question? The question is about vectors in $\mathbb{R}^n$ not $\mathbb{R}^N$.

(14) An $n \times n$ matrix A is invertible if ______________.

there is a matrix B such that $AB = BA = I$ where I is the $n \times n$ identity matrix.

Comments: Do not say *if it is non-singular*. The fact that A is non-singular if and only if it is invertible is a *theorem*, i.e., it is a *consequence* of the definitions of invertible and non-singular, not the definition itself.

(15) The rank of a matrix is ______________.

the dimension of its range **OR** the number of non-zero rows in the echelon form of the matrix.

Comments: Some incorrect answers:

- *the dimension of its column space* because that is a *consequence* of the definition, not the definition itself.
- *the dimension of the range*. But which range? The answer should say either *the dimension of the range of A* or *the dimension of its range*.
- *the number of non-zero rows*. In which matrix?

- *the dimension of the columns is the range*—but columns don't have dimension, only subspaces have dimension.

(16) A vector $\underline{w}$ is a linear combination of $v_1, \dots, v_n$ if _______________..

$\underline{w} = a_1\underline{v}_1 + \cdots + a_d\underline{v}_d$ for some $a_1, \dots, a_d \in \mathbb{R}$.
Comments: Do not give an example of a single linear combination—you are being asked for the *definition* of "linear combination" not for an example of a linear combination. One student said *any comnbination of* $a_1, \dots, a_d \in \mathbb{R}$ *where* $a_1\underline{v}_1 + \cdots a_d\underline{v}_d$ but that is just a garbled form of the definition and therefore incorrect.

(17) The row space of a matrix is _______________.

the linear span of its rows.
Comments: One student gave the incorrect answer *the subspace of its rows* which is not correct—it would have been correct to say *the subspace spanned by its rows.*

(18) The union of two sets A and B is the set $A \cup B =$ _______________.. .

$\{x \mid x \in A$ or $x \in B\}$.
Comments: Most people did poorly on this and the next definition. The notions of union and intersection apply to *all* sets, not just sets of vectors. There was nothing in the question saying that the elements in A and B are vectors. Thus an answer that began *the subspace ...* would be incorrect.

I saw $A + B$ and AB as answers to this and/or the next question. If you are given arbitrary sets A and B neither $A + B$ nor AB has any meaning.

The symbol for union is $\cup$ *not* U.

Some people omitted the symbols $\{$ and $\}$ and simply wrote down $x \in A$ or $x \in B$.

Someone wrote $\{x \mid w \in A$ or $w \in B\}$. That is wrong because it says "the set of all x such that w is in A or w is in B". But what is the relation between x and w? You need your answer to read (in math notation) "the set of all w such that w is in A or w is in B".

(19) The intersection of two sets A and B is the set $A \cap B =$ _______________..
.

$\{x \mid x \in A$ and $x \in B\}$.
Comments:

(20) The range of an $m \times n$ matrix A is $\{...\}$.

$\{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}$. **Comments:**

(21) The null space of an $m \times n$ matrix A is $\{...\}$.

$\{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = 0\}$. **Comments:**

(22) Two systems of linear equations are equivalent if ______________.

they have the same set of solutions.

Comments: Don't say *if the matrices are row equivalent.* First of all, *which* matrices do you mean (the matrices of coefficients) but, more importantly, the row equivalence is a *consequence* of the definition, and requires one to first define the elementary row operations. The definition of equivalence of linear systems precedes all that stuff. The definition of equivalence of linear systems comes almost immediately after defining what is meant by a system of linear equations.

The same objections apply to an answer like *if B can be obtained from A by elementary row operations.* In any case, what are A and B?

(23) Two matrices are row equivalent if ______________.

one can be obtained from the other by a sequence of elementary row operations **OR** they have the same row reduced echelon form.

Part C.

True or False—just write T or F

(1) If a square matrix has a row of zeroes its is singular.

T

Comments: There are several different ways to see this. If the i^{th} row of A is zero, then the i^{th} row of the product AB is zero for every matrix B, so AB can never be the identity matrix. Or, A^T has a column consisting entirely of zeroes so it its columns are linearly dependent (every set of vectors that contains the zero vector is linearly dependent), and it is therefore singular because a square matrix is non-singular if and only if its columns are linearly dependent.

(2) There are infinitely many choices of a and b that make the matrix $\begin{pmatrix} a & 1 \\ -2 & b \end{pmatrix}$ singular.

T

Comments:

(3) There are infinitely many choices of a and b that make the matrix $\begin{pmatrix} a & 1 \\ -2 & b \end{pmatrix}$ non-singular.

T
Comments:

(4) There is an invertible matrix A such that $AA^T = 0$.

F
Comments: If A had a inverse and $AA^T = 0$, then $0 = AA^T = A^{-1}(AA^T) = (A^{-1}A)A^T = IA^T = A^T$. But then $A = (A^T)^T = 0$ and that is absurd—the zero matrix does not have an inverse.

(5) (I make no assumptions about the matrix A in this question.) A solution to the equation $A\underline{x} = \underline{b}$ is given by $\underline{x} = A^{-1}\underline{b}$.

F
Comments: Because no assumptions about A are made, A need not have an inverse. It may not even be a square matrix.

(6) Consider a consistent system of linear equations in 5 unknowns. If there are 4 left-most 1s in its reduced echelon form, there are 4 dependent variables.

T
Comments:

(7) The matrix AA^T is always symmetric.

T
Comments:

(8) The matrix AA^T is always square.

T
Comments:

(9) There is an invertible matrix A such that $AA^T = I$.

T
Comments: e.g. $A = I$.

(10) If $BD = EB = I$, then $D = E$.

T
Comments: See question 6 in Part A.

(11) Every set of five vectors in $\mathbb{R}^4$ is linearly dependent.

T
Comments: If a subspace W has dimension p, then every set of $p + 1$ elements in W is linearly dependent. That is a theorem in section 3.5(?) the book. It is very important that you not only know the answer to this and the next question but you know *why* the answers are what they are.

(12) If a subset of $\mathbb{R}^4$ spans $\mathbb{R}^4$ it is linearly independent.

F
Comments: If a subspace W has dimension p, a set that spans W is linearly independent if and only if it has exactly p elements. That is a theorem in section 3.5(?) the book.

(13) A homogeneous linear system of 15 equations in 16 unknowns always has a non-zero solution.

T
Comments: See the section in the book about homogeneous linear systems.

(14) If S is a linearly dependent subset of $\mathbb{R}^n$ so is every subset of $\mathbb{R}^n$ that contains S.

T
Comments: It is very important that you know the answer to this and the next question.

(15) Every subset of a linearly independent set is linearly independent.

T
Comments:

(16) A square matrix is singular if and only if its transpose is singular.

T
Comments: First, a square matrix is either singular or non-singular. A matrix is non-singular if and only if it has an inverse. But if A has an inverse then $(A^{-1})^T$ is the inverse of A^T. Thus a matrix is non-singular if and only if its transpose is. Hence a matrix is singular if and only if its transpose is.

(17) $3I - 2I^2 - I^{-1}$ is singular.

T
Comments: The matrix is zero. The zero matrix is singular (look at the definition of "singular").

(18) $I + I^2 - 5I^{-1}$ is singular.

F
Comments: The matrix in question is $-3I$ so $-3I)\underline{x} = -3\underline{x}$ which is zero if and only if $\underline{x} = 0$.

(19) The set $W = \{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

T

Comments: It is very important that you not only know the answer to this and the next question but you know *why* the answers are what they are. The W in this question is a subspace because the set of solutions to *any* set of homogeneous equations is a subspace.

(20) The set $W = \{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a subspace of $\mathbb{R}^4$.

F
Comments: $0 \notin W$.

(21) The solutions to a system of homogeneous linear equations always form a subspace.

T
Comments: Look at the proof in the book – it is important to understand why this is true.

(22) The solutions to a system of linear equations always form a subspace.

F
Comments: Question 25 gives a counterexample. Do you know what I mean by *a counterexample*.

(23) If $\text{Sp}(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$, then $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$.

T
Comments: $\underline{u}$ is certainly in $\text{Sp}(\underline{u}, \underline{v}, \underline{w})$ because it is equal to $1.\underline{u} + 0.\underline{v} + 0\underline{w}$. The hypothesis that $\text{Sp}(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$ therefore implies that $\underline{u} \in Sp(\underline{v}, \underline{w})$. But $Sp(\underline{v}, \underline{w})$ is, by definition, all linear combinations of $\underline{v}$ and $\underline{w}$, so $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$

An important step towards mastering the material in this course is to be able to answer this and the next question *instantly*, and to know why the answers are what they are. In fact, I would go so far as to say that if you can't answer this and the next question *instantly* you are struggling.

(24) If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $Sp(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$.

T
Comments:

(25) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, then $\{\underline{v}_1 - 2\underline{v}_2, 2\underline{v}_2 - 3\underline{v}_3, 3\underline{v}_3 - \underline{v}_1\}$ is linearly dependent.

T
Comments: $1.(\underline{v}_1 - 2\underline{v}_2) + 1.(2\underline{v}_2 - 3\underline{v}_3) + 1.(3\underline{v}_3 - \underline{v}_1) = 0$.

(26) The null space of an $m \times n$ matrix is contained in $\mathbb{R}^m$.

F
Comments:

(27) The range of an $m \times n$ matrix is contained in $\mathbb{R}^n$.

F
Comments:

(28) If $\underline{a}$ and $\underline{b}$ belong to $\mathbb{R}^n$, then the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 0\}$ is a subspace of $\mathbb{R}^n$.

T
Comments: See comment about question 15.

(29) If $\underline{a}$ and $\underline{b}$ belong to $\mathbb{R}^n$, then the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 1\}$ is a subspace of $\mathbb{R}^n$.

F
Comments: $0 \notin W$

(30) If $\underline{a}$ and $\underline{b}$ belong to $\mathbb{R}^n$, then the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 1\} \cup \{\underline{0}\}$ a subspace of $\mathbb{R}^n$?

F
Comments: If $\underline{x} \in W$ and $\underline{x} \neq 0$, then $2\underline{x} \notin W$.

(31) If A and B are non-singular $n \times n$ matrices, so is AB.

T
Comments: If $AB\underline{x} = 0$, then $B\underline{x} = 0$ because A is non-singular, but $B\underline{x} = 0$ implies $\underline{x} = 0$ because B is non-singular.

(32) If A and B are non-singular $n \times n$ matrices, so is $A + B$.

F
Comments: For example, I and $-I$ are non-singular but there sum is the zero matrix which is singular.

(33) Let A be an $n \times n$ matrix. If the rows of A are linearly dependent, then A is singular.

T
Comments:

(34) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.

T
Comments: It is easy to check, and the book does show you this, that if A' is obtained from A by a *single* elementary row operation, then A obtained from A' by a single elementary row operation. Now just string together a

sequence of elementary row operations, and reverse each one of them to get back from B to A.

(35) There is a matrix whose inverse is $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$.

F
Comments: The inverse of a matrix A has an inverse, namely A. The given matrix is obviously not invertible because its columns (and rows) are not linearly dependent. So it cannot be the inverse of a matrix.

(36) If $A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

F
Comments: A is a 2×2 matrix and E is a 2×3 matrix. The product BA only exists when B is an $m \times 2$ matrix for some m and in that case BA is an $m \times 2$ matrix, so cannot equal a 2×3 matrix.

(37) Any linearly independent set of five vectors in $\mathbb{R}^5$ is a basis for $\mathbb{R}^5$.

T
Comments:

(38) The row space of the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

T
Comments: The rows are clearly linearly independent (if that is not clear to you then you do not understand linear (in)dependence) but any three linearly independent vectors form a basis for $\mathbb{R}^3$.

Perhaps it is better to answer this directly: if (a, b, c) is any vector in $\mathbb{R}^3$, then

$$(a, b, c) = \tfrac{1}{3}(3, 0, 0) + \tfrac{1}{2}(0, 2, 0) + \tfrac{1}{3}(0, 0, 3)$$

and it is clear that (a, b, c) can't be written as a linear combination of the rows in any other way.

(39) The subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is the same as the subspace spanned by $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

T

Comments: Each is a multiple of the other: the linear span of a single non-zero vector is the line through that vector and 0, i.e., consists of all multiples of the given vector.

(40) The subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}6\\4\\2\end{pmatrix}$ is the same as the subspace spanned by $\begin{pmatrix}3\\2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\\6\end{pmatrix}$.

T
Comments:

(41) For all matrices A and B, $\mathcal{N}(A) \subset \mathcal{N}(AB)$.

F
Comments:

(42) For all matrices A and B, $\mathcal{N}(B) \subset \mathcal{N}(AB)$.

T
Comments:

(43) For all matrices A and B, $\mathcal{N}(AB) \subset \mathcal{N}(A)$.

F
Comments:

(44) $\{24x \mid x \in \mathbb{Z}\} \subset \{8x \mid x \in \mathbb{Z}\}$

T
Comments: Every integer multiple of 24 is an integer multiple of 8.

(45) For all matrices A and B, $\mathcal{R}(AB) \subset \mathcal{R}(A)$.

T
Comments: This is no more complicated than the fact that every integer multiple of 6 is an even number. Let's prove that. If n is an integer multiple of 6, then $n = 6m$ for some m. But $6 = 2 \times 3$, so $n = 2 \times 3m$. We have just shown that

$$\{6x \mid x \in \mathbb{Z}\} \subset \{2x \mid x \in \mathbb{Z}\}.$$

Of course, we usually call $\{2x \mid x \in \mathbb{Z}\}$ the set of even numbers and call $\{6x \mid x \in \mathbb{Z}\}$ the set of integer multiples of 6.

If $\underline{x} \in \mathcal{R}(AB)$, then $\underline{x} = AB\underline{u}$ for some $\underline{u}$, so $\underline{x} = A(B\underline{u})$. But the last equation says that $\underline{x}$ is a multiple of A so $\underline{x} \in \mathcal{R}(A)$.

(46) For all matrices A and B, $\mathcal{R}(AB) = \mathcal{R}(BA)$.

F

Comments:

(47) For all matrices A and B, $\mathcal{R}(A) \subset \mathcal{R}(AB)$.

F
Comments:

Final comments

Be very careful with the words “a” and “the”. They mean different things.

Take care when using the words ”it” or ”its” so the reader knows what *it* is.

Write $\mathbb{R}$ not R. On the final I will deduct $\frac{1}{2}$ a point every time you write R instead of $\mathbb{R}$.

The symbol $\in$ means *is an element of* and $\subset$ means *is a subset of*. So we write $2 \in \mathbb{R}$ but $\{x \in \mathbb{R} \mid x^2 = 4\} \subset \mathbb{R}$. Sometimes we are casual and write $x, y \in S$ to mean that x and y belong to the set S, i.e., x and y are elements of S.

Several proposed definitions had all the right words in them but in the wrong order. It is your job to get them in the right order, and to give a grammatically correct sentence. Likewise, some people put extra words in their proposed definitions that made no sense. It is not my job to delete those words to make your definition correct. Other proposed definitions omitted essential words. It is not my job to insert the missing words to make your definition correct.

There were many errors in “mathematical grammar”. For example, the words *linear span of a matrix* make no sense: a set of vectors has a linear span, but not a matrix. I was more forgiving of these errors than I will be when grading the final. So, try to be more precise.

When taking an exam it is not sufficient to say enough to suggest that you have some idea of the correct answer and then leave me to give you the benefit of the doubt. You must write down the answer. All the answer. Not just enough to suggest that you *could* write down the answer on another occasion if asked to. The exam is the time to write down the answer. All of it.

You were asked for 15 definitions. The best score was 14 out of 15 correct.

Distinguish between singular and plural words. For example, don’t say *all linear combination*, say *all linear combinations*.