Math 308 **Midterm** **November 1, 2010**

- We will write $\underline{A}_1, \dots, \underline{A}_n$ for the columns of an $m \times n$ matrix A. We sometimes indicate this by writing $A = [\underline{A}_1, \underline{A}_2, \dots, \underline{A}_n]$.
- Several questions involve an unknown vector $\underline{x} \in \mathbb{R}^n$. We will write $x_1, \dots, x_n$ for the entries of $\underline{x}$; thus $\underline{x} = (x_1, \dots, x_n)^T$.
- The linear span of a set of vectors is denoted by $\mathrm{Sp}(\underline{v}_1, \dots, \underline{v}_n)$.

Scoring. On the True/False section you get $+2$ for each correct answer, -2 for each incorrect answer and 0 if you choose not to answer the question.

Part A.

Short answer questions

Scoring: 2 points per question. No partial credit.

(1) Let A be a 4×5 matrix and $\underline{b} \in \mathbb{R}^4$. Suppose the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & \mid & 2 \\ 0 & 0 & 1 & 3 & 0 & \mid & 2 \\ 0 & 0 & 0 & 0 & 1 & \mid & 0 \\ 0 & 0 & 0 & 0 & 0 & \mid & 0 \end{pmatrix}$$

Say which the independent and dependent variables are and write down all solutions to the equation $A\underline{x} = \underline{b}$.

The independent variables are x_2 and x_4 because the 2^{nd} and 4^{th} columns of the row-reduced echelon matrix do not contain a left-most 1. Then the solutions are

$$\begin{aligned} x_1 &= 2 - 2x_2 - x_4 \\ x_3 &= 2 - 3x_4 \\ x_5 &= 0. \end{aligned}$$

(2) Show that the set $\{(1,0,0,1),\ (4,-1,1,2),\ (1,-1,1,-1)\}$ is linearly dependent.

You should give an answer like $(4,-1,1,2) = 3(1,0,0,1) + (1,-1,1,-1)$ because that allows me to see at a glance that the three vectors are linearly dependent. Doing some unexplained calculations that convince you they are linearly dependent is not a good answer because I am still left to figure out how to turn your calculations into something which shows they are linearly dependent. So, after doing your calculations put them into effect by writing down an equation such as the one I just wrote down.

(3) Find integers a and b such that $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 4 & 2 \\ 3 & a & b \end{pmatrix}$ is singular.

Any integers a and b such that $a = b+3$ will work because then the columns become linearly dependent: the middle column is the sum of the other two.

(4) Find an equation for the plane in $\mathbb{R}^3$ that contains $(0,0,0)$, $(1,1,1)$ and $(1,2,2)$.

The plane is given by the equation $y = z$. You might see that immediately by noticing that the y- and z-coordinates of $(1,1,1)$ are the same and $(1,2,2)$ also has this property.

Comments: This question is one that typically arises in Math 126. You learn there that a plane in $\mathbb{R}^3$ is given by a single linear equation, i.e., an equation of the form $ax + by + cz = d$ where a, b, c, d are some numbers. In this question you are told that $(0,0,0)$ lies on the plane in question so d must equal zero. So you need to solve the system

$$\begin{aligned} a + b + c &= 0 \\ a + 2b + 2c &= 0. \end{aligned}$$

(5) This problem takes place in $\mathbb{R}^4$. Find two linearly independent vectors belonging to the plane that consists of the solutions to the system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 0 \\ x_1 - x_2 - x_3 + x_4 &= 0. \end{aligned}$$

Lots of answers: e.g., any two of $(1,1,-1,-1)$, $(4,3,2,1)$, $(6,5,0,-1)$, $(5,4,1,0)$.

Comments: A difficult question for most people, not so much because of the requirement that the solutions be linearly independent but because some of you are forgetting that the solutions to a system of equations in four unknowns are points in 4-space, $\mathbb{R}^4$. A vector, or point, in $\mathbb{R}^4$ with coordinates (a_1, a_2, a_3, a_4) lies on the plane in question if and only if it is a solution to both equations. Some people gave an answer consisting of one point that was a solution to the equation $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$ and another point that was a solution to $x_1 - x_2 - x_3 + x_4 = 0$.

(6) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

$A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$.

Comments: VERY, VERY important to know this. Here are examples of "answers" that are not good:

(a) $A_1\underline{x}_1 + \cdots + A_2\underline{x}_n$. No good because our notation for the i^{th} column of A is $\underline{A}_i$ not A_i; we are underlining A_i to emphasize it is a vector. Likewise, $\underline{x}$ is a vector whose entries are numbers, so we write x_1 for the first of those numbers, not $\underline{x}_1$ which suggests that $\underline{x}_1$ denotes a vector rather than a number.

(b) $x_1\underline{A}_1, \dots, x_n\underline{A}_n$. No good because this is a list of vectors, not a single vector: $x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$ is not the same thing as $x_1\underline{A}_1, \dots, x_n\underline{A}_n$.

(c) $\underline{A}x_1 + \cdots + \underline{A}x_n = \underline{A}x$. No good because the columns are denoted by $\underline{A}_i$ not just by $\underline{A}$, and on the right hand side $\underline{A}x$ should be $A\underline{x}$.

One other remark: although $A\underline{x} = \underline{A}_1x_1 + \cdots + \underline{A}_nx_n$ is technically correct you should not write this for the same reason we don't like to write $y2$ when we mean $2y$. The numbers x_i are usually written before the vectors $\underline{A}_i$. The convention is to write

$$3\begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \text{rather than} \qquad \begin{pmatrix}1\\2\\3\end{pmatrix}3.$$

(7) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix with columns $\underline{B}_1, \dots, \underline{B}_p$. Express the columns of AB in terms of A and the columns for B.

$AB = [A\underline{B}_1, \dots, A\underline{B}_p]$. Very important to know this.
VERY important to know this.

(8) Let $A = \begin{pmatrix}1 & 2 & 3\\1 & 1 & 1\\3 & 2 & 0\end{pmatrix}$. Illustrate the three elementary row operations by writing down three matrices below each of which is obtained by performing a single elementary row operation to A and beside each matrix describe in words the operation you performed:

(a)
(b)
(c)

Comments: There are three elementary row operations and I wanted you to identify those, not combinations of them. They are called elementary because they really are elementary. They are:

- switch the position of two rows;
- multiply a row by a non-zero number;
- replace row i by row i plus an arbitrary multiple of row j for any $j \neq i$.

(9) Give an example of a 4×3 matrix that is in echelon form but not in row-reduced echelon form.

(10) Given an example of a 4×5 matrix that is in row-reduced echelon form and has rank 2.

(11) Express the statement *"x is an element of A"* in symbols:

$x \in A$

(12) Express the statement *"B is a subset of A"* in symbols: $B \subset A$

(13) Let $\mathcal{S}$ be the set of matrices A such that the inverse of A is equal to the transpose of A. Use set notation to define $\mathcal{S}$ succinctly, i.e., without using words, just standard math symbols. Your answer should look like

$$\mathcal{S} = \{ \qquad | \qquad \}.$$

(14) Let A be a matrix with entry $A_{35} = 7$. Then the _______ entry in A^T is ____.

(15) The transpose of an $m \times n$ matrix has size _______________.

Part B.

Complete the following definitions. **You do not need to write the part I have already written.** Just complete the sentence. Make sure you give the definition rather than a theorem which is a consequence of the definition.

Scoring: 2 points per question. No partial credit.

Comment: I was asking for *definitions* not theorems.

(1) Two systems of linear equations are equivalent if _______________.

they have the same solutions.

(2) An $n \times n$ matrix A is non-singular if the only solution _______________.

to $A\underline{x} = \underline{0}$ is $xul = \underline{0}$.

(3) A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if _________

$\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ for some $a_1, \dots, a_n \in \mathbb{R}$.

(4) The linear span of $\{\underline{v}_1, \dots, \underline{v}_n\}$ consists of _________

all linear combinations of $\underline{v}_1, \dots, \underline{v}_n$.

(5) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent if the only solution to the equation _______________ is _______________.

$a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}$ is $a_1 = \cdots = a_n = 0$.

(6) An $n \times n$ matrix A is singular if there exists _______________ in $\mathbb{R}^n$ such that _______________.

a non-zero $\underline{x} \in \mathbb{R}^n$ such that $A\underline{x} = 0$.

(7) An $n \times n$ matrix A is non-singular if the only solution _______________.

to the equation $A\underline{x} = 0$ is $\underline{x} = 0$.

Comments: It is not suffficient to say $\underline{x} = 0$. You must name the equation whose only solution is $\underline{x} = 0$.

(8) An $n \times n$ matrix A is invertible if _______________.

there is a matrix B such that $AB = BA = I$.

Comments: It is important to write $AB = BA = I$ and not just $AB = I$. It is true that if $AB = I$, then $BA = I$ but this is a theorem whose proof is not trivial. So in the definition of invertible we always make it clear that we want both $AB = I$ and $BA = I$.

(9) The rank of a matrix is _______________.

the number of non-zero rows in rref$A$, the row-reduced echelon form of A.

(10) The union of two sets A and B is the set $A \cup B = \{\quad \mid \qquad \}$

$\{x \mid x \in A$ or $x \in B\}$.

Part C.

Complete the following sentences. Most of the sentences are theorems or consequences of theorems or definitions. **You do not need to write the part I have already written.** Just complete the sentence.

Scoring: 2 points per question. No partial credit.

(1) A homogeneous system of linear equations always has a non-zero solution if the number of unknowns is ______________.

bigger than the number of equations.

(2) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is in the linear span of ________

the columns of A.

Comments: Several people just said "A". But that is not correct because then the sentence reads "The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is in the linear span of A" but A does not have a linear span. We only speak of the linear span of a collection of vectors, not the linear span of a single matrix. Your answer MUST include the word "columns". But a proposed answer "the columns" is not correct because you must name the matrix whose columns you are talking about.

(3) A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.

a linear combination.

Comments: You should not say "the linear combination".

(4) Let A be an $n \times n$ matrix and $\underline{b} \in \mathbb{R}^n$. The equation $A\underline{x} = \underline{b}$ has a unique solution if the rank of A is ___________

n.

(5) The matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible if and only if ___________.

$wz - xy \neq 0$

(6) If the matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible its inverse is ___________.

(7) If $(1, 0, 1, 0, 1, 0)$ and $(6, 5, 4, 3, 2, 1)$ are solutions to the same system of linear equations so is ___________.

(8) The equation $A\underline{x} = \underline{b}$ has a unique solution if and only if
 (a) $\underline{b}$ is in the ___________ and
 (b) A is ___________.

(9) If $A\underline{u} = \underline{b}$, then the set of all solutions to the equation $A\underline{x} = \underline{b}$ consists of the vectors $\underline{u} + \underline{v}$ as $\underline{v}$ ranges over all ___________

all solutions to the equation $A\underline{x} = 0$. (Very important to know this.)

Part D.

True or False

Use your BUBBLES.

Make sure you have filled in the bubbles with your name and student ID.

Scoring: You get +2 for each correct answer and -2 for each incorrect answer, and 0 if you do not answer the question.

Comment: The scantron gives me a lot of information about which questions were difficult for students so I will focus on those. Before the stating the answer I will give a three numbers, e.g., **(45,15,40)** which mean the following: 45% of students gave the correct answer, 15% gave the wrong answer, and 40% did not answer the question.

My goal is to have around 70% of students give the correct answer. We have a long way to go to reach that goal.

(1) If $(1, 2, 3, 4, 5, 6)$ and $(6, 5, 4, 3, 2, 1)$ are solutions to a homogenous system of linear equations so is $(7, 7, 7, 7, 7, 7)$.

True.

Comment: Every linear combination of solutions to a homogenous system of linear equations is also a solution to that system.

Proof: If $\underline{v}_1, \dots, \underline{v}_p$ are solutions to $A\underline{x} = \underline{0}$ and $c_1, \dots, c_p \in \mathbb{R}$, then

$$A(c_1\underline{v}_1 + \cdots c_p\underline{v}_p) = c_1A\underline{v}_1 + \cdots + c_pA\underline{v}_p = c_1\underline{0} + \cdots + c_p\underline{0}_p = \underline{0}.$$

(2) If $(1, 2, 3, 4, 5, 6)$ and $(7, 7, 7, 7, 7, 7)$ are solutions to the a homogenous system of linear equations so is $(6, 5, 4, 3, 2, 1)$.

True.

Comment: Same comment as for the previous question.

(3) If a system of equations has more equations than unknowns it has no solutions.

False.

It might be the same equation repeated many times or, in the same spirit, the equations might be $x = 1$, $2x = 2$, and $5x = 5$. Three equations, one unknown, but a solution $x = 1$.

(4) If a system of equations has fewer equations than unknowns it always has a solution.

Hard (29,60,11) False.

It is true that a system of *homogeneous* equations having fewer equations than unknowns always has a non-trivial solution. But the question did not ask about homogeneous systems or non-trivial solutions. The following system of two equations in three unknowns has no solutions:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 3x_3 &= 5 \end{aligned}$$

has no solution.

(5) Every set of five vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

Hard (42,40,18) False.

The vectors

$$(1, 0, 0, 0),\ (2, 0, 0, 0),\ (3, 0, 0, 0),\ (4, 0, 0, 0),\ (5, 0, 0, 0),$$

do not span $\mathbb{R}^4$. They only span the line through the origin and $(1, 0, 0, 0)$. The point $(1, 1, 1, 1)$ in $\mathbb{R}^4$ is not a linear combination of the five vectors I listed.

(6) Every set of five vectors in $\mathbb{R}^4$ is linearly dependent.

Hard (40,20,40) True: this is a special case of an important theorem that we proved in class. That theorem said that any set of $\geq n+1$ vectors in $\mathbb{R}^n$ is linearly dependent. In fact, we will prove a more general theorem: given any r

vectors $\underline{v}_1, \dots, \underline{v}_r$ in $\mathbb{R}^n$ any set of $\geq r+1$ vectors in $\mathrm{Sp}(\underline{v}_1, \dots, \underline{v}_r)$ is linearly dependent.

(7) Every set of four vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.

Hard (51,9,40) False. The vectors

$$(1,0,0,0),\ (2,0,0,0),\ (3,0,0,0),\ (4,0,0,0),$$

do not span $\mathbb{R}^4$.

(8) Every set of four vectors in $\mathbb{R}^4$ is linearly independent.

Hard (60,4,36) False. The vectors

$$(1,0,0,0),\ (2,0,0,0),\ (3,0,0,0),\ (4,0,0,0),$$

are linearly dependent. Indeed, any two of those four vectors form a linearly dependent set. For example,

$$2(1,0,0,0) - (2,0,0,0) = \underline{0}.$$

(9) If S is a linearly independent set so is every subset of S.

(78,4,18) True.

Suppose $S = \{\underline{v}_1, \dots, \underline{v}_s\}$ and $a_1\underline{v}_1 + \cdots + a_r\underline{v}_r = 0$ for some $r \leq s$ and some numbers $a_1, \dots, a_r$. It follows that

$$a_1\underline{v}_1 + \cdots + a_r\underline{v}_r + 0\underline{v}_{r+1} + \cdots + 0\underline{v}_s = 0.$$

But S is linearly independent so $a_1 = a_2 = \cdots = a_r = 0$. This implies that $\{\{\underline{v}_1, \dots, \underline{v}_r\}$ is linearly independent.

(10) A square matrix having a column of zeroes is always singular.

(49,33,18) Hard. True.

A square matrix is singular if and only if its columns are linearly dependent. But any set of vectors that contains the zero vector is linearly dependent.

(11) There is an invertible 3×3 matrix A such that $A^3 + A = 0$.

This was not a good question! My apologies.

(12) There is an invertible 3×3 matrix A such that $A^3 + I = 0$.

(33,16,51) True: Just take $A = -I$. Consider the analogy with the fact that -1 is a solution to $x^3 + 1 = 0$.

(13) If $\mathrm{Sp}(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$, then $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$.

(89,2,9) True.

(14) If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $Sp(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$.

(91,0,9) True.

(15) For any vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, $Sp(\underline{u}, \underline{v}, \underline{w})$ contains $Sp(\underline{v}, \underline{w})$.

Hard (51,22,27) True.

This should be easy: every linear combination of $\underline{v}$ and $\underline{w}$ is a linear combination of $\underline{u}, \underline{v}, \underline{w}$ because

$$a\underline{v} + b\underline{w} = 0\underline{u} + a\underline{v} + b\underline{w}.$$

(16) If A is singular and B is non-singular then AB is always singular.

(42,16,42) True. You should use the result that a matrix is non-singular if and only if it has an inverse.

You are told that A is singular so there is a non-zero vector $\underline{x}$ such that $A\underline{x} = \underline{0}$. You are told that B is non-singular so it has an inverse B^{-1}. If AB were non-singular it would have an inverse, i.e., there would be a matrix C such that $I = (AB)C = C(AB)$. But $I = CAB$ implies $B^{-1} = CA$. It follows that $B^{-1}\underline{x} = CA\underline{x} = \underline{0}$. Now multiply both sides of the equality $B^{-1}\underline{x} = 0$ by B to deduce $\underline{x} = \underline{0}$ which contradicts the fact that $\underline{x}$ is non-zero. [Difficult question.]

(17) If A and B are non-singular so is AB.

(9,67,24) True. You need to use the result that a matrix is non-singular if and only if it has an inverse.

Because A and B are non-singular they have inverses. This implies AB has an inverse, namely $B^{-1}A^{-1}$. In the lectures we proved that AB has an inverse if A and B do and we carried out the computations

$$(AB)B^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$B^{-1}A^{-1}(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$$

Remember too that the operations "put on shoes" and "put on socks" both have inverses so the combined operation "put on socks then put on shoes" has an inverse.

As soon as possible you must fully integrate the fact that non-singularity and having an inverse are equivalent conditions on a square matrix.

(18) If A and B are non-singular so is $A + B$.

(56,11,33) False.

In order to show that a statement like this is false you need only find two non-singular matrices whose sum is singular. For example, I and $-I$ are non-singular (each is its own inverse) but their sum is the zero matrix which is certainly singular.

(19) If A is non-singular so is $A + I$.

(31,16,53) False.

In order to show that a statement like this is false you need only find a non-singular matrix A such that $A + I$ is singular. For example, $-I$ is non-singular (it is its own inverse) but $(-I) + I = 0$ which is certainly singular.

(20) The linear span of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is the same as the linear span of $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

(91,0,9) True. I was pleased to see almost everyone got this correct. You understand that if $\underline{u}$ and $\underline{v}$ are multiples of one another they have the same linear span. Our short notation for the linear span of a single vector is $\mathbb{R}\underline{u}$, all multiples of $\underline{u}$. Hence, if $\underline{u} = a\underline{v}$ for some non-zero $a \in \mathbb{R}$, then $\mathbb{R}\underline{u} = \mathbb{R}\underline{v}$.

(21) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$ have the same the linear span as $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

(53,27,20) True.

Comment: After your success on the previous question it was a surprise this question seemed so much harder because the principle underlying it is the same as the principle underlying the previous question. Perhaps you will understand this best if I use symbols rather than words. Let's write

$$\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \underline{v} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}.$$

Then the linear span of these two vectors is $\mathbb{R}\underline{u} + \mathbb{R}\underline{v}$. However,

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{2}\underline{v}$$

and

$$\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2\underline{u}$$

so the linear span of $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ is

$$\mathbb{R}(\frac{1}{2}\underline{v}) + \mathbb{R}(2\underline{u}) = \mathbb{R}\underline{v} + \mathbb{R}\underline{u} = \mathbb{R}\underline{u} + \mathbb{R}\underline{v}$$

which is the linear span of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$.

(22) There is a matrix whose inverse is $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$.

(40,27,33) False.

Comment: If there is a matrix A such that $A^{-1} = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$, then A would be the inverse of $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$. But $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$ cannot have an inverse because its columns are linearly dependent:

$$\begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

(23) If $A^{-1} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

Hard (47,20,33) True.

Comment: This should have been trivial: $B = EA^{-1}$ is a solution! I wish I knew what made this question difficult. You would be doing me and others a favor if you sent a short email telling me why it was difficult.

(24) If I is the identity matrix the equation $I\underline{x} = \underline{b}$ has a unique solution.

Medium (69,2,29) True.

Comment: This should have been trivial: $\underline{x} = \underline{b}$ is the only solution because the equation really is just $\underline{x} = \underline{b}$.

(25) If A is a non-singular matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.

Hard (53,11,36) True.

Comment: This is an important theorem that we stated and proved in class. You need to know it. You should also know that A being non-singular is the same thing as A having an inverse and then $\underline{x} = A^{-1}\underline{b}$ is the only solution.

(26) If A is an invertible matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.

Hard (44,16,40) True.

Comment: See the comment to the previous question. Questions 25 and 26 are essentially the same question.

(27) If A is a singular matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.

Hard (58,4,38) False.

(28) If the row-reduced echelon form of A is the identity matrix, then the equation $A\underline{x} = \underline{b}$ has a unique solution.

(69,2,29) True.

(29) If A is row-equivalent to the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

then the equation $A\underline{x} = \underline{b}$ has a unique solution.

(53,9,38) True.

(30) There is a 7×7 matrix that has no inverse.

(76,4,20) True. The zero matrix has no inverse.

(31) There is a 7×7 matrix that has a unique inverse.

(42,27,31) True. Every matrix that has an inverse has a unique inverse.

(32) There is a 7×7 matrix that has two inverses.

(69,2,29) False. Every matrix that has an inverse has a unique inverse. Interesting to compare the number of correct answers to this question with those to the previous two questions.

(33) Let A and B be $n \times n$ matrices. If $AB = I$, then $BA = I$.

(42,27,31) True. We proved this important result in class.

(34) There exists a 2×3 matrix A and a 3×2 matrix B such that AB is the 2×2 identity matrix.

(49,13,38) True. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In questions (35)-(40), A is a 4×4 matrix whose columns $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ have the property that $\underline{A}_1 + 2\underline{A}_2 = 3\underline{A}_3 + 4\underline{A}_4$.

(35) The columns of A span $\mathbb{R}^4$.

(24,31,44) False. The question tells us that the columns of A are linearly dependent:

$$\underline{A}_1 + 2\underline{A}_2 - 3\underline{A}_3 - 4\underline{A}_4 = 0 \tag{1}$$

so A is singular. It therefore has no inverse. But A has an inverse if and only if the equation $A\underline{x} = \underline{b}$ has a solution for all $\underline{b} \in \mathbb{R}^4$. We therefore conclude that there is some $\underline{b} \in \mathbb{R}^4$ for which $A\underline{x} = \underline{b}$ does not have a solution. The fact that it has no solution tells us that $\underline{b}$ is not in the linear span of the columns of A. Hence the columns of A do not span $\mathbb{R}^4$.

(36) A is singular.

(55,7,38) True . A is non singular if and only if its columns are linearly dependent and equation (1) shows us that the columns of A are linearly dependent.

(37) The columns of A are linearly dependent.

(53,11,36) True . Equation (1) shows us that the columns of A are linearly dependent.

(38) The rows of A are linearly dependent.

(36,7,58) True .The rows of A are linearly dependent if and only if the columns are because A has an inverse if and only if A^T does.

(39) The equation $A\underline{x} = 0$ has a non-trivial solution.

(56,2,42) True. See the next question.

(40) $A \begin{pmatrix} 1 \\ 2 \\ -3 \\ -4 \end{pmatrix} = 0.$

(33,13,53) True. That is exactly what equation (1) is telling us because for any matrix A, $A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n$.