Chapter 12: Linear Transformations

doesn't correspond to Paul's notes ← (EXPECTED TO READ)

Outline

§12.1 Definition / Examples / Properties of Linear Transformations
§12.2 Linear Transformations & matrices
(§12.3) Inverses & Isomorphisms (if time permits)

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§1. **Def'n** Let V be a subspace of $\mathbb{R}^n$
W be a subspace of $\mathbb{R}^m$

A linear transformation from V to W is a function $F: V \to W$ s.t.

① $F(\underline{u}+\underline{v}) = F(\underline{u}) + F(\underline{v})$ for all $\underline{u}, \underline{v} \in V$ and

② $F(a\underline{v}) = aF(\underline{v})$ for all $\underline{v} \in V$, $\in \mathbb{R}$.

— OR —

F is a linear transf if

(1+2) $F(a\underline{u}+b\underline{v}) = aF(\underline{u}) + bF(\underline{v})$
for all $\underline{u}, \underline{v} \in V$; $a, b \in \mathbb{R}$.

Def'n Two linear transformations F, G from V to W are equal
if $F(\underline{v}) = G(\underline{v})$ for all $\underline{v} \in V$

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Examples

① The zero transformation: "F_{zero}"

$$F_{zero}: V \to W$$
$$\underline{v}' \mapsto \underline{0} \quad \forall \underline{v}' \in V$$

Proof: Let $\underline{u}, \underline{v} \in V$ and $a, b \in \mathbb{R}$. Then

$F_{zero}(a\underline{u} + b\underline{v}) = \underline{0}$ because $a\underline{u} + b\underline{v}$ is a vector in V & F_{zero} sends all vectors in V to 0

On the other hand,

$$aF_{zero}(\underline{u}) + bF_{zero}(\underline{v}) = a\underline{0} + b\underline{0} = \underline{0} + \underline{0} = 0$$

Therefore

$$F_{zero}(a\underline{u} + b\underline{v}) = \underline{0} = aF_{zero}(\underline{u}) + bF_{zero}(\underline{v}).$$

so F_{zero} satisfies the definition of a linear transformation. ///

(2) The identity transformation: $F_{id} : V \to V$, $\underline{v}' \mapsto \underline{v}'$ $\forall v' \in V$

Check at home (using the defn !!) that F_{id} is a linear transformation.

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(3) Projection $F : V \longrightarrow W$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here V subspace of $\mathbb{R}^3$
W subspace of $\mathbb{R}^2$

we'll see more about this later geometrically.

To check that F is a linear transformation it's just algebra. (use the defn!)

Pf/ For $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \in V$ and $a, b \in \mathbb{R}$:

~~F(a~~

$$F\left(a\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+b\begin{bmatrix}x_1'\\x_2'\\x_3'\end{bmatrix}\right)=F\left(\begin{bmatrix}ax_1\\ax_2\\ax_3\end{bmatrix}+\begin{bmatrix}bx_1'\\bx_2'\\bx_3'\end{bmatrix}\right)$$

$$=F\left(\begin{bmatrix}ax_1+bx_1'\\ax_2+bx_2'\\ax_3+bx_3'\end{bmatrix}\right)=\begin{bmatrix}ax_1+bx_1'\\ax_2+ax_2'\end{bmatrix}$$

On the other hand,

$$aF\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)+bF\left(\begin{bmatrix}x_1'\\x_2'\\x_3'\end{bmatrix}\right)=a\begin{bmatrix}x_1\\x_2\end{bmatrix}+b\begin{bmatrix}x_1'\\x_2'\end{bmatrix}$$

$$=\begin{bmatrix}ax_1+bx_1'\\ax_2+bx_2'\end{bmatrix}$$

So $F\left(a\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+b\begin{bmatrix}x_1'\\x_2'\\x_3'\end{bmatrix}\right)=aF\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)+bF\left(\begin{bmatrix}x_1'\\x_2'\\x_3'\end{bmatrix}\right)$ ///.

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Non-Example

On notation?

① $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ [Here $V=W=\mathbb{R}^2$]

just place holders $\begin{bmatrix}x_1\\x_2\end{bmatrix}\longmapsto\begin{bmatrix}x_1^2\\x_1+x_2\end{bmatrix}$ square the first entry; add the two entrys

is not a linear transformation

Pf/. For $a,b\in\mathbb{R}$ and $\begin{bmatrix}x_1\\x_2\end{bmatrix},\begin{bmatrix}x_1'\\x_2'\end{bmatrix}\in\mathbb{R}^2$,

square the first entry

$$F\left(a\begin{bmatrix}x_1\\x_2\end{bmatrix}+b\begin{bmatrix}x_1'\\x_2'\end{bmatrix}\right)=F\left(\begin{bmatrix}ax_1+bx_1'\\ax_2+bx_2'\end{bmatrix}\right)=\begin{bmatrix}(ax_1+bx_1')^2\\(ax_1+bx_1')+(ax_2+bx_2')\end{bmatrix}$$

add the two entrys.

On the other hand

$$aF\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)+bF\left(\begin{bmatrix}x_1'\\x_2'\end{bmatrix}\right)=a\begin{bmatrix}x_1^2\\x_1+x_2\end{bmatrix}+b\begin{bmatrix}x_1'^2\\x_1'+x_2'\end{bmatrix}$$

$$=\begin{bmatrix}ax_1^2+bx_1'^2\\a(x_1+x_2)+b(x_1'+x_2')\end{bmatrix}$$

but the first entries of the images

$$(ax_1+bx_1')^2 \quad \text{and} \quad ax_1^2+bx_1'^2$$

are NOT equal.

so F is not a linear transformation. ///

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Note on notation

Linear transformations can be presented in a variety of ways.

① $F: V \longrightarrow W$ means $F(\underline{v}) = \underline{w}$
$\underline{v} \longmapsto \underline{w}$

② $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ means $F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1+x_2 \end{bmatrix}$
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1^2 \\ x_1+x_2 \end{bmatrix}$

using "maps-to" notation

using "formula" notation.

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... using matrices later....

Before that

Properties of linear transformations

Say $F: V \to W$ is a linear transformation

⓪ $F(\underline{0}) = \underline{0}$

① Respects linear combinations

If $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n \in V$ and $a_1, \ldots, a_n \in \mathbb{R}$, then

$$F(a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_n\underline{v}_n) = a_1F(\underline{v}_1) + a_2F(\underline{v}_2) + \cdots + a_nF(\underline{v}_n).$$

why? Induction (see Paul's notes; base case is 1).
($n=2$ is the definition of linear transf).

② Determined by action on basis

If $\{u_1, \ldots, u_n\}$ is a basis of V, and we know $F(\underline{u}_1), \ldots, F(\underline{u}_n)$, then we know $F(\underline{v})$ for all $\underline{v} \in V$.

Why? Since $\{\underline{u}_1, \ldots, \underline{u}_n\}$ is a basis of V,
$\underline{v} = a_1\underline{u}_1 + \cdots + a_n\underline{u}_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$.

Now
$$F(\underline{v}) = F(a_1\underline{u}_1 + \cdots + a_n\underline{u}_n)$$
$$= a_1\underbrace{F(\underline{u}_1)}_{\text{known}} + \cdots + a_n\underbrace{F(\underline{u}_n)}_{\text{known by assumption}} \quad \text{by Property ①}$$

So $F(\underline{v})$ is known.

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③ Composition of linear transformations is a linear transformation.
meaning
if $F: V \to W$ and $G: W \to X$ are linear transformations
then $(G \circ F): V \to W \to X$ is a linear transformation
(read proof in Paul's notes).

④ Linear combinations of lin. transfs are linear transformations

meaning –

if $F: V \to W$ and $G: V \to W$ are linear transformations then for all $a, b \in \mathbb{R}$

$$(aF + bG): V \to W$$
$$\underline{v} \mapsto (aF + bG)(\underline{v}) = aF(\underline{v}) + bG(\underline{v})$$

is a linear transformation.

Again read proof online.

(1:02)

$\boxed{\S 2}$ Linear transformation and Matrices

Here take $V = \mathbb{R}^n$ $\quad W = \mathbb{R}^m$ $\quad F: \mathbb{R}^n \to \mathbb{R}^m$. linear transformation.

Can ~~write~~ represent F as ~~as~~ a unique $m \times n$ matrix...

Theorem: Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a unique $m \times n$ matrix A so that

$$T(\underline{v}) = A \cdot \underline{v} \quad \text{for all } \underline{v} \in \mathbb{R}^n.$$

Read proof online!

Def'n Say that A is the matrix that represent the linear transformation F.

(1:06)

Examples

(1) $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \left(\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}\right)$

is a linear transformation

Need a 2×3 matrix A so that $F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} \overset{\text{need}}{=} \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

Can read off A:

$a_{11} = 1 \qquad a_{12} = -1 \qquad a_{13} = 0$

$a_{21} = 1 \qquad a_{22} = 1 \qquad a_{23} = 0$

So $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ the matrix corresponding to the linear transformation F

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Properties (corresponding to properties of linear transformations)

~~(3) Say linear transformations F, G correspond to matrices A, B~~

~~(3) Composition $G \circ F : V \to X$ corresponds to matrix multip~~

Properties (corresponding to properties of linear transformations)

Say linear transformations $\begin{cases} F, F' : \mathbb{R}^n \to \mathbb{R}^m \\ G : \mathbb{R}^m \to \mathbb{R}^p \end{cases}$

correspond to matrices $\begin{cases} A, A' \in Mat_{m \times n}(\mathbb{R}) \\ B \in Mat_{p \times m}(\mathbb{R}) \end{cases}$

Then

③ Composition $G \circ F : \mathbb{R}^n \to \mathbb{R}^m \to \mathbb{R}^p$

corresponds to

matrix multiplication $\underset{p \times m}{B} \cdot \underset{m \times n}{A} \in Mat_{p \times n}(\mathbb{R})$

④ Linear combination (for $a, b \in \mathbb{R}$:)

$aF + bF' : \mathbb{R}^n \to \mathbb{R}^m$

corresponds to

linear combination of matrices $aA + bA' \in Mat_{m \times n}(\mathbb{R})$.

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Next time:

- (Invertible linear transformations & matrices; isomorphisms)
- Rotations
- Orthogonal projections
- Invariant subspaces
- one to one & onto

Lots to do

If time permits

§3 Inverses & Isomorphisms

Defn Suppose $F: V \to W$ and $G: W \to V$ are linear transformations.

F and G are ~~a~~ inverses to one another if

① $F \circ G : W \to W$ is the identity id_W, and

② $G \circ F = id_V$

In this case F and G are isomorphisms and U and V are isomorphic vector spaces. (*)

with matrices ...

Defn Say F corresponds to matrix A and

G " " " B

with $AB = BA =$ the identity matrix

then (*) above applies.