November 21, 2011 (1230)

Example of computing a basis for a vector space....

* Recall that a basis $\{\underline{u}_1, \underline{u}_2, \dots \underline{u}_r\}$ for a vector space V is a spanning set for V that is linearly independent.

for all $\underline{v} \in V$: there are scalars $a_1, \dots, a_r \in \mathbb{R}$ so that $\underline{v} = a_1 \underline{u}_1 + \dots + a_r \underline{u}_r$

If $a_1 \underline{u}_1 + a_2 \underline{u}_2 + \dots + a_r \underline{u}_r = \underline{0}$ for some scalars $a_1, \dots, a_r \in \mathbb{R}$, then $a_1 = a_2 = \dots = a_r = 0$

* There may be many different bases for V, but the # of elements in the basis remains the same.

Example There are variety of ways to describe a vector space; let's compute the basis for a solution space V of say $\mathbb{R}^4$ with coordinates x, y, z, w.

Say V is defined by $\begin{cases} x + 2y + 3w = 0 \\ y + 2z + w = 0 \end{cases}$ (★)

In other words, for all vectors $\underline{v} = [x, y, z, w]^T$ in V: the coordinates satisfy (★). V is a vector space (check at home). To compute a basis for V:

① Write (★) in terms of matrix multiplication: $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

A = coeff. matrix

② Row reduce coefficient matrix: $A \sim \begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$

③ ... to get equivalent system of eqn's: $\begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} x = 4z - w \\ y = -2z - w \end{cases}$ (★) (reduced)

④ Identify free (indep) variables & write rest of variables (dep) in terms of the free ones: (1245)

free: z and w the rest: $x = 4z - w$ and $y = -2z - w$

⑤ Write general solution and factor to get basis

general: $\begin{pmatrix} 4z - w \\ -2z - w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow \boxed{\text{Basis of } V = \left\{ \underline{u}_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \underline{u}_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}}$

Note that $2 = \#$ free variables

& all bases for V will have 2 elts

Back to linear transformations —

Recall that a linear transformation from V to W is a function $F: V \to W$ so that $F(a\underline{u}+b\underline{v}) = aF(\underline{u}) + bF(\underline{v})$ for all $a, b \in \mathbb{R}$ and $\underline{u}, \underline{v} \in V$.

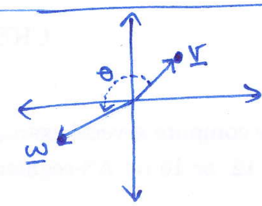
... can be given by a matrix A with $F(\underline{v}) = A\underline{v}$ ....

Today

① Rotations in the plane ② Orthogonal Projections ← (Important Examples)

③ Invariant subspaces ④ One-to-one / onto ← (Important terminology).

① Rotation in the plane $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



To say: how do we define F (or a 2×2 matrix A) so that $\underline{v}$ gets rotated by angle θ to $\underline{w}$?? COUNTER CLOCKWISE

Call such an lin. transf $\boxed{F_\theta}$.

What we should expect:

ⓐ $F_{\theta=0}$ is the identity transformation

ⓑ $F_{2\pi k}$ is also the identity transformation ; k integer

ⓒ $F_\theta \circ F_\psi = F_{\theta+\pi}$

ⓓ Rotation of θ in the clockwise direction should be ; $F_{-\theta}$

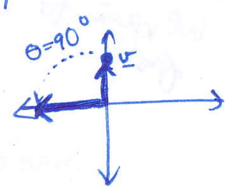
Solution F_θ is represented by matrix $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Let's check some examples:

ⓐ Take $\underline{v} = [v_1; v_2]^T$ in $\mathbb{R}^2$. $F_{\theta=0}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ should be $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$:

$$F_{\theta=0}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \checkmark\checkmark$$

ⓑ Take $\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. $F_{\cancel{90}°=\frac{\pi}{2}}$ should send $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$:



$$F_{\frac{\pi}{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \checkmark\checkmark$$

* Note that the determinant of $A_\theta = 1$ as $\cos^2\theta + \sin^2\theta = 1$

* You can recover formulas for $\underbrace{\sin(\theta+\psi) \quad \cos(\theta+\psi)}$, $\underbrace{\sin(-\theta) \text{ and } \cos(-\theta)}$

· by using the fact that $A_\theta A_\psi = A_{\theta+\psi}$

· $A_\theta^{-1} = A_{-\theta}$ and the formula to compute inverses of 2×2 matrices

§2 Orthogonal Projections Let V be a subspace of $\mathbb{R}^n$, the orthogonal to V is defined as:

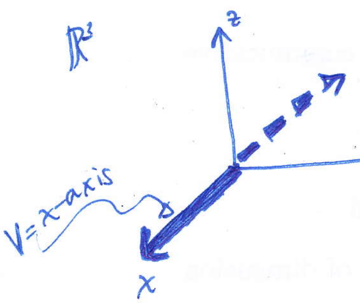
$$V^\perp = \{ \underline{w} \in \mathbb{R}^n \mid \underline{w}^T \underline{v} = 0 \quad \text{for } \underline{v} \in V \}$$ (sometimes called "V perp")

To say $V^\perp$ consists of all vectors that are perpendicular to every vector of V.

** $V^\perp$ is a subspace of $\mathbb{R}^n$ ** (check at home)

Example Let V be the 1-dimensional subspace of $\mathbb{R}^3$, the x-axis.

In other words $V = \{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \mid x \in \mathbb{R}^3 \}$ or is defined as the solution space of $\begin{cases} y = 0 \\ z = 0 \end{cases}$



Now:

$$V^\perp = \{ \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \in \mathbb{R}^3 \mid [x' \ y' \ z'] \cdot \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = 0 \}$$

$$\Updownarrow$$

$x \cdot x' = 0$ but x could be nonzero $\Rightarrow x' = 0$

$$\therefore V^\perp = \{ \begin{bmatrix} 0 \\ y' \\ z' \end{bmatrix} \in \mathbb{R}^3 \mid y', z' \in \mathbb{R} \}$$

the y'-z' plane (Here the " ' " don't matter just use these to distinguish between $\underline{v}$ & $\underline{w}$)

We really didn't need to go through the algebra of finding $V^\perp$ as geometrically we see that the set of vectors in the y-z plane of $\mathbb{R}^3$ are precisely those perpendicular to the x-axis.

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§3 Invariant Subspaces

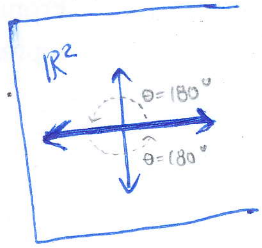
$U \subset V$
$F: V \to V$

Def'n When given a linear transformation F from V to itself, an invariant subspace U of V under F is a U so that $F(U) \subset U$.

To say U is invariant if for all $\underline{u} \in U$, we have that $F(\underline{u}) \in U$.

Example Consider $F_{\theta = \pi} : \mathbb{R}^2 \to \mathbb{R}^2$ rotation by 180° counterclockwise.

Check that the subspace U spanned by $\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$ (the x-axis) is an invariant subspace of $\mathbb{R}^2$.



Pf/ a general elt of U is $\begin{bmatrix} a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for some $a \in \mathbb{R}$.

Now $F_\pi \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \end{bmatrix} = -a \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in U$. ///

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§4 One-to-one and onto:

Adjectives for functions in general
(Recall that a linear transf. is an example of a function)

Def'n A function from a set X to a set Y $\qquad f: X \to Y$

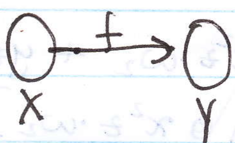
① is <u>one-to-one</u> if $f(x) = f(x')$ only when $x = x'$

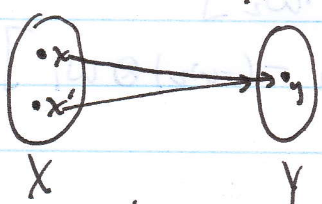
—OR—

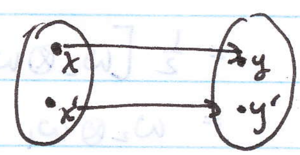
if $x \neq x'$ forces $f(x) \neq f(x')$.

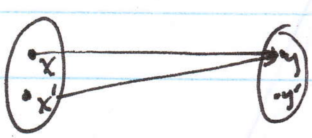
② is <u>onto</u> if every element $y \in Y$ equals $f(x)$ for some $x \in X$.
In other words,
The <u>range of f</u> is the set $\{y \in Y \mid f(x) = y \text{ for some } x \in X\}$
So f is onto if Range of $f = Y$

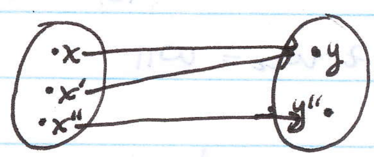
<u>Diagrams</u>


representing $f: X \to Y$


<u>not</u> one-to-one


one-to-one


<u>not</u> onto


<u>onto</u>

<u>Explicit Examples</u>

$f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is <u>not</u> one-to-one because $2 \neq -2$ but $f(2) = 4 = f(-2)$
is <u>not</u> onto because nothing gets mapped to -5

Same principles apply to linear transformations —