Math 308 **Pop Quiz No. 1**

Complete the sentence/definition:

(1) If a system of linear equations has ≥ 2 solutions, then _________ solutions.

(2) A system of linear equations can have either
 (a) a _________ solution, or
 (b) _________ solutions, or
 (c) _________ solutions.

(3) An $m \times n$ system of linear equations consists of
 (a) ___ linear equations
 (b) in ___ unknowns.

(4) If an $m \times n$ system of linear equations is converted into a single matrix equation $A\underline{x} = \underline{b}$, then
 (a) A is a ___$\times$ ___ matrix
 (b) $\underline{x}$ is a ___$\times$ ___ matrix
 (c) $\underline{b}$ is a ___$\times$ ___ matrix

(5) A system of linear equations is <u>consistent</u> if _______________

(6) A system of linear equations is <u>inconsistent</u> if ______________

(7) An $m \times n$ matrix has
 (a) ___ columns and
 (b) ___ rows.

(8) Let A and B be sets. Then
 (a) $\{x \mid x \in A \text{ or } x \in B\}$ is called the ________ of A and B;
 (b) $\{x \mid x \in A \text{ and } x \in B\}$ is called the ________ of A and B;

(9) Write down the symbols for the integers, the rational numbers, the real numbers, and the complex numbers, and indicate by using the notation for *<u>subset</u>* which is contained in which.

(10) Let E be set of all even numbers, O the set of all odd numbers, and T the set of all integer multiples of 3. Then
 (a) $E \cap O =$________ (use a single symbol for your answer)
 (b) $E \cup O =$________ (use a single symbol for your answer)
 (c) $E \cap T =$________ (use words for your answer)

Math 308 **Answers to Pop Quiz No. 1**

Complete the sentence/definition:

(1) If a system of linear equations has ≥ 2 solutions, then it has infinitely many solutions.

(2) A system of linear equations can have either
  (a) a unique solution, or
  (b) no solutions, or
  (c) infinitely many solutions.

(3) An $m \times n$ system of linear equations consists of
  (a) m linear equations
  (b) in n unknowns.

(4) If an $m \times n$ system of linear equations is converted into a single matrix equation $A\underline{x} = \underline{b}$, then
  (a) A is an $m \times n$ matrix
  (b) $\underline{x}$ is a $n \times 1$ matrix
  (c) $\underline{b}$ is a $m \times 1$ matrix

(5) A system of linear equations is consistent if it has a solution

(6) A system of linear equations is inconsistent if does not have a solution

(7) An $m \times n$ matrix has
  (a) n columns and
  (b) m rows.

(8) Let A and B be sets. Then
  (a) $\{x \mid x \in A \text{ or } x \in B\}$ is called the union of A and B;
  (b) $\{x \mid x \in A \text{ and } x \in B\}$ is called the intersection of A and B;

(9) Write down the symbols for the integers, the rational numbers, the real numbers, and the complex numbers, and indicate by using the notation for *subset* which is contained in which.

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

(10) Let E be set of all even numbers, O the set of all odd numbers, and T the set of all integer multiples of 3. Then
  (a) $E \cap O = \phi$
  (b) $E \cup O = \mathbb{Z}$
  (c) $E \cap T =$ all integer multiples of 6.

Math 308 **Pop Quiz No. 2**

(1) (a) the intersection of two sets A and B is denoted by _________
(b) the union of two sets A and B is denoted by _________

(2) Using the notation for intersection and the set notation $\{\cdots \mid \cdots\}$ write out "A is the set of all integers that are divisible by both 3 and 5". Your answer should begin
$$A = \{\cdots .$$

(3) Using the notation for union and the set notation $\{\cdots \mid \cdots\}$ write out "B is the set of all integers that are divisible by either 3 or 5". Your answer should begin
$$B = \{\cdots .$$

(4) Describe in geometric language the set $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Your answer should begin "A is the ..."

(5) The system of equations $A\underline{x} = \underline{b}$ is homogeneous if _________

(6) I want a geometric description of the solutions: the set of solutions to a 2×3 system of linear equations is either
(a) _________
(b) or _________
(c) or _________
(d) or _________

(7) The set of solutions to a 3×3 system of linear equations is one of the four possibilities in the previous answer or _________.

(8) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form _________ where _________

(9) In the previous question let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. If $A\underline{u} = \underline{b}$, then
$$T = \{\ldots \mid \ldots\}.$$
Your answer should involve $\underline{u}$ and S and the symbol $\in$ and some more.

(10) We only apply the words *singular* and *non-singular* to matrices that are _________

(11) A matrix A is singular if _________

(12) A matrix A is non-singular if _________

(13) A matrix is non-singular if and only if it is _________

Math 308 **Answers to Pop Quiz No. 2**

(1) (a) the intersection of two sets A and B is denoted by $\underline{A \cap B}$. (The symbol $\cap$ is *not* the letter n.)
(b) the union of two sets A and B is denoted by $\underline{A \cup B}$. (The symbol $\cup$ is *not* the letter u.)

(2) Using the notation for intersection and the set notation $\{\cdots | \cdots\}$ write out "A is the set of all integers that are divisible by both 3 and 5". Your answer should begin

$$A = \{n \in \mathbb{Z} \mid 3 \text{ divides } n\} \cap \{n \in \mathbb{Z} \mid 5 \text{ divides } n\}.$$

(3) Using the notation for union and the set notation $\{\cdots | \cdots\}$ write out "B is the set of all integers that are divisible by either 3 or 5". Your answer should begin

$$B = \{n \in \mathbb{Z} \mid 3 \text{ divides } n\} \cup \{n \in \mathbb{Z} \mid 5 \text{ divides } n\}$$

(4) Describe in geometric language the set $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. A is the circle of radius one centered at the origin.

(5) The system of equations $A\underline{x} = \underline{b}$ is homogeneous if $\underline{\underline{b} = \underline{0}}$.

(6) I want a geometric description of the solutions: the set of solutions to a 2×3 system of linear equations is either
(a) the empty set ϕ,
(b) or a line,
(c) or a plane
(d) or all of $\mathbb{R}^3$.

(7) The set of solutions to a 3×3 system of linear equations is one of the four possibilities in the previous answer or a point.

(8) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form $\underline{\underline{u} + \underline{v}}$ where $\underline{v}$ is a solution to $A\underline{x} = \underline{0}$ OR you could just say $\underline{A\underline{v} = \underline{0}}$.

(9) In the previous question let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. If $A\underline{u} = \underline{b}$, then

$$T = \{\underline{u} + \underline{v} \mid \underline{v} \in S\}.$$

(10) We only apply the words *singular* and *non-singular* to matrices that are square.

(11) A matrix A is singular if $A\underline{x} = \underline{0}$ has a non-trivial solution.

(12) A matrix A is non-singular if the only solution to $A\underline{x} = \underline{0}$ is the trivial solution.

(13) A matrix is non-singular if and only if it is invertible.

Math 308 **Pop Quiz No. 2.5**

(1) The system of equations $A\underline{x} = \underline{b}$ is homogeneous if _________
(2) I want a geometric description of the solutions: the set of solutions to a 2×3 system of linear equations is either
 (a) _________
 (b) or _________
 (c) or _________
 (d) or _________
(3) The set of solutions to a 3×3 system of linear equations is one of the four possibilities in the previous answer or _________.
(4) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form _________ where _________
(5) In the previous question let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. If $A\underline{u} = \underline{b}$, then

$$T = \{\ldots \mid \ldots\}.$$

Your answer should involve $\underline{u}$ and S and the symbol $\in$ and some more.
(6) A matrix E is in echelon form if
 (a) _________
 (b) _________
 (c) _________
(7) A matrix is in row reduced echelon form if it is _________ and _________ .
(8) In this question use R_i to denote the i^{th} row of a matrix. The three elementary row operations are
 (a) _________
 (b) _________
 (c) _________
(9) If $(A|\underline{b})$ is the augmented matrix for an $m \times n$ system of linear equations and is in reduced echelon form, then the system of linear equations is inconsistent if _________
(10) The system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

is homogeneous if _________
(11) True or False: The echelon form of a matrix is unique.
(12) True or False: The row reduced echelon form of a matrix is unique.
(13) True or False: Given a matrix A there is a unique matrix in echelon form that is equivalent to A.
(14) True or False: Given a matrix A there is a unique matrix in reduced echelon form that is equivalent to A.
(15) A solution to a homogeneous system of equations is non-trivial if _________

Math 308 **Answers to Pop Quiz No. 2.5**

(1) The system of equations $A\underline{x} = \underline{b}$ is homogeneous if $\underline{b} = \underline{0}$.
(2) I want a geometric description of the solutions: the set of solutions to a 2×3 system of linear equations is either
 (a) the empty set ϕ,
 (b) or a line,
 (c) or a plane
 (d) or all of $\mathbb{R}^3$.
(3) The set of solutions to a 3×3 system of linear equations is one of the four possibilities in the previous answer or a point.
(4) Let $\underline{u}$ be a solution to the equation $A\underline{x} = \underline{b}$. Then every solution to $A\underline{x} = \underline{b}$ is of the form $\underline{u} + \underline{v}$ where $\underline{v}$ is a solution to $A\underline{x} = \underline{0}$ OR you could just say $A\underline{v} = \underline{0}$.
(5) In the previous question let S denote the set of solutions to the equation $A\underline{x} = \underline{0}$ and T the set of solutions to the equation $A\underline{x} = \underline{b}$. If $A\underline{u} = \underline{b}$, then

$$T = \{\underline{u} + \underline{v} \mid \underline{v} \in S\}.$$

(6) A matrix is in echelon form if
 (a) all the rows of zeros are at the bottom of E, and
 (b) the left-most non-zero entry in each non-zero row is 1 (we call it a leading 1), and
 (c) every leading 1 is to the right of the leading 1s in the rows above it.
(7) A matrix is in row reduced echelon form if it is in echelon form and all other entries in a column that contains a leading 1 are zero.
(8) In this question use R_i to denote the i^{th} row of a matrix. The three elementary row operations are
 (a) swap R_i with R_j;
 (b) multiply R_i by a non-zero number;
 (c) replace R_i by $R_i + R_j$
(9) If $(A|\underline{b})$ is the augmented matrix for an $m \times n$ system of linear equations and is in reduced echelon form, then the system of linear equations is inconsistent if there is a row with all entries 0 except the right-most entry.
(10) The system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

 is homogeneous if $b_1 = b_2 = \cdots = b_m = 0$.
(11) True or False: The echelon form of a matrix is unique. FALSE
(12) True or False: The row reduced echelon form of a matrix is unique. TRUE
(13) True or False: Given a matrix A there is a unique matrix in echelon form that is equivalent to A. FALSE
(14) True or False: Given a matrix A there is a unique matrix in row reduced echelon form that is equivalent to A. TRUE
(15) A solution to a homogeneous system of equations is non-trivial if it is non-zero.

Math 308 **Pop Quiz No. 3**

Complete the sentence/definition:

(1) Two systems of linear equations are equivalent if ________

(2) A matrix E is in echelon form if
 (a) ________
 (b) ________
 (c) ________

(3) A matrix is in row reduced echelon form if it is ________ and ________ .

(4) In this question use R_i to denote the i^{th} row of a matrix. The three elementary row operations are
 (a) ________
 (b) ________
 (c) ________

(5) If $(A|\underline{b})$ is the augmented matrix for an $m \times n$ system of linear equations and is in reduced echelon form, then the system of linear equations is inconsistent if ________

(6) The system of linear equations

$$\begin{aligned} a_{11}x_1 \,+\, a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 \,+\, a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

is homogeneous if ________

(7) True or False: The echelon form of a matrix is unique.

(8) True or False: The row reduced echelon form of a matrix is unique.

(9) True or False: Given a matrix A there is a unique matrix in echelon form that is equivalent to A.

(10) True or False: Given a matrix A there is a unique matrix in reduced echelon form that is equivalent to A.

(11) A solution to a homogeneous system of equations is non-trivial if ________

(12) A system of linear equations is homogeneous if it is of the form ________

(13) A matrix is invertible if and only if it is ________

Math 308 **Answers to Pop Quiz No. 3**

Complete the sentence/definition:

(1) Two systems of linear equations are equivalent if they have the same solutions.

(2) A matrix is in echelon form if

(a) all the rows of zeros are at the bottom of E, and

(b) the left-most non-zero entry in each non-zero row is 1 (we call it a leading 1), and

(c) every leading 1 is to the right of the leading 1s in the rows above it.

(3) A matrix is in row reduced echelon form if it is in echelon form and all other entries in a column that contains a leading 1 are zero.

(4) In this question use R_i to denote the i^{th} row of a matrix. The three elementary row operations are

(a) swap R_i with R_j;

(b) multiply R_i by a non-zero number;

(c) replace R_i by $R_i + R_j$

(5) If $(A|\underline{b})$ is the augmented matrix for an $m \times n$ system of linear equations and is in reduced echelon form, then the system of linear equations is inconsistent if there is a row with all entries 0 except the right-most entry.

(6) The system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

is homogeneous if $b_1 = b_2 = \cdots = b_m = 0$.

(7) True or False: The echelon form of a matrix is unique. FALSE

(8) True or False: The row reduced echelon form of a matrix is unique. TRUE

(9) True or False: Given a matrix A there is a unique matrix in echelon form that is equivalent to A. FALSE

(10) True or False: Given a matrix A there is a unique matrix in row reduced echelon form that is equivalent to A. TRUE

(11) A solution to a homogeneous system of equations is non-trivial if it is non-zero.

(12) A system of linear equations is homogeneous if it is of the form $A\underline{x} = \underline{0}$.

(13) A matrix is invertible if and only if it is non-singular.

Math 308 **Pop Quiz No. 4**

If A is an $m \times n$ matrix denote its columns by $\underline{A}_1, \underline{A}_2, \dots, \underline{A}_n$. If $\underline{x}$ is an $n \times 1$ matrix write it as $\underline{x} = (x_1, \dots, x_n)^T$.

Complete the sentence/definition:

(1) An $n \times n$ matrix A is non-singular if the only _________.
(2) A vector $\underline{w}$ is a <u>linear combination</u> of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if there are _______________.
(3) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a _______________.
(4) The <u>linear span</u> of $\{\underline{v}_1, \dots, \underline{v}_n\}$ consists of _______________.
(5) Express $A\underline{x}$ as a linear combination of the columns of A.
(6) The rank of a matrix is the number _________.
(7) The rank of an $m \times n$ matrix is $\leq$ _________.
(8) Let A be an $m \times n$ matrix having rank r. If $r = n$, then the equation $A\underline{x} = \underline{b}$ has _________.
(9) $\mathbb{R}^n$ denotes the set of all _________.
(10) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is <u>linearly dependent</u> if the equation _______________ has a solution such that _______________
(11) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is <u>linearly independent</u> if the only solution to_______________
(12) Write down 3 linearly independent vectors in $\mathbb{R}^4$.
(13) The independent variables in the system of linear equations

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 3 & 4 \\ 0 & 1 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

are ________.
(14) In the previous question, express one of the dependent variables in terms of the independent variables.
(15) **True or False:**
   (a) $\{(1,2),(2,3)\}$ is linearly dependent.
   (b) Every vector in $\mathbb{R}^2$ is a linear combination of $\{(1,2),(2,3)\}$.
   (c) A matrix is row equivalent to a unique row reduced echelon matrix.
   (d) An $m \times n$ system of homogeneous equations has a unique solution if $m < n$.
   (e) Professor Smith can find 5 linearly independent vectors in $\mathbb{R}^4$.

Math 308 **Answers to Pop Quiz No. 3.5**

(1) If a system of equations has more equations than unknowns it has no solutions.
(2) If a system of homogeneous equations has more equations than unknowns it has no solutions.
(3) If a system of equations has fewer equations than unknowns it always has a solution.
(4) If a system of homogeneous equations has fewer equations than unknowns it always has a solution.

Math 308 **Answers to Pop Quiz No. 4**

If A is an $m \times n$ matrix denote its columns by $\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_n$. If $\underline{x}$ is an $n \times 1$ matrix write it as $\underline{x} = (x_1, \ldots, x_n)^T$.

Complete the sentence/definition:

(1) An $n \times n$ matrix A is non-singular if the only solution to the equation $A\underline{x} = 0$ is $\underline{x} = 0$.
(2) A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \ldots, \underline{v}_n\}$ if there are numbers $a_1, \ldots, a_n$ such that $\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$.
(3) The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.
(4) The linear span of $\{\underline{v}_1, \ldots, \underline{v}_n\}$ consists of vectors of the form $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ where $a_1, \ldots, a_n$ range over all of $\mathbb{R}$.
(5) Express $A\underline{x}$ as a linear combination of the columns of A.

$$A\underline{x} = x_1\underline{A}_1 + \cdots + x_n\underline{A}_n.$$

(6) The rank of a matrix is the number non-zero rows in its row-reduced echelon form.

Comment: Later in the course we will also see that the rank of A is the dimension of the range of A; i.e., $\dim\{A\underline{x} \mid \underline{x} \in \mathbb{R}^n\}$. The range of A is also equal to the linear span of the columns of A so the rank of A is equal to the dimension of the linear span of the columns of A. We will also see that the rank of A is equal to the maximal number of linearly independent rows of A, and equal to the maximal number of linearly independent columns of A.

(7) The rank of an $m \times n$ matrix is $\leq$ both m and n.
(8) Let A be an $m \times n$ matrix having rank r. If $r = n$, then the equation $A\underline{x} = \underline{b}$ has a unique solution.
(9) $\mathbb{R}^n$ denotes the set of all $n \times 1$ matrices.

Comment: Sometimes one says that $\mathbb{R}^n$ consists of all n-tuples of real numbers. By an *n-tuple of real numbers* we simply mean an ordered sequence $(a_1, \ldots, a_n)$ where each $a_i \in \mathbb{R}$. This is consistent with saying that $\mathbb{R}^2$ consists of all ordered pairs (x, y) of real numbers, and that $\mathbb{R}^3$ consists of all ordered pairs (x, y, z) of real numbers.

Frequently when one is more familiar with linear algebra one often just writes $\mathbb{R}^n$ without specifying whether one means all $1 \times n$ matrices or all $n \times 1$ matrices.

(10) A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly dependent if the equation $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ has a solution such that at least one of the a_is is non-zero.
(11) A set of vectors $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly independent if the only solution to the equation $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0$ is $a_1 = \cdots = a_n = 0$.
(12) Write down 3 linearly independent vectors in $\mathbb{R}^4$.

There are lots of correct answers. A simple one is to take $\underline{e}_1 = (1, 0, 0, 0)$, $\underline{e}_2 = (0, 1, 0, 0)$, and $\underline{e}_3 = (0, 0, 1, 0)$.

(13) The independent variables in the system of linear equations

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 3 & 4 \\ 0 & 1 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

are $\underline{x_3,\ x_6, \text{ and } x_7}$.

(14) In the previous question, express one of the dependent variables in terms of the independent variables.

$$x_5 = 3 - x_6 + 2x_7.$$

(15) True or False:

(a) $\{(1, 2), (2, 3)\}$ is linearly dependent. **False**

(b) Every vector in $\mathbb{R}^2$ is a linear combination of $\{(1, 2), (2, 3)\}$. **True**

(c) A matrix is row equivalent to a unique row reduced echelon matrix. **True**

(d) An $m \times n$ system of homogeneous equations has a unique solution if $m < n$. **False**

(e) Professor Smith can find 5 linearly independent vectors in $\mathbb{R}^4$. **False**

Math 308 **Pop Quiz No. 5**

Throughout V is a vector space.

(1) A non-empty subset W of V is a subspace if and only if
  (a)
  (b)

(2) Is the empty set a vector space? Give reasons.

(3) $C(\mathbb{R})$ is the vector space consisting of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.
  (a) What is the definition of $+$ in $C(\mathbb{R})$?
  (b) What is the zero element in $C(\mathbb{R})$?
  (c) What is the definition of scalar multiplication in $C(\mathbb{R})$?

(4) A polynomial with real coefficients is a function from $\mathbb{R}$ to $\mathbb{R}$. It is also continuous so the set of polynomials with real coefficients is a subset of $C(\mathbb{R})$. Is this subset of $C(\mathbb{R})$ a subspace? Give reasons for your answer in terms of your answer to question 1.

(5) Suppose V is a vector space and $\underline{u}$, $\underline{v}$, and $\underline{w}$, are elements of V. We define

$$\mathbb{R}\underline{u} + \mathbb{R}\underline{v} + \mathbb{R}\underline{w} = \{a\underline{u} + b\underline{v} + c\underline{w} \mid a, b, c \in \mathbb{R}\}.$$

Is $\mathbb{R}\underline{u} + \mathbb{R}\underline{v} + \mathbb{R}\underline{w}$ a subspace of V? Give reasons for your answer in terms of your answer to question 1.

(6) We write $M_2(\mathbb{R})$ for the set of all 2×2 matrices with real entries. Let W be the subset of $M_2(\mathbb{R})$ consisting of the matrices whose diagonal entries add up to zero. Is W a subspace of V? Give reasons for your answer in terms of your answer to question 1.

(7) Let W be the subset of $M_2(\mathbb{R})$ consisting of the matrices whose determinant is zero. Is W a subspace of V? Give reasons for your answer in terms of your answer to question 1.

(8) Let W be the subset of $M_2(\mathbb{R})$ consisting of the matrices whose determinant is 1. Is W a subspace of V? Give reasons for your answer in terms of your answer to question 1.

(9) What are all the subspaces of $\mathbb{R}^2$?

(10) What are all the subspaces of $\mathbb{R}$?

(11) Which subspace of V is contained in every subspace of V?

(12) Which subspace of V contains every subspace of V?

(13) What is the connection between subspaces and solutions of systems of linear equations?

(14) Professor Smith can find 7 different subspaces of $\mathbb{R}^3$. True or False?

Math 308 **Answers to Pop Quiz No. 5**

(1) A non-empty subset W of V is a subspace if and only if
   (a) $\underline{u}+\underline{v}$ is in W for all $\underline{u}$ and $\underline{v}$ in W, and
   (b) $c\underline{u}$ is in W for all $c \in \mathbb{R}$ and $\underline{u} \in W$.

(2) Is the empty set a vector space? Give reasons.

   No, a vector space is required to be a non-empty set.

(3) $C(\mathbb{R})$ is the vector space consisting of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.
   (a) What is the definition of $+$ in $C(\mathbb{R})$?

   If f and g are in $C(\mathbb{R})$ we define their sum $f+g$ to be the function whose value at a point $x \in \mathbb{R}$ is $f(x)+g(x)$; in short, $(f+g)(x) = f(x)+g(x)$ for all $x \in \mathbb{R}$.
   The majority of people did not understand this question. Read it again—it asks for the definition of $+$ so you must say what $f+g$ is when f and g are elements in $C(\mathbb{R})$.

   (b) What is the zero element in $C(\mathbb{R})$?

   The zero element in $C(\mathbb{R})$ is the function that takes the value zero at all points of $\mathbb{R}$. It is usual to denote that function by the symbol zero; i.e., 0 is the function defined by $0(x) = 0$ for all $x \in \mathbb{R}$. If you just said the zero element is 0 you have not said enough—the 0 on the left of the equality $0(x) = 0$ is the *function* zero whereas the 0 on the right of the equality $0(x) = 0$ is the *number* zero so, if you just said the zero element is 0, it is not clear that you mean something different from the number 0. I know, it is all a little confusing—the symbol 0 is overworked—it means many different things so we must always make it clear to the reader what meaning we are associating to it when we use it.

   (c) What is the definition of scalar multiplication in $C(\mathbb{R})$?

   If f is in $C(\mathbb{R})$ and $l \in \mathbb{R}$ we define λf to be the function whose value at a point $x \in \mathbb{R}$ is $\lambda \times f(x)$, the product of the numbers λ and $f(x)$; in short, $(\lambda f)(x) = \lambda f(x)$ for all $x \in \mathbb{R}$.

(4) A polynomial with real coefficients is a function from $\mathbb{R}$ to $\mathbb{R}$. It is also continuous so the set of polynomials with real coefficients is a subset of $C(\mathbb{R})$. Is this subset of $C(\mathbb{R})$ a subspace? Give reasons for your answer in terms of your answer to question 1.

   It is a subspace. If f and g are polynomials so is $f+g$; if f is a polynomial and $\lambda \in \mathbb{R}$, then the function λf is a polynomial.

   It is good to give some details; for example, if $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x_i$ where all a_i and b_i are in $\mathbb{R}$, then

$$(f+g)(x) = \sum_{i=0}^{n} (a_i + b_i) x^i$$

   is also a polynomial with real coefficients. Therefore condition (a) in question 1 holds for the set of all polynomials with real coefficients. If $c \in \mathbb{R}$, then

$$(cf)(x) = \sum_{i=0}^{n} (c a_i) x^i$$

which is again a polynomial with real coefficients. Therefore condition (b) in question 1 holds for the set of all polynomials with real coefficients.

It is not enough to give a single example as some people did. Conditions (a) and (b) in question 1 involve the truth of something *for all* elements in W, not just one or two or some, but *all*.

(5) Suppose V is a vector space and $\underline{u}$, $\underline{v}$, and $\underline{w}$, are elements of V. We define

$$\mathbb{R}\underline{u} + \mathbb{R}\underline{v} + \mathbb{R}\underline{w} = \{a\underline{u} + b\underline{v} + c\underline{w} \mid a, b, c \in \mathbb{R}\}.$$

Is $\mathbb{R}\underline{u} + \mathbb{R}\underline{v} + \mathbb{R}\underline{w}$ a subspace of V? Give reasons for your answer in terms of your answer to question 1.

It is a subspace. Elements in this set are the linear combinations of $\underline{u}$, $\underline{v}$, and $\underline{w}$. If we add two of these, say $a\underline{u} + b\underline{v} + c\underline{w}$ and $a'\underline{u} + b'\underline{v} + c'\underline{w}$ we get another linear combination, namely

$$(a + a')\underline{u} + (b + b')\underline{v} + (c + c')\underline{w}.$$

If we multiply the linear combination $a\underline{u}+b\underline{v}+c\underline{w}$ by the number λ we again get a linear combination, namely $(\lambda a)\underline{u}+(\lambda b)\underline{v}+(\lambda c)\underline{w}$. This shows that conditions (a) and (b) in question 1 are satisfied so we conclude that $\mathbb{R}\underline{u} + \mathbb{R}\underline{v} + \mathbb{R}\underline{w}$ is a subspace of V. It is the linear span of $\underline{u}$, $\underline{v}$, and $\underline{w}$, and the linear span of any set of vectors in any vector space is a subspace.

(6) We write $M_2(\mathbb{R})$ for the set of all 2×2 matrices with real entries. Let W be the subset of $M_2(\mathbb{R})$ consisting of the matrices whose diagonal entries add up to zero. Is W a subspace of V? Give reasons for your answer in terms of your answer to question 1.

It is a subspace. The elements in W are the matrices of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

where a, b, c range over all real numbers. The sum of two such elements is

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix}$$

is

$$\begin{pmatrix} a + a' & b + b' \\ c + c' & -a - a' \end{pmatrix}$$

which is a matrix whose diagonal entries add to zero; i.e., the sum of two elements in W belongs to W so condition (a) in question 1 is satisfied. Condition (b) also holds because

$$\lambda \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & -\lambda a \end{pmatrix}$$

is a matrix whose diagonal entries add up to zero. Since conditions (a) and (b) in question 1 are satisfied so we conclude that W is a subspace of $M_2(\mathbb{R})$.

Several people gave the answer *it is a subspace because it satisfies conditions (a) and (b) in question 1.* That is no good. When I say "Give reasons" I mean give reasons why it satisfies (a) and (b). Saying it is a subspace because it satisfies (a) and (b) in question 1 is really just saying *It is a subspace because it is a subspace.* Likewise, saying *it is a subspace because when you add two* *elements of W you get an element of W* is also *saying,* rather than *showing,* it satisfies (a) in question 1.

There was some confusion about the word *"diagonal"*. We almost always mean *the* diagonal, the entries that are labelled $a_{11}, a_{22}, \dots, a_{nn}$.

(7) Let W be the subset of $M_2(\mathbb{R})$ consisting of the matrices whose determinant is zero. Is W a subspace of V? Give reasons for your answer in terms of your answer to question 1.

No it is not a subspace. Condition (a) in question 1 fails to hold because although

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

belong to W their sum does not: their sum is the identity matrix whose determinant is 1. (Condition (b) does hold: a multiple of a matrix with determinant zero has determinant zero, but this is irrelevant to answering the question. Why?)

(8) Let W be the subset of $M_2(\mathbb{R})$ consisting of the matrices whose determinant is 1. Is W a subspace of V? Give reasons for your answer in terms of your answer to question 1.

No it is not a subspace. Condition (b) in question 1 fails to hold because although the matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

belongs to W, $2I$ does not belong to W.

Many people made the same error in questions 7 and 8. To show a particular subset W of a vector space is *not* a subspace you must show that either condition (a) or condition (b) in question 1 fails. Since both (a) and (b) require something to be true *for all* elements in W (and all $c \in \mathbb{R}$ in (b)) you must show (a) or (b) fails *for some* elements in W. It is just like this: to show the statement *"all dogs are black"* is false I must show you (at least) one dog that is not black. Two things: I need only *one* dog and that one *must be not black*. Compare your answer to (7) and (8) to my answers.

Another problem with the answers was that some of you left the reader to do the work. For example, in question 7 some wrote the following: if $\det(A) = \det(B) = 0$ it is not necessarily true that $\det(A+B) = 0$. Although that is a true statement you should verify it by writing down *explicit* matrices A and B that illustrate its truth.

(9) What are all the subspaces of $\mathbb{R}^2$?

Every line through the origin, and $\mathbb{R}^2$ itself, and $\{\underline{0}\}$.

(10) What are all the subspaces of $\mathbb{R}$?

There are only two, $\mathbb{R}$ and $\{0\}$.

Some people said *"any real number"* but a real number is not a subspace. The *set of all real numbers* is a subspace, but a single number is not. Also, it is not good to write $\{\underline{0}\}$ when you mean $\{0\}$, the set consisting of the real number 0.

(11) Which subspace of V is contained in every subspace of V?

The zero subspace, $\{\underline{0}\}$.

(12) Which subspace of V contains every subspace of V?

V does.

(13) What is the connection between subspaces and solutions of systems of linear equations?

The set of all solutions to a homogeneous system of m equations in n unknowns is a subspace of $\mathbb{R}^n$. Every subspace of $\mathbb{R}^n$ is of this form.

Some people said *"The solutions are subspaces of a homogeneous system of linear equations."* That is not correct. Equations to do not subspaces. A subspace is a subset of a vector space, not a subset of a system of linear equations.

(14) Professor Smith can find 7 different subspaces of $\mathbb{R}^3$. True or False? Why?

True because $\mathbb{R}^3$ contains infinitely many subspaces. It only contains 4 *kinds* of subspaces, lines through the origin, planes through the origin, $\mathbb{R}^3$, and $\{\underline{0}\}$, but $\mathbb{R}^3$ contains infinitely many different lines through the origin and infinitely many different planes through the origin.

Comments on how to prove something is or is not a subspace. Let V be a vector space and W a subset of V. Then W may or may not be a subspace of V depending on what W is. If you want to show W is a subspace you must show that (a) and (b) in question 1 are satisfied. Since (a) is a statement about *all* elements $\underline{u}$ and $\underline{v}$ in W you must show it is true *for all* $\underline{u}, \underline{v}$ *in* W. Thus in question 6, for example, you must take two completely arbitrary elements in W; but all you know about an element in W is that it is a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the property that $a + d = 0$.

Some people answered question 6 by saying it is a subspace because its satisfies (a) and (b). That's all they said. In effect, they are saying W is a subspace because it is a subspace. Or, to put it more bluntly, they are saying W is a subspace because I say it is a subspace. But the question says *"give reasons"*.

On the other hand, if you aim to show that W is

Math 308 **Pop Quiz No. 6**

(1) A vector $\underline{w}$ is a linear combination of vectors $\underline{v}_1, \dots, \underline{v}_n$ if _________ for some _________

(2) Let S be a set of vectors. The linear span of S is the set of all _________. (Use words in this answer and set-theoretic notation in the next answer.)

(3) Let S be a set of vectors. The linear span of S is the set

$$\{\cdots\cdots \mid \underline{v}_1, \dots, \underline{v}_n \in S \quad \text{and} \quad \cdots\}.$$

(4) Vectors $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent if the only solution to the equation _________ is _________.

(5) If V is a vector space we say that a subset $S \subset V$ spans V if every element in V is _________.

(6) A subset B of a vector space V is a basis for V if _________.

(7) All bases for a vector space have _________.

(8) The dimension of a vector space V is _________.

(9) True or False: If $\underline{u}$ and $\underline{v}$ are non-zero linearly dependent vectors, then $\underline{u}$ is a multiple of $\underline{v}$ and $\underline{v}$ is a multiple of $\underline{u}$. Why?

(10) True or False: If $\underline{u}$ is a multiple of $\underline{v}$, then $\underline{u}$ and $\underline{v}$ are linearly dependent vectors. Why?

(11) Every subset of

$$\{(1,1,1,1),\ (1,2,3,4),\ (1,-1,1,-1),\ (4,1,2,1),\ (-1,2,3,4),\ (1,0,1,0)\}$$

consisting of TWO vectors is linearly independent because _________

(12) If $\underline{u}$, $\underline{v}$, and $\underline{w}$, are elements of a vector space V, then $\{\underline{u}-\underline{v}, \underline{v}-\underline{w}, \underline{w}-\underline{u}\}$ is linearly dependent because _________. (Your answer should consist of a single equation.)

(13) Give an example of a basis for $\mathbb{R}^2$ consisting of 2 vectors, neither of which lies on the x- or y-axes.

(14) If V is spanned by 7 elements, then the dimension of V is _________.

Math 308 **Answers to Pop Quiz No. 6**

(1) A vector $\underline{w}$ is a linear combination of vectors $\underline{v}_1, \dots, \underline{v}_n$ if $\underline{\underline{w} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n}$ for some $\underline{a_1, \dots, a_n \in \mathbb{R}}$.

It is also acceptable to say *for some numbers* $a_1, \dots, a_n$. I did not accept the answer "if $\underline{w}$ is in the linear span of $\underline{v}_1, \dots, \underline{v}_n$" because the definition of linear span comes *after* the definition of linear combination; indeed, one needs the notion of a linear combination to define the linear span—see the next question.

(2) Let S be a set of vectors. The linear span of S is the set of all <u>linear combinations of elements in S</u>.

(3) Let S be a set of vectors. The linear span of S is the set

$$\{a_1\underline{v}_1 + \cdots + a_n\underline{v}_n \mid \underline{v}_1, \dots, \underline{v}_n \in S \quad \text{and} \quad a_1, \dots, a_n \in \mathbb{R}\}.$$

(4) Vectors $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent if the only solution to the equation $\underline{a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = 0}$ is $\underline{a_1 = \cdots = a_n = 0}$.

(5) If V is a vector space we say that a subset $S \subset V$ spans V if every element in V is <u>a linear combination of vectors in S</u>.

(6) A subset B of a vector space V is a basis for V if <u>it is linearly independent and spans V</u>.

(7) All bases for a vector space have <u>the same number of elements</u>.

(8) The dimension of a vector space V is <u>the number of elements in a basis for it</u>.

(9) True or False: If $\underline{u}$ and $\underline{v}$ are non-zero linearly dependent vectors, then $\underline{u}$ is a multiple of $\underline{v}$ and $\underline{v}$ is a multiple of $\underline{u}$. Why?

True. Because $\underline{u}$ and $\underline{v}$ are linearly dependent $a\underline{u} + b\underline{v} = 0$ for some a and b, at least one of which is non-zero. If a is zero, then b is non-zero and $b\underline{v} = \underline{0}$ which implies that $\underline{v} = \underline{0}$; but $\underline{u}$ is not zero so we conclude a is *not* zero. A similar argument shows that b is not zero. Therefore $\underline{u} = -a^{-1}b\underline{v}$ and $\underline{v} = -b^{-1}a\underline{u}$; i.e., $\underline{u}$ and $\underline{v}$ are multiples of each other.

(10) True or False: If $\underline{u}$ is a multiple of $\underline{v}$, then $\underline{u}$ and $\underline{v}$ are linearly dependent vectors. Why?

True. If $\underline{u} = c\underline{v}$, then $\underline{u} - c\underline{v} = 0$ and because the coefficient of $\underline{u}$ in this expression is not zero $\underline{u}$ and $\underline{v}$ are linearly dependent.

(11) Every subset of

$$\{(1,1,1,1),\ (1,2,3,4),\ (1,-1,1,-1),\ (4,1,2,1),\ (-1,2,3,4),\ (1,0,1,0)\}$$

consisting of TWO vectors is linearly independent because <u>no vector in S is a multiple of any other element in S.</u>

(12) If $\underline{u}$, $\underline{v}$, and $\underline{w}$, are elements of a vector space V, then $\{\underline{u} - \underline{v}, \underline{v} - \underline{w}, \underline{w} - \underline{u}\}$ is linearly dependent because $\underline{(\underline{u} - \underline{v}) + (\underline{v} - \underline{w}) + (\underline{w} - \underline{u}) = 0}$.

(13) Give an example of a basis for $\mathbb{R}^2$ consisting of 2 vectors, neither of which lies on the x- or y-axes.

There are infinitely many answers to this question. I'll let you figure out one that suits you.

(14) If V is spanned by 7 elements, then the dimension of V is $\underline{\le 7}$.

Math 308 **Pop Quiz No. 7**

(1) What is the dimension of the subspace of $\mathbb{R}^5$ consisting of the solutions of the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0?$$

Why?

(2) Write down a non-zero point in $\mathbb{R}^5$ of the form $(0,0,0,-,-)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

(3) Write down a non-zero point in $\mathbb{R}^5$ of the form $(0,0,-,0,-)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

(4) Write down a non-zero point in $\mathbb{R}^5$ of the form $(0,-,0,0,-)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

(5) Write down a non-zero point in $\mathbb{R}^5$ of the form $(-,0,0,0,-)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

(6) Are the vectors in your answers to the last four questions linearly independent? Why?

(7) Explain the following joke. On the linear algebra exam the professor writes down two vectors and asks if they are linearly dependent. The student answers "the first is but second is not."

(8) What is the dimension of the subspace of $\mathbb{R}^5$ consisting of the solutions of the equations

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 0?$$

Why?

(9) Find equations for a plane in $\mathbb{R}^4$ that contains $(0,0,0,0)$ and $(1,0,1,0)$ and $(0,1,2,3)$.

(10) Do the vectors $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$, span $\mathbb{R}^4$? Why?

(11) Do the vectors $(1,0,0,0)$, $(0,2,0,0)$, $(0,0,3,0)$, and $(0,0,0,4)$, span $\mathbb{R}^4$? Why?

(12) Do the vectors $(1,1,0,0)$, $(0,1,1,0)$, $(0,0,1,1)$, and $(1,1,1,1)$, span $\mathbb{R}^4$? Why?

(13) Are the vectors $(1,1,0,0)$, $(0,1,1,0)$, $(0,0,1,1)$, and $(1,1,1,1)$, linearly dependent? Why?

(14) Are the vectors $(1,2,3,4)$, $(5,6,7,8)$, $(12,11,10,9)$, $(2,4,8,16)$, and $(1,1,1,1)$, linearly dependent. Why?

Math 308 **Answers to Pop Quiz No. 7**

(1) What is the dimension of the subspace of $\mathbb{R}^5$ consisting of the solutions of the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0?$$

Why?

Four.

As with any system of linear equations you can obtain all solutions by performing elementary row operations to get the system of equations into row-reduced echelon form and then producing independent and dependent variables. In this question we can make $x_1, \ldots, x_4$ the independent variables and x_5 the only dependent variable. Thus $x_1, \ldots, x_4$ can take any values and x_5 must equal $-5x_1 + 4x_2 - 3x_3 + 2x_4$. The set of solutions therefore consists of the vectors

$$\begin{aligned} \underline{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ -5x_1 + 4x_2 - 3x_3 + 2x_4 \end{pmatrix} \\ &= x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -5 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -3 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \end{aligned}$$

as x_1, x_2, x_3, x_4 run over all of $\mathbb{R}$. The vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

are a basis for the set of all solutions to the given equation. The dimension of the space of solutions is equal to the number of elements in a basis for it, hence four.

(2) Write down a non-zero point in $\mathbb{R}^5$ of the form $(0, 0, 0, -, -)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

There are infinitely many answers to this question. One is $(0, 0, 0, 1, 2)$ and others are $c(0, 0, 0, 1, 2)$ where $c \neq 0$.

(3) Write down a non-zero point in $\mathbb{R}^5$ of the form $(0, 0, -, 0, -)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

There are infinitely many answers to this question. One is $(0, 0, 1, 0, -3)$ and others are $c(0, 0, 1, 0, -3)$ where $c \neq 0$.

(4) Write down a non-zero point in $\mathbb{R}^5$ of the form $(0, -, 0, 0, -)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

There are infinitely many answers to this question. One is $(0, 1, 0, 0, 4)$ and others are $c(0, 1, 0, 0, 4)$ where $c \neq 0$.

(5) Write down a non-zero point in $\mathbb{R}^5$ of the form $(-, 0, 0, 0, -)$ that is a solution to the equation

$$5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 = 0.$$

There are infinitely many answers to this question. One is $(1, 0, 0, 0, -5)$ and others are $c(1, 0, 0, 0, -5)$ where $c \neq 0$.

(6) Are the vectors in your answers to the last four questions linearly independent? Why?

Yes. If

$$a(1, 0, 0, 0, -5) + b(0, 1, 0, 0, 4) + c(0, 0, 1, 0, -3) + d(0, 0, 0, 1, 2) = (0, 0, 0, 0, 0)$$

then $(a, b, c, d, -5a + 4b - 3c + 2d) = (0, 0, 0, 0, 0)$ so $a = b = c = d = 0$.

(7) Explain the following joke. On the linear algebra exam the professor writes down two vectors and asks if they are linearly dependent. The student answers "the first is but second is not."

Ha, ha, ...

It is similar to this: I ask Mary if Ann and Peter are related to each other and Mary answers "Ann is but Peter isn't". The questions "are the vectors ... and ... linearly dependent" is a question about the relation between the vectors and can only be answered "yes" or "no".

(8) What is the dimension of the subspace of $\mathbb{R}^5$ consisting of the solutions of the equations

$$\begin{aligned} 5x_1 - 4x_2 + 3x_3 - 2x_4 + x_5 &= 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0? \end{aligned}$$

Why?

Three.

This is like question one. We can take $x_1, \dots, x_3$ to be the independent variables and x_4 and x_5 the dependent variable. Thus $x_1, \dots, x_3$ can take any values and following the same reasoning as in question 1 the dimension for the space of solutions is equal to the number of independent variables, namely 3.

It is also good to see this in terms of the formula $\operatorname{rank}(A) + \operatorname{nullity}(A) = 5$ where A is the 2×5 matrix of coefficients in the system of equations. The two rows of A are obviously linearly independent so $\operatorname{rank}(A) = 2$. Therefore $\operatorname{nullity}(A) = 3$. But nullity is, by definition, the dimension of $\mathcal{N}(A)$, the null space of A and $\mathcal{N}(A) = \{\underline{x} \in \mathbb{R}^5 \mid A\underline{x} = \underline{0}\}$ which is the set of solutions to the equations $A\underline{x} = \underline{0}$, i.e., the set of solutions to the two equations given in the question.

(9) Find equations for a plane in $\mathbb{R}^4$ that contains $(0, 0, 0, 0)$ and $(1, 0, 1, 0)$ and $(0, 1, 2, 3)$.

There are infinitely many possible answers to this question. A plane in $\mathbb{R}^4$ containing $(0,0,0,0)$ is a subspace of dimension 2, which is equal to $4-2$, so the plane is the set of solutions to a system of two homogeneous linear equations, i.e., two equations of the form $ax_1+bx_2+cx_3+dx_4=0$, neither of which is a multiple of the other. But, as I said, there are infinitely many choices of a,b,c,d.

To ensure that $(1,0,1,0)$ and $(0,1,2,3)$ lie on the plane you must ensure that when you plug them into the formulas $ax_1+bx_2+cx_3+dx_4$ you get zero.

One obvious equation having both $(1,0,1,0)$ and $(0,1,2,3)$ as solutions is $3x_2-x_4=0$. A little less obvious is $x_1+2x_2-x_3=0$.

It follows that $(1,0,1,0)$ and $(0,1,2,3)$ also lie on the 3-plane

$$a(x_1+2x_2-x_3)+b(3x_2-x_4)=0$$

for all choices of $(a,b)\in\mathbb{R}^2$ for which $(a,b)\neq(0,0)$.

(10) Do the vectors $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$, span $\mathbb{R}^4$? Why?

Yes. The given vectors are linearly independent and $\mathbb{R}^4$ is spanned by every set of four linearly independent vectors in $\mathbb{R}^4$.

(11) Do the vectors $(1,0,0,0)$, $(0,2,0,0)$, $(0,0,3,0)$, and $(0,0,0,4)$, span $\mathbb{R}^4$? Why?

Yes. The given vectors are linearly independent and $\mathbb{R}^4$ is spanned by every set of four linearly independent vectors in $\mathbb{R}^4$.

Alternatively, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$ are non-zero multiples of $(1,0,0,0)$, $(0,2,0,0)$, $(0,0,3,0)$, and $(0,0,0,4)$, and vice-versa, so

$$\mathbb{R}(1,0,0,0)+\mathbb{R}(0,1,0,0)+\mathbb{R}(0,0,1,0)+\mathbb{R}(0,0,0,1)$$

is equal to

$$\mathbb{R}(1,0,0,0)+\mathbb{R}(0,2,0,0)+\mathbb{R}(0,0,3,0)+\mathbb{R}(0,0,0,4).$$

(12) Do the vectors $(1,1,0,0)$, $(0,1,1,0)$, $(0,0,1,1)$, and $(1,1,1,1)$, span $\mathbb{R}^4$? Why?

No. Since $(1,1,0,0)+(0,0,1,1)=(1,1,1,1)$ the four vectors are linearly dependent. Therefore their linear span has dimension <4.

(13) Are the vectors $(1,1,0,0)$, $(0,1,1,0)$, $(0,0,1,1)$, and $(1,1,1,1)$, linearly dependent? Why?

Yes, $(1,1,0,0)+(0,0,1,1)=(1,1,1,1)$.

(14) Are the vectors $(1,2,3,4)$, $(5,6,7,8)$, $(12,11,10,9)$, $(2,4,8,16)$, and $(1,1,1,1)$, linearly dependent. Why?

Yes, any five vectors in $\mathbb{R}^4$ are linearly dependent. We proved that every subset of $\mathrm{Span}\{\underline{v}_1,\dots,\underline{v}_p\}$ having $\geq p+1$ elements is linearly dependent.

Math 308 **Pop Quiz No. 8**

(1) Let P be a 2-dimensional subspace of $\mathbb{R}^4$. Suppose $\underline{u}$ and $\underline{v}$ lie on P. Does every linear combination of $\underline{u}$ and $\underline{v}$ belong to P?

(2) Find two linearly independent vectors belonging to the plane that consists of the solutions to the system of equations

$$\begin{aligned} 4x_1 - 3x_2 + 2x_3 - x_4 &= 0 \\ x_1 - x_2 - x_3 + x_4 &= 0. \end{aligned}$$

(3) Find a basis for the plane in the previous question.

(4) Is $(1,2,3,4)$ a linear combination of the two vectors in your answer to the previous question? Explain why.

(5) Find a basis for the line $x_1 + 2x_2 = x_1 - 2x_3 = x_2 - x_4 = 0$ in $\mathbb{R}^4$.

(6) Find a basis B for the vector space consisting of the polynomials of degree ≤ 4 with the property that $f(1) = 2$ for all $f \in B$.

(7) If

$$A^{-1} = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 3 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

what is $(A^T)^{-1}$ equal to?

(8) Let A be any $n \times n$ matrix. Write out an equation that proves A^T has an inverse if A does.

(9) Yes or No: Does your answer to the previous question use the fact that the transpose of the identity matrix is the identity matrix?

(10) True or False: A square matrix is invertible if and only if its rows are linearly independent.

(11) True or False: A square matrix is invertible if and only if its columns are linearly independent.

Math 308 **Pop Quiz No. 9**

(1) Every set of five vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.
(2) Every set of five vectors in $\mathbb{R}^4$ is linearly dependent.
(3) Every set of four vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.
(4) Every set of four vectors in $\mathbb{R}^4$ is linearly independent.
(5) If S is a linearly independent set so is every subset of S.
(6) If $\mathrm{Span}(\underline{u}, \underline{v}, \underline{w}) = \mathrm{Span}(\underline{v}, \underline{w})$, then $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$.
(7) If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $Sp(\underline{u}, \underline{v}, \underline{w}) = Sp(\underline{v}, \underline{w})$.
(8) For any vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, $Sp(\underline{u}, \underline{v}, \underline{w})$ contains $Sp(\underline{v}, \underline{w})$.
(9) The linear span of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is the same as the linear span of $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.
(10) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$ have the same the linear span as $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.
(11) Does the phrase *"$\underline{x}$ is a linear span of $\underline{v}_1, \dots, \underline{v}_n$"* does not make sense. Explain why.

Math 308 **Pop Quiz No. 10**

(1) The four vectors

$$(1,1,1,1), \quad (2,4,8,16), \quad (3,9,27,81), \quad \left(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{8},\tfrac{1}{16}\right)$$

are linearly independent. Does it follows that

$$(1,1,1,1), \quad (2,4,8,16), \quad (3,9,27,81)$$

are also linearly independent? Explain your answer

(2) If $\mathrm{Span}(\underline{u},\underline{v},\underline{w}) = \mathrm{Span}(\underline{v},\underline{w})$, then $\{\underline{u},\underline{v},\underline{w}\}$ is linearly dependent.

Math 308 **Pop Quiz No. 11**

In questions (1)-(6), A is a 4×4 matrix whose columns $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ have the property that $\underline{A}_1 + 2\underline{A}_2 = 3\underline{A}_3 + 4\underline{A}_4$.

(1) The columns of A span $\mathbb{R}^4$.
(2) A is singular.
(3) The columns of A are linearly dependent.
(4) The rows of A are linearly dependent.
(5) The equation $A\underline{x} = 0$ has a non-trivial solution.
(6) $A\begin{pmatrix} 1 \\ 2 \\ -3 \\ -4 \end{pmatrix} = 0.$

Math 308 **Pop Quiz No. 12**

(1) There is a matrix whose inverse is $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$.

(2) If $A^{-1} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.

(3) If I is the identity matrix the equation $I\underline{x} = \underline{b}$ has a unique solution.

(4) If A is a non-singular matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.

(5) If A is an invertible matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.

(6) If A is a singular matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.

(7) If the row-reduced echelon form of A is the identity matrix, then the equation $A\underline{x} = \underline{b}$ has a unique solution.

(8) If A is row-equivalent to the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

then the equation $A\underline{x} = \underline{b}$ has a unique solution.

(9) There is a 7×7 matrix that has no inverse.

(10) There is a 7×7 matrix that has a unique inverse.

(11) There is a 7×7 matrix having more than one inverse.

(12) Let A and B be $n \times n$ matrices. If $AB = I$, then $BA = I$.

(13) There exists a 2×3 matrix A and a 3×2 matrix B such that AB is the 2×2 identity matrix.

(14) There exists a 2×3 matrix A and a 3×2 matrix B such that BA is the 3×3 identity matrix.

Math 308 **Pop Quiz No. 13**

(1) Find an equation for the plane in $\mathbb{R}^3$ that contains $(0,0,0)$, $(1,1,1)$ and $(1,2,2)$.

(2) This problem takes place in $\mathbb{R}^4$. Find two linearly independent vectors belonging to the plane that consists of the solutions to the system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 0 \\ x_1 - x_2 - x_3 + x_4 &= 0. \end{aligned}$$

(3) Let A be an $m \times n$ matrix with columns $\underline{A}_1, \dots, \underline{A}_n$. Express $A\underline{x}$ as a linear combination of the columns of A.

(4) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix with columns $\underline{B}_1, \dots, \underline{B}_p$. Express the columns of AB in terms of A and the columns for B.