Remarks on Definitions **July 23, 2009**

I received the following email from a student today:

`So i just want clarification on some definition to see they acceptable for the exam. If not, what can be changed?`

`1) Invertible Let matrix A be an (n x n) matrix. It is invertible if` $A * A^{-}1 = I$ `can be satisfied`

`2) Rank The rank of a system of linear equation is the dimension of the range of A`

`3) Basis Let W be a nonzero subspace in Rn. A basis for W is a linearly independent spanning set for W.`

`4) Dimension Let W be a subspace in Rn. If the basis for W is <v1, vp> then W has the dimension of n dim(W)=p`

`5) Row/column space Let A be an (m x n) matrix. Row space a set of all possible linear combination of the row/column vectors of matrix A.`

`6) Linear combination The sum of the scalar multiple of its column/row vectors.`

`7) Linear span Let W be a subspace in Rn and let S=w1, w2..wm a subset of W. S spans W if vectors w in W can be expressed as a linear combination of S`

This provides me the opportunity to say something that is probably of use to several other people, so let me do that.

(1) Invertible

My Definition. An $n \times n$ matrix A is invertible if there is a matrix B such that $AB = BA = I$ where I is the $n \times n$ identity matrix.

Comments. The student's attempted definition doesn't make it clear that one is seeking a matrix whose product with A *in both orders* is equal to I. Also the meaning of the phrase $A * A^{-}1 = I$ *can be satisfied* is a little unclear. What does *satisfied* mean? Also the word *matrix* is used twice in the first sentence of the student's attempted definition – that is overkill and creates more work for the reader. By combining the two sentences into one the word "it" can be eliminated. When using "it" you must always make it 100% clear what "it" refers to.

After defining an invertible matrix as I do, the next step in developing the theory and language of inverses is to prove that such a B is unique. After doing that we are justified is saying that we will denote the B by A^{-1} and call it the inverse of A.

Notice we are not justified in saying *the* inverse or writing A^{-1} until we have proved there is only one B such that $AB = BA = I$.

(2) Rank

My Definition. The rank of a matrix is the dimension of its range.

My Definition. The rank of a system of linear equations is the rank of the matrix of coefficients.

My Definition. The rank of a system of linear equations is the dimension of the range of the matrix of coefficients.

Comments. The student does not tell us what A is in his/her definition. The word *equation* should be plural, *equations*.

(3) Basis

Comments. The student's definition is good. The only place where one might take issue is the requirement that W be non-zero. It is nice to be able to say that *every* vector space has a basis without having to make the zero vetor space a special case. In order to do that we adopt the convention that the empty set is linearly independent and the linear span of the empty set is $\{0\}$, the subspace consisting of the zero vector. Then we can say

My Definition. A basis for a subspace W is a linearly independent set that spans W.

(4) Dimension

My Definition. The dimension of a subspace is the number of elements in a basis for it.

Comments. My definition is shorter and needs no notation. The parentheses $<$ and $>$ used by the student should be the symbols $\{$ and $\}$ since $\langle v_1, \dots, v_p \rangle$ denotes the linear span of $v_1, \dots, v_p$ not the set $\{v_1, \dots, v_p\}$. The student's definition does not accomodate the zero subspace. We want a definition of dimension that allows us to say that the zero subspace has dimension zero. Otherwise the student's definition is OK, but a little clunky. Better to try an have something that reads smoothly. Finally, my definition will work for infinite dimensional subspaces — we don't need to worry about that in this course, but eventually one does and then the definition should be stated as *The* dimension *of a subspace is the cardinality of a basis for it.* This allows for the fact that there are different sizes of infinity.

(5) Row/column space

My Definition. The row space of a matrix is the linear span of its rows.

Comments. It is unnecessary to state the size of the matrix. It is unnecessary to give the matrix a name. "Row space a set" should be replaced by "The row space is the set.." The word "possible" is not necessary.

(6) Linear combination

My Definition. If $a_1, \dots, a_n \in \mathbb{R}$ and $v_1, \dots, v_n \in \mathbb{R}^n$, we call $a_1v_1 + \cdots + a_nv_n$ a linear combination of $v_1, \dots, v_n$

Comments. The student's attempted definition is way off the mark. We only speak of a linear combination of vectors. When the student says "*the* scalar multiple" I want to ask *which* scalar multiple. The mention of *column/row vectors* seems to suggest that linear combinations only arise in the context of a matrix. Finally the use of the word "its" is confusing because the student does not tell us what "it" is. Also, I suspect that whatever the student meant the word *multiple* should be plural, *multiples.*

(7) Linear span

My Definition. A subset S of a subspace W spans W if every element in W is a linear combination of elements in S.

Comments. There is no need to label the elements of S. The student needs to make it clear that *every* w in W must be a linear combination of elements in S. One should not write $S = w_1, w_2, \dots, w_m$ but $S = \{w_1, w_2, \dots, w_m\}$. The phrase "can be expressed as" could be replaced by "is".

It was not clear to me whether when the student wrote Linear span whether he/she was wanting to give a definition of *linear span* or wanting to define what it means to say that a subset of W spans W. The proposed definition seems to suggest the latter but for completeness I also define the phrase *linear span* below

My Definition. The linear span of a set of vectors consists of all linear combinations of elements in the set.

OR

My Definition. The linear span of $\{v_1, v_2, \dots\}$ is the set

$$\langle v_1, v_2 \dots, \rangle := \{a_1 v_1 + a_2 v_2 + \cdots + a_p v_p \mid a_a, \dots, a_p \in \mathbb{R} \text{ and } p \geq 1\}.$$