- We will write $\underline{A}_1, \dots, \underline{A}_n$ for the columns of an $m \times n$ matrix A.
- Several questions involve an unknown vector $\underline{x} \in \mathbb{R}^n$. We will write $x_1, \dots, x_n$ for the entries of $\underline{x}$; thus $\underline{x} = (x_1, \dots, x_n)^T$.
- The null space and range of a matrix A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively.
- The linear span of a set of vectors is denoted by $\mathrm{Sp}(\underline{v}_1, \dots, \underline{v}_n)$.
- We will write $\underline{e}_1, \dots, \underline{e}_n$ for the standard basis for $\mathbb{R}^n$. Thus $\underline{e}_i$ has a 1 in the i^{th} position and zeroes elsewhere.
- In order to save space I will often write elements of $\mathbb{R}^n$ as row vectors, particularly in questions about linear transformations. For example, I will write $T(x, y) = (x + y, x - y)$ rather than

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}.$$

Part A.

True or False.

Scoring. You get +1 for each correct answer, −1 for each incorrect answer and 0 if you choose not to answer the question. Use your BUBBLES: A=True. B=False. Fill in bubble A if you think it is True, bubble B if you think it is False, and fill in nothing if you do not want to answer it.

(1) If a system of equations has fewer equations than unknowns it always has a solution.
(2) Every set of five vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.
(3) Every set of five vectors in $\mathbb{R}^4$ is linearly dependent.
(4) Every set of four vectors in $\mathbb{R}^4$ is linearly dependent.
(5) Every set of four vectors in $\mathbb{R}^4$ spans $\mathbb{R}^4$.
(6) A square matrix having a row of zeroes is always singular.
(7) If a square matrix does not have a column of zeroes it is non-singular.
(8) For any vectors $\underline{u}$, $\underline{v}$, and $\underline{w}$, $Sp(\underline{u}, \underline{v}, \underline{w})$ contains $Sp(\underline{v}, \underline{w})$.
(9) If A is singular and B is non-singular then AB is always singular.
(10) If A and B are non-singular so is AB.
(11) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$ have the same the linear span as $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.
(12) There is a matrix whose inverse is $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$.
(13) If $A^{-1} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ there is a matrix B such that $BA = E$.
(14) If A is a non-singular matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.
(15) If A is an invertible matrix the equation $A\underline{x} = \underline{b}$ has a unique solution.

(16) If A is row-equivalent to the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

then the equation $A\underline{x} = \underline{b}$ has a unique solution.
(17) There exists a 3×2 matrix A and a 2×3 matrix B such that AB is the 3×3 identity matrix.
(18) There exists a 2×3 matrix A and a 3×2 matrix B such that AB is the 2×2 identity matrix.
(19) The dimension of a subspace is the number of elements in it.
(20) Every subset of a linearly dependent set is linearly dependent.
(21) Every subset of a linearly independent set is linearly independent.
(22) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, then $\{\underline{v}_1 + 3\underline{v}_2, 3\underline{v}_2 + \underline{v}_3, \underline{v}_3 - \underline{v}_1\}$ is linearly dependent.
(23) Let A and B be $n \times n$ matrices. Suppose 2 is an eigenvalue of A and 3 is an eigenvalue of B. Then 6 is an eigenvalue of AB.
(24) Let A and B be $n \times n$ matrices. Suppose 2 is an eigenvalue of A and 3 is an eigenvalue of B. Then 5 is an eigenvalue of $A + B$.
(25) Let A and B be $n \times n$ matrices. If $\underline{x}$ is an eigenvector for both A and B it is also an eigenvector for AB.
(26) Let A and B be $n \times n$ matrices. If $\underline{x}$ is an eigenvector for both A and B it is also an eigenvector for $A + B$.
(27) If A is an invertible matrix, then $A^{-1}\underline{b}$ is a solution to the equation $A\underline{x} = \underline{b}$.
(28) The linear span $\mathrm{Sp}\{\underline{u}_1, \ldots, \underline{u}_r)$ is the same as the linear span $\mathrm{Sp}(\underline{v}_1, \ldots, \underline{v}_s)$ if and only if every $\underline{u}_i$ is a linear combination of the $\underline{v}_j$s and every $\underline{v}_j$ is a linear combination of the $\underline{u}_i$s.
(29) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and V is a subspace of $\mathbb{R}^n$, then $T(V)$ is a subspace of $\mathbb{R}^m$.
(30) The row reduced echelon form of a square matrix is the identity if and only if the matrix is invertible.
(31) Let A be an $n \times n$ matrix. If the columns of A are linearly dependent, then A is singular.
(32) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.
(33) There is a matrix whose inverse is $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix}$.
(34) There is a matrix whose inverse is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$
(35) The column space of the matrix $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$ is a basis for $\mathbb{R}^3$.

(36) The subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\6\\2\end{pmatrix}$ is the same as the subspace spanned by $\begin{pmatrix}2\\3\\1\end{pmatrix}$ and $\begin{pmatrix}2\\4\\6\end{pmatrix}$.

(37) Two systems of m linear equations in n unknowns have the same row reduced echelon form if and only if they have the same solutions.

(38) If $\underline{u}$ and $\underline{v}$ are $n \times 1$ column vectors then $\underline{u}^T \underline{v} = \underline{v}^T \underline{u}$.

(39) If $A^2 = B^2 = C^2 = I$, then $(ABAC)^{-1} = CABA$.

(40) Let A be a non-singular 5×5 matrix and $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ a subset of $\mathbb{R}^5$. Then $\{A\underline{u}_1, A\underline{u}_2, A\underline{u}_3\}$ is linearly independent if and only if $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is.

(41) If W is a subspace of $\mathbb{R}^n$ that contains $\underline{u} + \underline{v}$, then W contains $\underline{u}$ and $\underline{v}$.

(42) There is a 5×5 matrix having eigenvalues 1 and 2 and no others.

(43) There is a 5×5 matrix having eigenvalues $1, 2, 3, 4, 5$ and no others.

(44) There is a 5×5 matrix having eigenvalues $1, 2, 3, 4, 5, 6, 7$ and no others.

(45) A 5×5 matrix can't have more than 5 eigenvectors.

(46) A 5×5 matrix has exactly 5 eigenvalues.

(47) The vector $\begin{pmatrix}2\\3\end{pmatrix}$ is an eigenvector for the matrix $\begin{pmatrix}1 & 3\\3 & 2\end{pmatrix}$.

(48) The vector $\begin{pmatrix}3\\2\end{pmatrix}$ is an eigenvector for the matrix $\begin{pmatrix}1 & 3\\3 & 2\end{pmatrix}$.

(49) If $\begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector for a matrix so is $\begin{pmatrix}10\\20\end{pmatrix}$.

(50) If $\underline{u}$ and $\underline{v}$ are eigenvectors for A is is $\underline{u} + 2\underline{v}$.

(51) If $\underline{u}$ is an eigenvector for A and B it is an eigenvector for $A + 2B$.

(52) The vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is an eigenvector for the matrix $\begin{pmatrix}1 & 0 & 0\\2 & 0 & 0\\3 & 0 & 0\end{pmatrix}$.

(53) The vector $\begin{pmatrix}0\\2\\3\end{pmatrix}$ is an eigenvector for the matrix $\begin{pmatrix}1 & 0 & 0\\2 & 0 & 0\\3 & 0 & 0\end{pmatrix}$.

(54) The number 0 is an eigenvalue for the matrix $\begin{pmatrix}1 & 0 & 0\\2 & 0 & 0\\3 & 0 & 0\end{pmatrix}$.

(55) The range of a matrix is its columns.

(56) The formula $T(a, b, c) = 0$ defines a linear transformation $\mathbb{R}^3 \to \mathbb{R}$.

(57) The formula $T(a, b, c) = 1$ defines a linear transformation $\mathbb{R}^3 \to \mathbb{R}$.

(58) The formula $T(a, b, c) = \sin(a) + \sin(b) + \sin(c)$ defines a linear transformation $\mathbb{R}^3 \to \mathbb{R}$.

(59) The formula $T(a, b, c) = (a, b, 1)$ defines a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$.

(60) The vectors $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}3\\0\\-3\end{pmatrix}$ form an orthogonal basis for $\mathbb{R}^3$.

(61) $\{\underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4\}$ is a subspace of $\mathbb{R}^4$.

(62) $\{\underline{x} \in \mathbb{R}^5 \mid x_1 - x_2 = x_3 + x_4 = 1\}$ is a subspace of $\mathbb{R}^5$.

(63) The solutions to a system of homogeneous linear equations is a subspace.

(64) The solutions to a system of linear equations is a subspace.

(65) The set $W = \{\underline{x} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 \mid x_1^2 = x_2^2\}$ is a subspace.
(66) The null space of A is equal to its 0-eigenspace.
(67) The linear span of a matrix is its set of columns.
(68) $U \cup V$ is a subspace if U and V are.
(69) U^{-1} is a subspace if U is.
(70) Similar matrices have the same eigenvalues.
(71) Similar matrices have the same eigenvectors.
(72) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, then T is invertible if and only if its nullity is zero.
(73) If $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation, then T is invertible if and only if its nullity is zero.
(74) A matrix is linearly independent if its columns are different.
(75) If A is a 3×5 matrix, then the inverse of A is a 5×3 matrix.
(76) If A is a 2×2 matrix it is possible for $\mathcal{R}(A)$ to equal $\mathcal{N}(A)$.
(77) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation it is possible for $\mathcal{R}(T)$ to equal $\mathcal{N}(T)$.
(78) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation it is possible for $\mathcal{R}(T)$ to equal $\mathcal{N}(T)$.
(79) If $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation it is possible for $\mathcal{R}(T)$ to equal $\mathcal{N}(T)$.
(80) If A is a 2×2 matrix it is possible for $\mathcal{R}(A)$ to be the parabola $y = x^2$.
(81) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$. The null space of T is $\{(0, 0, 0, a) \mid \text{where } a \text{ is a real number}\}$.
(82) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$. The null space of T is $\{(x_4, 0, 0, 0) \mid \text{where } x_4 \text{ is a real number}\}$.
(83) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x_1, x_2) = (x_2, 0)$. The null space of T is $\{(t, 0) \mid t \text{ is a real number}\}$.
(84) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x_1, x_2) = (x_2, 0)$. The null space of T is $\{(1, 0)\}$.
(85) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_2)$. The nullspace of T has many bases; one of them is the set $\{(1, 0, 0, 0), (0, 0, 1, 1)\}$.
(86) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_2)$. The nullspace of T has many bases; one of them is the set $\{(0, 0, 1)\}$.
(87) The set $\{(1, 0, 0, 0), (0, 0, 1, 1)\}$ is a basis for the range of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by $T(x_1, x_2, x_3) = (x_1, 0, x_2, x_2)$.
(88) The smallest subspace containing subspaces V and W is $V + W$.
(89) No linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ is onto.
(90) No linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ is onto.
(91) No linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ is one-to-one.
(92) No linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ is one-to-one.
(93) A linear transformation is invertible if and only if its nullity is zero.
(94) A linear transformation is one-to-one if and only if its nullity is zero.

In the next 6 questions, A is a 4×4 matrix whose columns $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ have the property that $\underline{A}_1 - \underline{A}_2 = \underline{A}_3 - \underline{A}_4$.

(95) The columns of A span $\mathbb{R}^4$.
(96) A is singular.
(97) The columns of A are linearly dependent.
(98) The rows of A are linearly dependent.

(99) The equation $A\underline{x} = 0$ has a non-trivial solution.

(100) $A \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = 0.$

Part B.

Complete the definitions and theorems by completing the sentences.
Scoring: 2 points per question. No partial credit.

Systems of linear equations

(1) **Definition:** Two systems of linear equations are equivalent if ______________.
(2) **Theorem:** Two systems of linear equations are equivalent if their row reduced echelon forms are ______________.
(3) **Definition:** Let A be an $m \times n$ matrix and let E be the row-reduced echelon matrix that is row equivalent to it. If $x_1, \dots, x_n$ are the unknowns in the system of equations $A\underline{x} = \underline{b}$, then x_j is a dependent variable if and only if ________.
(4) **Theorem:** A homogeneous system of linear equations always has a non-zero solution if the number of unknowns is ______________.
(5) **Theorem:** The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is in the linear span of ________
(6) **Theorem:** Let A be an $n \times n$ matrix and $\underline{b} \in \mathbb{R}^n$. The equation $A\underline{x} = \underline{b}$ has a unique solution if the rank of A is ___________
(7) **Theorem:** Let A be an $n \times n$ matrix and $\underline{b} \in \mathbb{R}^n$. The equation $A\underline{x} = \underline{b}$ has a unique solution if and only if A is ___________.
(8) **Theorem:** If $A\underline{u} = \underline{b}$, then the set of all solutions to the equation $A\underline{x} = \underline{b}$ consists of the vectors $\underline{u} + \underline{v}$ as $\underline{v}$ ranges over all ___________
(9) **Theorem:** The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a ________ of the columns of A.
(10) **Theorem:** Let A be an $m \times n$ matrix and let E be the row-reduced echelon matrix that is row equivalent to it. Then the non-zero rows of E are a basis for ________.

Linear combinations and Linear spans

(1) **Definition:** A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if _________
(2) **Theorem:** A vector $\underline{w}$ is a linear combination of $\{\underline{v}_1, \dots, \underline{v}_n\}$ if $\mathrm{Sp}(\underline{w}, \underline{v}_1, \dots, \underline{v}_n) =$ _________
(3) **Definition:** The linear span of $\{\underline{v}_1, \dots, \underline{v}_n\}$ consists of _________
(4) **Definition:** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent if the only solution to the equation ______________ is ______________.
(5) **Theorem:** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent if the dimension of $\mathrm{Sp}(\underline{v}_1, \dots, \underline{v}_n)$ ______________
(6) **Theorem:** A set of vectors is linearly dependent if and only if one of the vectors is _________ of the others.

Subspaces

(1) **Definition:** A subset W of $\mathbb{R}^n$ is a subspace if it satisfies the following three conditions: ______________.

(2) **Theorem:** If V and W are subspaces of $\mathbb{R}^n$ so are ______________ and ______________.
(3) **Definition:** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_d\}$ is a basis for a subspace V of $\mathbb{R}^n$ if ______________
(4) **Definition:** The dimension of a subspace V of $\mathbb{R}^n$ is ______________.
(5) **Definition:** A set of vectors $\{\underline{v}_1, \dots, \underline{v}_d\}$ is orthogonal if _________.
(6) **Definition:** We call $\{\underline{v}_1, \dots, \underline{v}_d\}$ an orthonormal basis for a subspace V if _________.
(7) **Theorem:** If $\{\underline{v}_1, \dots, \underline{v}_d\}$ is an orthogonal basis for W, then {_________ } is an orthonormal basis for W.

Matrices

(1) If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then AB exists if and only if ______________ and in that case AB is a ______________ matrix.
(2) If A is an $m \times n$ matrix and $\underline{x} \in \mathbb{R}^n$, then $A\underline{x}$ is a linear combination of the columns of A, namely $A\underline{x} =$______________.
(3) **Definition 1:** An $n \times n$ matrix A is non-singular if the only solution ______________ and is singular if it is not non-singular.
(4) **Definition 2:** An $n \times n$ matrix A is singular if there exists ______________ in $\mathbb{R}^n$ such that ______________ and is non-singular otherwise.
(5) **Theorem:** An $n \times n$ matrix A is non-singular if its columns ______________.
(6) **Theorem:** An $n \times n$ matrix A is non-singular if and only if it has ______________.
(7) **Theorem:** An $n \times n$ matrix A is singular if its columns ______________.
(8) **Theorem:** An $n \times n$ matrix A is singular if its range ______________.
(9) **Theorem:** An $n \times n$ matrix A is non-singular if the equation $A\underline{x} = \underline{b}$ ______________.

Invertible matrices and determinants

(1) **Definition:** An $n \times n$ matrix A is invertible if ______________.
(2) **Theorem:** An $n \times n$ matrix is invertible if and only if it is ___________.
(3) **Theorem:** An $n \times n$ matrix is invertible if and only if its ___________ is non-zero.
(4) **Theorem:** An $n \times n$ matrix is invertible if and only if its row-reduced echelon form is ___________.
(5) **Theorem:** The matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible if and only if ________$\neq 0$.
(6) **Theorem:** If the matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is invertible its inverse is ___________.
(7) **Definition:** Let A be an $n \times n$ matrix. The characteristic polynomial of A is ______________
(8) **Theorem:** Let A be an $n \times n$ matrix. If B is obtained from A by
   (a) replacing row i by row i + a multiple of row $k \neq i$, then $\det B =$?
   (b) swapping two rows of A, then $\det B =$?
   (c) multiplying a row in A by $c \in \mathbb{R}$, then $\det B =$?

Rank and Nullity

(1) **Definition:** The rank of a matrix A is the number of non-zero ______________.
(2) **Theorem:** The rank of a matrix is equal to the dimension of ______________.

(3) **Definition:** The rank of a linear transformation T is equal to ________.

Eigenvalues and eigenvectors

(1) **Definition:** Let A be an $n \times n$ matrix. We call $\lambda \in \mathbb{R}$ an eigenvalue of A if ______________
(2) **Definition:** Let A be an $n \times n$ matrix. A non-zero vector $\underline{x} \in \mathbb{R}^n$ is an eigenvector for A if ______________
(3) **Definition:** Let λ be an eigenvalue for the $n \times n$ matrix A. The λ-eigenspace for A is the set

$$E_\lambda := \{\_\_ \mid \_\_\_\_\_\_\_\_\}.$$

(4) **Theorem:** Let λ be an eigenvalue for A. The λ-eigenspace of A is a subspace of $\mathbb{R}^n$ because it is equal to the null space of ________.
(5) **Theorem:** Consequently, the λ-eigenspace of A is non-zero if and only if the matrix ________ is singular.
(6) **Theorem:** If $\{\underline{v}_1, \dots, \underline{v}_n\}$ are eigenvectors for an $n \times n$ matrix A having n different eigenvalues, then ______________
(7) **Theorem:** The eigenvalues of a matrix A are the zeroes of ______________.
(8) **Theorem:** Let $\lambda_1, \dots, \lambda_r$ be different eigenvalues for a matrix A. If $\underline{v}_1, \dots, \underline{v}_r$ are non-zero vectors such that $\underline{v}_i$ is an eigenvector for A with eigenvalue λ_i, then $\{\underline{v}_1, \dots, \underline{v}_r\}$ is ________.

Linear transformations

(1) **Definition:** Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$. A function $T : V \to W$ is a linear transformation if ________
(2) **Definition:** The range of a linear transformation $T : V \to W$ is

$$\mathcal{R}(T) := \{\_\_ \mid \_\_\_\_\_\_\_\_\}.$$

(3) **Definition:** The null space of a linear transformation $T : V \to W$ is

$$\mathcal{N}(T) := \{\_\_ \mid \_\_\_\_\_\_\_\_\}.$$

(4) **Theorem:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a unique ________ matrix A such that ________ for all ________. We call A the matrix that represents T.
(5) **Theorem:** The j^{th} column of the matrix representing T is ________.
(6) **Theorem:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Then $\dim \mathcal{R}(T) + \dim \mathcal{N}(T) =$ ________.
(7) **Theorem:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^\ell$ be linear transformations. If A represents S and B represents T, then ________ represents the composition ________.
(8) **Theorem:** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. If A represents S and B represents T, then ________ represents $S + T$.

Other topics

(1) **Definition:** Two $n \times n$ matrices A and B are similar if ______________
(2) **[4 points]**
Theorem: If A and B are similar they have the same
(a) ______________
(b) ______________
(c) ______________

(d) ______________

(3) **Definition:** An $n \times n$ matrix A is diagonalizable if ______________

(4) **Theorem:** Let A be an $n \times n$ matrix. If $\mathbb{R}^n$ has a basis consisting of ______________, then A is diagonalizable.

(5) **Theorem:** Let A be an $n \times n$ matrix. If A has ______________ different ______________ it is diagonalizable.

Part C.

Some of these questions involve a little calculation.

Scoring: Each question is worth 3 points.

(1) The matrix representing the linear transformation $T(x, y) = (-y, x - 2y)$ is ________

(2) Let S and T be the linear transformation $T(x, y) = (x + 2y, x - y)$ and $S(x, y) = (-x, 2x)$. Then $ST(x, y)$= ________.

(3) The matrix $\begin{pmatrix} 1 & a \\ -a & 0 \end{pmatrix}$ has a real eigenvalue if and only if

$$\_\_\_\_ \leq a \leq \_\_\_\_$$

(4) Let A and B be invertible $n \times n$ matrices. Simplify the following expression as much as possible:

$$(ABA)^T \left((AB)^T\right)^{-1} A^T (B^{-1}A^T)^{-1} B$$

(5) $(1, 1, 1, 1)^T$ and $(1, 2, 1, 2)^T$ are solutions to the two (different!) equations ________ and ________

(6) The vectors $(1, 1, 1, 1)^T$ and $(1, 2, 1, 2)^T$ belong to the 2-dimensional subspace of $\mathbb{R}^4$ consisting of solutions to the two equations ________

(7) The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = \begin{cases} (x + y, x - y) & \text{if } x \geq 0 \text{ and } y \geq 0 \\ (x - y, x + y) & \text{otherwise} \end{cases}$$

is not a linear transformation because ________

(8) Find two linearly independent vectors that lie on the plane in $\mathbb{R}^4$ given by the equations

$$x_1 - x_2 + x_3 - 4x_4 = 0$$
$$x_1 - x_2 + x_3 - x_4 = 0$$

(9) Is $(1, 2, 1, 0)$ a linear combination of the vectors in your answer to the previous equation? Why?

(10) Find a basis for the line $x_1 - 2x_2 = 2x_2 + x_3 = 3x_1 - x_4 = 0$ in $\mathbb{R}^4$.