Math 308 **Final** **August 22, 2003**

Instructions. There are four parts to the exam. Part A consists of questions for which you can receive partial credit. Part B consists of questions that should be answered either **T**=*true* or **F**=*false*. You get

- 2 points for each correct answer,
- -2 points for each incorrect answer, and
- 0 points for no answer at all.

The questions in Part C will be graded 0,1, or 5 points, 5 for a solid definition, 1 for almost solid and 0 otherwise. In Part D you are asked to prove some things. You should give a rigorous argument using a mixture of mathematics and words. Use grammatically correct sentences. Your mathematics should also be grammatically correct. The style of your proofs should be very similar to the kinds of proofs you find in the book, and the kind of proofs I have been writing on the board this quarter. When using the word "it" make it clear what "it" refers to. If I must struggle to discern your meaning you will get few points.

Notation: I write $\underline{u}$, $\underline{v}$, etc. for vectors. For example, $\underline{0}$ denotes the zero vector. The letter I is used to denote the $n \times n$ identity matrix.

Part A.

(1) Fill in the $*$s to give an example of a singular matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & * \end{pmatrix}$$

(2) Fill in the $*$s to give an example of a singular matrix

$$A = \begin{pmatrix} * & 1 & 0 \\ 2 & * & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(3) Fill in the $*$s to give an example of a non-singular matrix

$$A = \begin{pmatrix} * & 1 \\ 2 & * \end{pmatrix}$$

(4) If

$$A^{-1} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

compute the inverse of $Q = A^T C$ **without computing** Q.

(5) If $\underline{u}^T = (1,1,1)$, and $\underline{v}^T = (1,0,1)$, write down a vector $\underline{w}$ such that $\{\underline{u}, \underline{v}, \underline{w}\}$ is a basis for $\mathbb{R}^3$.

(6) If $\underline{u}^T = (1,1,1)$, and $\underline{v}^T = (1,0,1)$, write down a non-zero vector $\underline{w}$, different from $\underline{u}$ and $\underline{v}$ such that $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly dependent.

(7) Find all solutions to the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 \\ 3x_1 + 2x_2 + x_3 &= 10 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

(8) Suppose that $\underline{a}$ and $\underline{b}$ are two vectors in $\mathbb{R}^n$. Prove the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 0\}$ is a subspace of $\mathbb{R}^n$.

(9) Suppose that $\underline{a}$ and $\underline{b}$ are two non-zero vectors in $\mathbb{R}^n$. Prove the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 1\}$ is not a subspace of $\mathbb{R}^n$.

(10) Suppose that $\underline{a}$ and $\underline{b}$ are two non-zero vectors in $\mathbb{R}^n$. Is the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 1\} \cup \{\underline{0}\}$ a subspace of $\mathbb{R}^n$?

(11) Compute A^{-1} when

$$(AB)^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$$

(12) If the following system of equations in the unknowns x, t, w is written as a matrix equation $A\underline{x} = \underline{b}$, what are A, $\underline{x}$, and $\underline{b}$?

$$2x + 3w - 5t = -1$$
$$t + 1 - x - w = 0$$
$$2x + 3t = 5 - w + 2t$$

(13) Write down the system of linear equations you need to solve in order to find the parabola $y = ax^2 + bx + c$ passing through the points $(2, 1)$, $(-1, 3)$, $(3, 3)$.

(14) Compute the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}.$$

Check your answer—you will get no points for an incorrect answer.

(15) If

$$B = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 1 \\ 3 & 6 & 4 & 2 & 2 & 1 \\ 2 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & -3 & 6 & 1 \end{pmatrix}$$

is the augmented matrix for a system of linear equations, how many equations and how many unknowns does that system have.

(16) If

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 2 \\ 0 & 0 & 4 & -3 \end{pmatrix}$$

is the coefficient matrix for a system of linear equations, how many equations and how many unknowns does that system have?

(17) Find a reduced echelon matrix that is equivalent to the matrix A in question 14.

(18) Suppose that J is a square matrix such that $J \neq 2I$. If $J^2 = 2J$ explain why J is singular.

(19) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

(20) Check your answer to the previous question.

(21) If A is a non-singular matrix such that $A^2 = A$, what is A?

(22) What conditions on m and n ensure that a system of m homgeneous equations in n unknowns has a solution?

(23) What does it mean for a system of linear equations to be inconsistent? Note: I am not asking you how to recognize such a system, simply asking what it means.
(24) List all the subspaces of $\mathbb{R}^1$.
(25) List all the subspaces of $\mathbb{R}^2$.
(26) List all the subspaces of $\mathbb{R}^3$.
(27) List all the subspaces of the zero vector space.
(28) List all orthogonal linear transformations from the zero vector space to itself.
(29) List all linear transformations $T : \mathbb{R} \to \mathbb{R}$.
(30) List all orthogonal linear transformations $T : \mathbb{R} \to \mathbb{R}$.
(31) If $T : \mathbb{R} \to \mathbb{R}^2$ is a non-zero linear transformation, describe the image of T.
(32) If $T : \mathbb{R}^2 \to \mathbb{R}$ is a non-zero linear transformation, describe the null space of T.
(33) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Explain how to obtain the $m \times n$ matrix A such that $T(\underline{x}) = A\underline{x}$ for all $\underline{x} \in \mathbb{R}^n$.
(34) Fix $a \in \mathbb{R}$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $T(\underline{x}) = a\underline{x}$. What is the matrix A such that $T(\underline{x}) = A\underline{x}$ for all $\underline{x} \in \mathbb{R}^n$?
(35) Give a short description of $\{\frac{2}{5}x^2 \mid x \in \mathbb{R}\}$.
(36) Give a short description of $\{-\frac{2}{5}x^2 \mid x \in \mathbb{R}\}$.
(37) Give a short description of $\{\frac{2}{5}x \mid x \in \mathbb{R}\}$.
(38) Give an example of a 3×2 matrix A having rank two and nullity one.
(39) Give an example of a 3×2 matrix A having rank one and nullity two.
(40) Give an example of a 2×3 matrix A having rank two and nullity one.
(41) Give an example of a 2×3 matrix A having rank one and nullity two.
(42) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation whose range is a line, what can you say about the null space of T?
(43) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation whose null space is a line, what can you say about the range of T?
(44) Let $\{\underline{u}, \underline{v}, \underline{w}\}$ be a basis for $\mathbb{R}^3$. Show that $\{\underline{u} + 2\underline{v}, \underline{v} + 2\underline{w}, \underline{w} + 2\underline{u}\}$ is also a basis for $\mathbb{R}^3$.
(45) Let $\{\underline{u}, \underline{v}, \underline{w}\}$ be a basis for $\mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T(\underline{u}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad T(\underline{v}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad T(\underline{w}) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Suppose that $\underline{e}_1 = \underline{u} + 2\underline{v}$, $\underline{e}_2 = \underline{v} + 2\underline{w}$, and $\underline{e}_3 = \underline{w} + 2\underline{u}$, where $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Write down the matrix A such that $T(\underline{x}) = A\underline{x}$ for all $\underline{x} \in \mathbb{R}^3$.
(46) Suppose that A is an 3×4 matrix and $\underline{b} \in \mathbb{R}^4$. Suppose that the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

Write down all solutions to the equation $A\underline{x} = \underline{b}$.

(47) Suppose that A is an 3×4 matrix and $\underline{b} \in \mathbb{R}^4$. Suppose that the augmented matrix $(A \mid \underline{b})$ can be reduced to

$$\begin{pmatrix} 1 & -1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Write down all solutions to the equation $A\underline{x} = \underline{b}$.

(48) Let $\underline{v} \in \mathbb{R}^m$ and $\underline{w} \in \mathbb{R}^n$. Write V for the linear span of $\underline{v}$ and W for the linear span of $\underline{w}$. Write down a non-zero linear transformation $T : V \to W$.

(49) Write down two different bases for the linear span of $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

(50) Suppose that A is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}.$$

Explain in one grammatically correct sentence why the eigenvalues of A are 1, 2, and 3.

(51) If 2 is an eigenvalue of A write down one eigenvalue of $A^5 + 2A + I$.

(52) If A is a square matrix such that $A^2 = A$, simplify $(I - A)^2$, and $(I - A)^{15}$.

(53) Suppose that $\{\underline{u}, \underline{v}\}$ is an orthonormal set in $\mathbb{R}^n$. Let A be the $n \times n$ matrix $\underline{u}\underline{u}^T + \underline{v}\underline{v}^T$. Do a calculation to simplify A^2 as much as possible.

(54) Write down a 2×2-matrix whose only eigenvalue is 3.

(55) State the relationship between the determinant of a matrix and the question of whether that matrix is non-singular.

(56) State the theorem that tells you about the determinant of a product of matrices.

(57) Suppose that $\{\underline{w}_1, \ldots, \underline{w}_d\}$ is a basis for a subspace W of $\mathbb{R}^m$, and let n be an integer that is at least as big as m. What $m \times n$ matrix will have range equal to W?

(58) Let C be an $m \times n$ matrix and $\underline{v} \in \mathbb{R}^m$. Write down the equation one has to solve in order to find a vector $\underline{x} \in \mathbb{R}^n$ such that $B\underline{x}$ is as close as possible to $\underline{b}$. Also explain why there will be more than one such $\underline{x}$ if the null space of B is non-zero.

(59) Let W be the subspace of $\mathbb{R}^4$ spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

Write down a 4×3 matrix whose range is W.

(60) Let W be the subspace of $\mathbb{R}^4$ spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

Write down a 4×2 matrix whose range is W.

(61) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$T(\underline{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad T(\underline{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Give a formula for $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(62) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_1 - x_2 \end{pmatrix}.$$

Write down the matrix A with the property that $A\underline{x} = T(\underline{x})$ for all $\underline{x} \in \mathbb{R}^2$.

(63) If W is a subspace of $\mathbb{R}^m$ and $\underline{b} \in \mathbb{R}^m$ does not belong to W write down the equation one must solve to find $\underline{w} \in W$ that is closest to $\underline{b}$.

(64) The equation you need to solve in order to find the line $y = mt + c$ that is a least-squares best fit to the points

$$(t_1, y_1), \dots, (t_n, y_n)$$

is of the form $CD\underline{w} = B\underline{v}$ where B, C, D are certain matrices and $\underline{w}$ and $\underline{v}$ are certain vectors. Say exactly what $B, C, D, \underline{v}, \underline{w}$ are in terms of the data $m, c, t_1, \dots, t_n, y_1, \dots, y_n$.

(65) Write down the 2×2 matrix that is rotation by an angle of $\frac{1}{4}\pi$ in the counter-clockwise direction.

(66) Write down the 2×2 matrix that is rotation by an angle of θ in the counter-clockwise direction.

(67) State the theorem relating linear transformations and multiplication by matrices.

(68) State the Gram-Schmidt theorem.

(69) In your own words, say what the Gram-Schmidt theorem is used for.

(70) How does one obtain an orthonormal basis from an orthogonal basis $\{\underline{v}_1, \dots, \underline{v}_d\}$?

(71) Give an example of a 2×2 matrix that is not diagonalizable.

(72) Give an example of a 2×2 matrix such that the corresponding linear transformation is orthogonal but is not a rotation.

(73) Give an example of an orthogonal linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is not a rotation.

(74) Write down the 2×2 matrix for which the corresponding linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = 2x$. Hint: take $\underline{u}$ to be a point on the line and compute $T(\underline{u})$, take $\underline{v}$ to be a vector perpendicular to the line and compute $T(\underline{v})$, write $\underline{e}_1$ and $\underline{e}_2$ as linear combinations of $\underline{u}$ and $\underline{v}$, then compute $T(\underline{e}_1)$ and $T(\underline{e}_2)$. Proceed from there. You can ignore the hint if you want!

(75) Write down a 3×3 matrix A whose range is the intersection of the planes $2x + 3y - z = 0$ and $x + y + z = 0$.

(76) Write down a 3×3 matrix A whose range is the plane $2x + 3y - z = 0$.

(77) Write down a 3×3 matrix A whose null space is the line $x + z = y + z = 0$.

(78) Write down two linearly independent vactors lying on the plane $2x+3y-z = 0$.

(79) Let A and B be $n \times n$ matrices. Let the columns of B be $\underline{u}_1, \ldots, \underline{u}_n$. What is the third column of AB?

(80) Write down a 2×2 matrix whose null space is the line through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(81) Suppose that A and B are $n \times n$ matrices having a common eigenvector $\underline{x}$ of eigenvalues 2 and 7 respectively. Then $\underline{x}$ is also an eigenvector for $A^3B - B^2 - 4A + 2I$; what is its eigenvalue for this matrix?

(82) List all the non-singular 1×1 matrices.

Part B. True/false

(1) The inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ is $\begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$.

(2) The product MN of the matrices

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$$

is NOT defined.

(3) Let A be the product of the two matrices in the previous question (only one product makes sense so this is unambiguous). Then the 21-entry in A is -3.

(4) The rank of a system of m linear equations in n unknowns is always $\leq n$.

(5) The rank of a system of m linear equations in n unknowns is always $\leq m$.

(6) Every subset $\{\underline{v}_1, \dots, \underline{v}_5\} \subset \mathbb{R}^6$ is linearly dependent.

(7) Every subset $\{\underline{v}_1, \dots, \underline{v}_5\} \subset \mathbb{R}^4$ is linearly dependent.

(8) Every subset $\{\underline{v}_1, \dots, \underline{v}_4\} \subset \mathbb{R}^4$ is linearly independent.

(9) A homogeneous linear system of 15 equations in 16 unknowns always has a non-zero solution.

(10) A homogeneous linear system of 16 equations in 16 unknowns always has a non-zero solution.

(11) A homogeneous linear system of 17 equations in 16 unknowns always has a non-zero solution.

(12) A homogeneous linear system of 16 equations in 16 unknowns always has a unique non-zero solution.

(13) The matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

has an inverse.

(14) An $n \times n$ matrix is singular if and only if A^T is singular.

(15) An $n \times n$ matrix is singular if and only if the rows of A^T are linearly independent.

(16) The rank of a matrix A is equal to the dimension of the linear span of its columns.

(17) $3I - 5I^2 + 2I^{-1}$ is singular.

(18) $3I - 5I^2 + 7I^{-1}$ is singular.

(19) Suppose that $n > 1$. There are vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$ such that $\underline{v}^T\underline{u} = 3$.

(20) Suppose that $n > 1$. There are vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$ such that $\underline{v}\underline{u}^T = 3I$.

(21) The set $W = \{\underline{x} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 \mid x_1 - x_2 = x_3 + x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

(22) The set $W = \{\underline{x} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 \mid x_1, x_2, x_3, x_4^2 \text{ are whole numbers}\}$ is a subspace of $\mathbb{R}^4$.

(23) The set of all solutions to a system of homogeneous equations is always a subspace.

(24) The set of all solutions to the system of equations

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 &= 1 \\ 2x_1 - 2x_2 + 3x_3 &= 2 \\ 5x_1 + 3x_2 + 2x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 3 \end{aligned}$$

is a subspace of $\mathbb{R}^3$.

(25) The set of all solutions to the system of equations

$$\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 2x_1 - 2x_2 + 3x_3 &= 2 \\ 3x_1 + 4x_2 + 2x_4 &= 3 \end{aligned}$$

is a subspace of $\mathbb{R}^4$.

(26) The set of all solutions to the system of equations

$$\begin{aligned} 5x_1 + 3x_2 + 2x_4 &= 0 \\ 2x_1 + 4x_2 - 3x_3 &= 0 \\ 2x_1 - 2x_2 + 3x_3 &= 0 \\ 3x_1 + 4x_2 + 2x_3 &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^4$.

(27) Suppose that $n > 1$. There are vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$ such that $\underline{v}^T \underline{u} = 3I$.

(28) Suppose that $n > 1$. There are vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$ such that $\underline{v}\underline{u}^T = 3$.

(29) If $\langle \underline{u}, \underline{v}, \underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle$, then $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$.

(30) If $\underline{u}$ is a linear combination of $\underline{v}$ and $\underline{w}$, then $\langle \underline{u}, \underline{v}, \underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle$.

(31) No matter what values the $*$s have the matrix

$$\begin{pmatrix} 1 & 2 & * & * \\ 0 & 2 & * & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

is singular.

(32) No matter what values the $*$s have the matrix

$$\begin{pmatrix} 1 & 2 & * & * \\ 0 & 2 & * & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

is non-singular.

(33) No matter what values the $*$s have the matrix

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 0 & * & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

is non-singular.

(34) The null space of an $m \times n$ matrix is contained in $\mathbb{R}^m$.

(35) The null space of an $m \times n$ matrix is contained in $\mathbb{R}^n$.

(36) The range of an $m \times n$ matrix is contained in $\mathbb{R}^n$.
(37) The range of an $m \times n$ matrix is contained in $\mathbb{R}^m$.
(38) If A is an $m \times n$ matrix, then $\operatorname{rank} A + \operatorname{nullity} A = n$.
(39) Suppose that $\underline{a}$ and $\underline{b}$ are two non-zero vectors in $\mathbb{R}^n$. Then either $\underline{a}$ or $\underline{b}$ belongs to the set $W = \{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{b}^T\underline{x} = 0\}$.
(40) Suppose that V and W are subspaces of $\mathbb{R}^n$. If $V \subset W$, then $V \cup W$ is a subspace of $\mathbb{R}^n$.
(41) The 1-dimensional subspaces of $\mathbb{R}^n$ are the lines through the origin.
(42) The 2-dimensional subspaces of $\mathbb{R}^n$ are the planes through the origin.
(43) If L is a line in $\mathbb{R}^m$ and L' is a line in $\mathbb{R}^n$, then there is a non-zero linear transformation $T : L \to L'$.
(44) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range is a plane and whose null space is a line.
(45) There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose range is a line and whose null space is a line.
(46) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ whose range is a plane and whose null space is a plane.
(47) There is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ whose range is a line and whose null space is a plane.
(48) There is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose range is a line and whose null space is a plane.
(49) The null space of an $m \times n$ matrix is a subspace.
(50) If $0 \neq \underline{u} \in \mathbb{R}^n$ and $n = 1$, then there is a non-zero vector $\underline{v} \in \mathbb{R}^n$ such that $\underline{u}.\underline{v} = 0$.
(51) If $0 \neq \underline{u} \in \mathbb{R}^n$ and $n > 1$, then there is a non-zero vector $\underline{v} \in \mathbb{R}^n$ such that $\underline{u}.\underline{v} = 0$.
(52) Let A and B be $n \times n$ matrices. If AB is singular, so is A.
(53) Let A and B be $n \times n$ matrices. If A is singular, so is BA.
(54) Let A and B be $n \times n$ matrices. If AB is non-singular, both A and B are non-singular.
(55) Let A be an $n \times n$ matrix. If the rows of A are linearly dependent, then A is non-singular.
(56) Let A be an $n \times n$ matrix. If A is row equivalent to I, then A is non-singular.
(57) The only non-singular 1×1 matrix is 0.
(58) The only singular 1×1 matrix is 1.
(59) If A and B are non-singular $n \times n$ matrices, so is $A + B$.
(60) If A and B are $m \times n$ matrices such that B can be obtained from A by elementary row operations, then A can also be obtained from B by elementary row operations.
(61) Let A be an $n \times n$ matrix. If A is non-singular, then $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^n$.
(62) There is a matrix A whose inverse is $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$.
(63) There is a matrix A whose inverse is $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

(64) The following set is a basis for $\mathbb{R}^4$:

$$\left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\}.$$

(65) The following set spans $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}4\\-3\\3\end{pmatrix}, \begin{pmatrix}5\\0\\3\end{pmatrix} \right\}.$$

(66) The following set is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}.$$

(67) The following spans $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}.$$

(68) If

$$A^{-1} = \begin{pmatrix}3 & 1\\0 & 2\end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix}2 & 3 & 1\\1 & 0 & 2\end{pmatrix}$$

there is a matrix C such that $CA = E$.

(69) If

$$A^{-1} = \begin{pmatrix}3 & 1\\0 & 2\end{pmatrix} \qquad \text{and} \qquad E = \begin{pmatrix}2 & 3 & 1\\1 & 0 & 2\end{pmatrix}$$

there is a matrix B such that $AB = E$.

(70) Any linearly independent subset $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5\}$ of $\mathbb{R}^5$ is a basis for $\mathbb{R}^5$.

(71) If an $n \times n$ matrix has an inverse, then the columns of its inverse are a basis for $\mathbb{R}^n$.

(72) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ and A is a non-singular $n \times n$ matrix. then $\{A\underline{v}_1, A\underline{v}_2, A\underline{v}_3\}$ is linearly independent.

(73) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$ and A is any $m \times n$ matrix, then $\{A\underline{v}_1, A\underline{v}_2, A\underline{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^m$.

(74) The 21-entry in the product of $\begin{pmatrix}-1 & 2\\-1 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 2\\3 & 0\\-1 & 4\end{pmatrix}$ is 4.

(75) The set $\{\frac{2}{3}x \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}$.

(76) The set $\{\frac{2}{3}x \mid x \in \mathbb{R}\}$ is not equal to $\mathbb{R}$.

(77) $\{0\}$ and $\mathbb{R}$ are the only subspaces of $\mathbb{R}$.

(78) $\{0\}$ and $\mathbb{R}^2$ are the only subspaces of $\mathbb{R}^2$.

(79) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\underline{x}) = A\underline{x}$ where

$$A = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

is an orthogonal transformation.

(80) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\underline{x}) = A\underline{x}$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is an orthogonal transformation.

(81) A non-zero matrix always has a non-zero eigenvalue.

(82) If 0 is an eigenvalue for A, then A is singular.

(83) Let $\underline{u}$ and $\underline{v}$ be non-zero vectors in $\mathbb{R}^n$ and let U and V be the subspaces they span. The formula $T(a\underline{u}) = \frac{1}{2}a\underline{v}$ defines a linear transformation $T : U \to V$.

(84) If $T : \mathbb{R} \to \mathbb{R}^n$ is a non-zero linear transformation then the image of T is always a line.

(85) If L is a line in $\mathbb{R}^n$ there is a non-zero linear transformation $T : \mathbb{R} \to L$.

(86) There is a linear transformation $\mathbb{R} \to \mathbb{R}^3$ whose range is a plane.

(87) The points $(x, y, z)^T$ in $\mathbb{R}^3$ such that $2x + z = 0$ form a subspace of $\mathbb{R}^3$.

(88) The points $(x, y, z)^T$ in $\mathbb{R}^3$ such that $2x - y + z = 5$ form a subspace of $\mathbb{R}^3$.

(89) If U and V are subspaces of $\mathbb{R}^n$ so is $U \cup V$.

(90) If U and V are subspaces of $\mathbb{R}^n$ so is $U + V$.

(91) If U and V are subspaces of $\mathbb{R}^n$ so is $U \cap V$.

(92) The only subspaces of $\mathbb{R}^1$ are $\{0\}$ and $\mathbb{R}^1$.

(93) The only subspaces of $\mathbb{R}^2$ are $\{0\}$, $\mathbb{R}^2$, and $\{a\underline{v} \mid a \in \mathbb{R}\}$ for each non-zero $\underline{v} \in \mathbb{R}^2$.

(94) The subspaces of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $-\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ are different.

(95) The subspaces of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ are different.

(96) If $\underline{u}$ is a non-zero element of $\mathbb{R}^n$, the sets $\{a\underline{u} \mid a \in \mathbb{R}\}$ and $\{b\underline{u} \mid b \in \mathbb{R}\}$ are equal.

(97) If $\underline{u}$ and $\underline{v}$ are elements of $\mathbb{R}^n$, then the linear span of $\underline{u}$ and $\underline{v}$ is equal to $\{3a\underline{u} - 5b\underline{v} \mid a, b \in \mathbb{R}\}$.

(98) If $\underline{u}$ and $\underline{v}$ are linearly independent elements of $\mathbb{R}^n$, then $\{3a\underline{u} - 5b\underline{v} \mid a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$ of dimension two.

(99) If $\underline{u}$ and $\underline{v}$ are linearly independent elements of $\mathbb{R}^n$ and $\underline{w} = \underline{u} + \underline{v}$, then $\{3a\underline{u} - 5b\underline{v} + 7c\underline{w} \mid a, b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$ of dimension three.

(100) If $\underline{v} \in \mathbb{R}^m$ and $\underline{w} \in \mathbb{R}^n$, then the rule $T(a\underline{v}) = \frac{1}{2}a\underline{w}$ is a linear transformation.

(101) If $\{\underline{u}_1, \dots, \underline{u}_k\}$ is a linearly independent set and A is any matrix, then $\{A\underline{u}_1, \dots, A\underline{u}_k\}$ is also linearly independent set.

(102) If $\{\underline{u}_1, \dots, \underline{u}_k\}$ is a linearly independent set and A is any invertible matrix, then $\{A\underline{u}_1, \dots, A\underline{u}_k\}$ is also linearly independent set.

(103) The set $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}\}$ is orthogonal.

(104) If -1 is an eigenvalue for a matrix A, then the matrix $A^2 + 2A + I$ must be singular.

(105) There is a 3×3 matrix with eigenvalues 1, 2, and 3 having corresponding eigenvectors

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \begin{pmatrix}3\\2\\1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}4\\4\\4\end{pmatrix}.$$

(106) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1 + x_2\\x_2\\x_1 - x_2\end{pmatrix}.$$

The rank of T is 1.

(107) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1 + x_2\\x_2\\x_1 - x_2\end{pmatrix}.$$

The nullity of T is 1.

(108) Similar matrices have the same eigenvectors.

(109) The matrix

$$\begin{pmatrix}1 & 2 & 3 & 4\\0 & -1 & 3 & 4\\0 & 0 & 1 & 4\\0 & 0 & 0 & -1\end{pmatrix}$$

does not have a characteristic polynomial.

(110) The characteristic polynomial of

$$\begin{pmatrix}1 & 2 & 3 & 4\\0 & -1 & 3 & 4\\0 & 0 & 1 & 4\\0 & 0 & 0 & -1\end{pmatrix}$$

is $(1+t)^2(1-t)^2$.

(111) The eigenvalues of

$$\begin{pmatrix}1 & 2 & 3 & 4\\0 & 2 & 3 & 4\\0 & 0 & 3 & 4\\0 & 0 & 0 & 4\end{pmatrix}$$

are 1, 2, 3, and 4.

(112) $\underline{e}_1, \underline{e}_2, \underline{e}_3$, and $\underline{e}_4$ are eigenvectors for

$$\begin{pmatrix}1 & 2 & 3 & 4\\0 & 2 & 3 & 4\\0 & 0 & 3 & 4\\0 & 0 & 0 & 4\end{pmatrix}.$$

(113) If $\underline{u}$ and $\underline{v}$ are elements of $\mathbb{R}^n$, then the linear span of $\underline{u}$ and $\underline{v}$ and $3\underline{u} - 6\underline{v}$ has dimension three.

(114) $\det(A) = -\det(A^T)$.

(115) If 3 is an eigenvalue of A, then 27 is an eigenvalue of A^3.

(116) If 3 is an eigenvalue of A, then 27 is an eigenvalue of $9A$.

(117) If 3 is an eigenvalue of A, then 27 is an eigenvalue of $A + 26I$.

(118) If 3 is an eigenvalue of A, then -3 is an eigenvalue of $I - A$.
(119) If 3 is an eigenvalue of an invertible matrix A, it is also an eigenvalue of A^{-1}.
(120) A 2×3 matrix has no eigenvalues at all.
(121) The formula
$$T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b - 1 \\ a \end{pmatrix}$$
defines a linear transformation.
(122) If 0 is an eigenvalue of A then A is singular.
(123) If D is a diagonal matrix and C an invertible matrix of the same size then CDC^{-1} is diagonalizable.
(124) Every diagonal matrix is diagonalizable.
(125) The only matrix that is similar to $3I$ is $3I$ itself.
(126) If A and B are diagonalizable matrices of the same size so is $A + B$.
(127) If A and B are diagonal matrices of the same size, then $2A + 3B$ and $A^2 - 3B^2$ are always diagonal.
(128) A sum of orthogonal matrices is orthogonal.
(129) The eigenvalues of $S^{-1}AS$ are the same as those of A.
(130) The eigenvalues of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ are 1, 2, and 3.
(131) If W is a subspace of $\mathbb{R}^m$ and n is an integer $\geq m$, then there is an $m \times n$ matrix whose range is W.
(132) If W is a subspace of $\mathbb{R}^m$ and n is any integer, then there is an $n \times m$ matrix whose range is W.
(133) If W is a subspace of $\mathbb{R}^m$ and n is an integer $\geq m$, then there is an $n \times m$ matrix whose range is W.
(134) If W is a subspace of $\mathbb{R}^m$ and n is any integer, then there is an $m \times n$ matrix whose range is W.
(135) If W is a subspace of $\mathbb{R}^m$ and n is any integer, then there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ whose range is W.
(136) Suppose that $T : \mathbb{R}^m \to \mathbb{R}^n$ has eigenvalues λ and μ, and let $\underline{u}$ and $\underline{v}$ be eigenvectors for A with eigenvalues λ and μ respectively. Then $\underline{u} + \underline{v}$ is an eigenvector for A with eigenvalue $\lambda + \mu$.
(137) Suppose that $T : \mathbb{R}^m \to \mathbb{R}^n$ has eigenvalues $\lambda \neq 0$, $\mu \neq 0$, and $\lambda + \mu$. Suppose that $\underline{u}$, $\underline{v}$, and $\underline{w}$ are eigenvectors for A with eigenvalues λ, μ, and $\lambda + \mu$ respectively. Then $\{\underline{u}, \underline{v}, \underline{w}\}$ is lineraly independent.
(138) If an $n \times n$ matrix has n distinct eigenvalues, then its columns form a basis for $\mathbb{R}^n$.
(139) An $n \times n$ matrix having n distinct eigenvalues is ALWAYS diagonalizable.

Part C.

(1) Define what it means for a matrix to be in echelon form.
(2) Define what it means for a matrix to be in reduced echelon form.
(3) Define what it means for a matrix to be invertible.
(4) Define the inverse of a matrix.
(5) Define the rank of a system of linear equations.
(6) Define linear independence.
(7) Define linear dependence.
(8) Define singular.
(9) Define non-singular.
(10) Define a basis.
(11) Define dimension.
(12) Define the range of a matrix.
(13) Define the null space of a matrix.
(14) Define the row space of a matrix.
(15) Define the column space of a matrix.
(16) Define subspace.
(17) Define linear span.
(18) Define the nullity of a matrix.
(19) Define the rank of a matrix.
(20) What are the three elementary row operations on a matrix?
(21) What is the definition of equivalence for systems of linear equations?
(22) Define eigenvalue.
(23) Define eigenvector.
(24) Define what it means for two non-zero vectors to be orthogonal.
(25) Let $\underline{u}$ be a non-zero vector in $\mathbb{R}^n$. Define the line through $\underline{u}$ and 0.
(26) Define a linear transformation.
(27) Define an orthonormal basis for a subspace W of $\mathbb{R}^n$.
(28) Define an orthogonal matrix.
(29) Define the null space of a linear transformation $T : V \to W$.
(30) Define the range of a linear transformation $T : V \to W$.
(31) Define what it means for two matrices to be similar.
(32) Define the determinant of an $n \times n$ matrix—you can assume that the determinant of an $(n-1) \times (n-1)$ has already been defined and you should express the determinant of an $n \times n$ matrix in terms of the determinant of various $(n-1) \times (n-1)$ matrices.
(33) Define the characteristic polynomial of a matrix.
(34) Define a plane in $\mathbb{R}^n$.
(35) Define a line in $\mathbb{R}^n$.
(36) Define the eigenvalues of a matrix.
(37) Define the eigenvectors of a matrix.

Part D.

(1) Prove that a homogeneous system of m linear equations in n unknowns always has a non-zero solution if $m < n$.

(2) Let $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ be a set of non-zero vectors in $\mathbb{R}^n$ such that $\underline{v}_i^T \underline{v}_j = 0$ whenever $i \neq j$. Show that the set is linearly independent. (*Hint:* write $\underline{0} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3$ and compute $\underline{0}^T\underline{0}$.)

(3) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are any vectors in $\mathbb{R}^n$, show that $\{\underline{v}_1 - \underline{v}_2, \underline{v}_2 - \underline{v}_3, \underline{v}_3 - \underline{v}_1\}$ is linearly dependent.

(4) If $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a linearly dependent set in $\mathbb{R}^n$, show that $\{\underline{v}_1 + \underline{v}_2, \underline{v}_2 + \underline{v}_3, \underline{v}_3 + \underline{v}_1\}$ is linearly dependent.

(5) Prove that a set that spans a subspace $W \subset \mathbb{R}^n$ is a basis if and only if it is linearly independent.

(6) Prove that a system of linear equations having more than one solution has infinitely many solutions.

(7) Prove that the set of all solutions to the equation $A\underline{x}$ is a subspace.

(8) If W is the set of solutions to the equation $A\underline{x} = \underline{0}$ and $\underline{u}$ is a solution to the equation $A\underline{x} = \underline{b}$ then $\underline{u} + W$ is the set of all solutions to the equation $A\underline{x} = \underline{b}$. (Say what the notation $\underline{u} + W$ means in your proof.

(9) If U and V are subspaces of $\mathbb{R}^n$ show that $U + V$ is also a subspace.

(10) Prove that $(AB)^T = B^T A^T$.

(11) If A is a non-singular matrix and $BA = CA$, prove that $B = C$.

(12) Prove that the equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.

(13) Prove that if $n > m$ then every subset $\{\underline{v}_1, \dots, \underline{v}_n\}$ of $\mathbb{R}^m$ is linearly dependent.

(14) Prove that a matrix can have at most one inverse.

(15) If I and J are both $n \times n$ identity matrices, prove that $I = J$. You should start your proof by giving the definition of the identity matrix.

(16) If A is a non-singular $n \times n$ matrix, the equation $A\underline{x} = \underline{b}$ has a unique solution for every $\underline{b} \in \mathbb{R}^n$. You can use the fact that a matrix has an inverse if and only if it is non-singular (although in the course we prove these two results in the reverse order!).

(17) Prove that the union of two distinct planes in $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^3$.

(18) Let L be a line through the origin and P a plane through the origin in $\mathbb{R}^5$. Suppose that L is not contained in P. Show that $L \cup P$ is not a subspace of $\mathbb{R}^5$. What happens if $L \subset P$?

(19) Prove that the range of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

(20) If A and B are matrices such that the product AB makes sense prove that the nullspace of AB is contained in the null space of B.

(21) If A and B are matrices such that the product AB makes sense prove that the range of AB is contained in the range space of A.

(22) If A and B are matrices of the same size prove that the null space of $A+B$ contains the intersection of the nullspaces of A and B. Is the same true with "range" in place of "nullspace"?

(23) If $n > 1$ and $\underline{u}, \underline{v} \in \mathbb{R}^n$, prove that $\underline{u}\underline{v}^T$ is singular. What happens if $n = 1$?

(24) Suppose that $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a linearly independent set of vectors in $\mathbb{R}^n$. Let A be a non-singular $n \times n$ matrix. Show that $\{A\underline{v}_1, A\underline{v}_2, A\underline{v}_3\}$ is linearly independent.

(25) Suppose that $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$. Let A be any $m \times n$ matrix. Show that $\{A\underline{v}_1, A\underline{v}_2, A\underline{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^m$.

(26) If A is a square matrix such that $A^{13} = 0$, show that A is singular.

(27) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation whose range is a line, prove that the null space of T is a plane.

(28) Prove that $||\underline{x}.\underline{y}|| \leq ||\underline{x}|| \, ||\underline{y}||$ by using the fact that $\underline{x} - c\underline{y}||^2 \geq 0$ for all $c \in \mathbb{R}$. Hint: do the case when $\underline{y} = 0$ first, then assume $\underline{y} \neq 0$ and substitute $c = (\underline{x}.\underline{y})/||\underline{y}||^2$ into the inequality.

(29) Let $\underline{x}, \underline{y} \in \mathbb{R}^n$ prove that $||\underline{x} + \underline{y}|| \leq ||\underline{x}|| + ||\underline{y}||$ by expanding $||\underline{x} + \underline{y}||$ and using the fact that $||\underline{x}.\underline{y}|| \leq ||\underline{x}|| \, ||\underline{y}||$.

(30) Show that two non-zero vectors $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^n$ are linearly independent if and only if the line through 0 and $\underline{u}$ is different from the line through 0 and $\underline{v}$.

(31) If $\{\underline{u}, \underline{v}, \underline{w}\}$ is a basis for W, show that $\{2\underline{u}, -3\underline{v}, 5\underline{w}\}$ is also a basis for W.

(32) If $\underline{u}$ is a multiple of $\underline{v}$ show that $\{\underline{u}, \underline{v}\}$ is linearly dependent.