Math 308 **The language of sets** **October 2, 2009**

Every area of human enquiry develops its own technical language. Those languages are often impenetrable to outsiders. There are good reasons for this. We want precise terminology. We want words that will distinguish things that are different. Those things may appear the same to an outsider but the experts in the field need to distinguish them.

For example, there are wonderful words to distinguish colors with a precision that most of us don't need. Violet, lavender, indigo, lavender indigo, patriarch, purple, lilac, mauve (named after the Dutch painter Anton Mauve), fandango, heliotrope, plum, royal purple, and many more, are all variations on what you might simply call purple or violet. You don't need to make these fine distinctions but people in the fashion industry, design, painting, etc., do.

Math is no exception in this regard. In Math 308 I want to use some of that language, the language of sets, and this note is a brief guide to some of it. Set-theoretic notation and language permeates mathematics from 300-level courses onwards so if you want to do any math courses beyond this one you better get used to it.

1. Sets

A set is a collection of things. Not necessarily mathematical things: there is the set of US presidents, past and present, the set of people who are alive, the set of living women, the set of living men, the set of living mothers, the set of living daughters, the set of living sons, and so on. Mathematical examples include the set of whole numbers (called integers), the set of prime numbers, the set of even numbers, the set of odd numbers, the set of squares, and so on.

Usually we use an upper case letter to denote a set and when specifying the set we use curly parentheses. For example, we could write

$$
\begin{aligned}
A &= \{\text{living mothers}\}\\
P &= \{\text{prime numbers}\}\\
E &= \{\text{even numbers}\}\\
O &= \{\text{odd numbers}\}
\end{aligned}
$$

Some sets have special names and we will use those names/symbols. For example,

$$
\begin{aligned}
\mathbb{N} &= \{\text{the natural numbers}\} = \{0,1,2,3, \ldots \}\\
\mathbb{Z} &= \{\text{integers}\} = \{\text{whole numbers}\}\\
\mathbb{Q} &= \{\text{rational numbers}\} = \{\text{fractions}\}\\
\mathbb{R} &= \{\text{real numbers}\}\\
\mathbb{C} &= \{\text{complex numbers}\}
\end{aligned}
$$

1.1. **Elements.** The things that belong to a set are called its elements. For example, 3 is an element of the set P above because it is a prime number. We often use lower case letters for the elements of a set. When x is an element of a set X, we write $x \in X$, and read this as x *is an element of* X or x *belongs to* X or X *contains* x or, simply, x *is in* X.

We use the symbol $\varnothing$ to denote the set having no elements at all and call it the empty set. It might seem a little odd to talk about the empty set and to have a special symbol for it, but think of the parallel with the symbol we use for zero, 0. It is quite interesting to read about the history of zero on Wikipedia. Check it out!

1.2. **Containment.** We say that a set X is contained in a set Y if every element of X is an element of Y. More formally, we say X is a subset of Y if it is contained in Y and write

$$X \subset Y$$

to denote this situation. Thus the symbol "$\subset$" is read as *is a subset of* or *is contained in*. For example,

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Of course, we also have $X \subset X$ for every set X. And $\varnothing \subset X$ for all sets X.

A set is completely determined by its elements. If two sets have the same elements they are equal. If they don't have the same elements the sets are not the same. Often in mathematics one wants to show two sets X and Y, often described in different ways, are the same. It is quite common to prove the equality of two sets X and Y by proving that $X \subset Y$ and $Y \subset X$. In other words, $X = Y$ if and only if $X \subset Y$ and $Y \subset X$

1.3. **Intersection and union.** There are two important operations on sets, intersection and union. They bear some resemblance to addition and multiplication of numbers.

The intersection of sets X and Y is the set denoted $X \cap Y$ consisting of the elements that are in both X and Y. Using our examples above, we have, for example,

$$P \cap E = \{2\}$$

because 2 is the only even prime number. You may know that there are infinitely many primes, so $P \cap O$ is an infinite set, i.e., it has infinitely many elements. Notice we wrote $P \cap E = \{2\}$ *not* $P \cap E = 2$. There is an important difference—the intersection of two sets is a set, and the number 2 is different from the set whose only element is 2.

The union of sets X and Y is the set denoted $X \cup Y$ consisting of the elements that are in either X or Y. For example, $\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}$. Likewise,

$$O \cup E = \mathbb{Z}$$

because every number is either even or odd. Notice that

$$O \cap E = \varnothing$$

because there are no numbers that are both even and odd.

If X is a subset of Y it is clear that $X \cap Y = X$ and $X \cup Y = Y$.

1.4. **Some basic properties.** You already know the basic properties of the arithmetic operations $+$ and $\times$. For example, there are the associative rules,

$$a + (b + c) = (a + b) + c \qquad \text{and} \qquad a \times (b \times c) = (a \times b) \times c$$

which implies that the expressions $a + b + c$ and $a \times b \times c$ are unambiguous. You also know that

$$a + b = b + a \qquad \text{and} \qquad a \times b = b \times a.$$

Slightly more sophisticated is the distributive rule

$$a \times (b + c) = a \times b + a \times c$$

which involves both operations, addition and multiplication. And zero has two special properties

$$0 \times a = 0 \qquad \text{and} \qquad 0 + a = a$$

for all numbers a.

There are analogous properties for the set operations $\cup$ and $\cap$. For any sets X, Y, and Z,

$$\begin{aligned} X \cup Y &= Y \cup X \\ X \cap Y &= Y \cap X \\ X \cup (Y \cup Z) &= (X \cup Y) \cup Z \\ X \cap (Y \cap Z) &= (X \cap Y) \cap Z \\ X \cap (Y \cup Z) &= (X \cap Y) \cup (X \cap Z) \\ X \cup (Y \cap Z) &= (X \cup Y) \cap (X \cup Z) \\ \varnothing \cap X &= \varnothing \\ \varnothing \cup X &= X. \end{aligned}$$

Mathematicians like it when there are similarities like this between the arithmetic operations $+$ and $\times$ and the set operations $\cup$ and $\cap$. Of course, there are some significant differences too. For example, $X \cap X = X \cup X = X$. One other difference is that there are *two* distributive laws for $\cap$ and $\cup$ but only one distributive law for $+$ and $\times$ ($+$ does not distribute across $\times$).

All these properties are easy to check. The only ones that might require some care are the distributive laws. You should try to prove them yourself. The strategy to use is that I mentioned earlier for showing two sets are equal: show each is a subset of the other.

There is also a similarity between $\subset$ and $\leq$. It is a good exercise for you to write down some of the similarities.

1.5. **One more symbol, "such that".** We already mentioned the set of rational numbers $\mathbb{Q}$. Of course, you already know what fractions are but let's define them using set notation:

$$\mathbb{Q} = \left\{ \tfrac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \right\}. \tag{1}$$

The vertical symbol $\mid$ should be read as "such that". Thus, the mathematical sentence (1) should be read as follows: $\mathbb{Q}$ is the set of all numbers $\frac{a}{b}$ such that a and b are integers and b is not zero.

Another common notation for "such that" is the colon. Using the colon the above sentence would be

$$\mathbb{Q} = \left\{ \tfrac{a}{b} \ : \ a, b \in \mathbb{Z},\ b \neq 0 \right\}. \tag{2}$$

In Chapter 3 of the Linear Algebra book you will find lots of sets where the colon notation is used in the definition of the set.

1.6. **A self-test.** How many elements in $\varnothing$? How many elements in $\{1\}$? How many elements in $\{1, \{1\}\}$? How many elements in $\{1, 2, 3\}$? How many elements in $\{1, 2, 3, 4, 4\}$? How many elements in $\{\varnothing, \{1\}, \{\{1\}\}, 1, 0\}$? The answers are 0,1,2,3,4,5. Do you understand why?

1.7. **Potential confusion.** Some people confuse the symbols $\in$ and $\subset$. For example, $2 \in \mathbb{Z}$, but $\{2\} \subset \mathbb{Z}$; the statement $2 \in \mathbb{Z}$ is the same as the statement "2 is an integer"; the statement $\{2\} \subset \mathbb{Z}$ is the same as the statement "the set whose only element is 2 is a subset of the set of integers". It is correct to write $\{2, 3\} \subset \mathbb{Z}$ but not correct to write $\{2, 3\} \in \mathbb{Z}$.

A more mundane example. Let F be the set of all fruits. On my desk I have an apple and an orange. The apple is an element of F. The orange is an element of F. The set consisting of the apple and orange on my desk is a subset of F.

Let $A = \{1, \{1\}\}$. Then $1 \in A$ and $\{1\} \in A$ and $\{1\} \subset A$, but it is not true that $1 \subset A$. Do you get it? If so, great!