Chapter 3

Prime ideals and Spec

The theory of prime ideals follows the commutative theory quite closely with some variations because our rings need not be commutative.

In section 3.1 we define prime ideals; the definition is in terms of ideals rather than elements. The annihilator of a simple module is prime. In an artinian ring, prime ideals are maximal, they are in bijection with the isomorphism classes of simple modules, and there is a finite number of them. Section 3.2 introduces the spectrum, $\operatorname{Spec} A$, of a ring and its Zariski topology; for non-commutative rings, the rule $A \to \operatorname{Spec} A$ is not a functor but if we only allow *central* ring homomorphisms it becomes a functor.

The inclusion $Z(A) \to A$ of the center is a central homomorphism and the map $\operatorname{Spec} A \to \operatorname{Spec} Z(A)$ is examined in section 3.3; if A is a fnite module over its center this map is surjective. The remaining sections of this chapter concern irreducible components, minimal primes, and the radical $\sqrt{J}$ of an ideal.

3.1 Prime ideals

Definition 1.1 An ideal $\mathfrak{p}$ in a ring A is prime if whenever I and J are two-sided ideals such that $IJ \subset \mathfrak{p}$, then either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$. The set of prime ideals in A is denoted by

$$\operatorname{Spec} A$$

and called the spectrum of A.

A ring A is prime if $\{0\}$ is a prime ideal. ◊

Check that an ideal $\mathfrak{p}$ is prime if and only if the quotient $A/\mathfrak{p}$ is a prime ring. Indeed, if I is a two-sided ideal of A there is a natural bijection

$$\begin{aligned} \operatorname{Spec}(A/I) &\longleftrightarrow \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I\} \\ \mathfrak{p}/I &\longleftrightarrow \mathfrak{p}. \end{aligned}$$

Lemma 1.2 *Let $\mathfrak{p}$ be a two-sided ideal of A. The following conditions on $\mathfrak{p}$ are equivalent:*

1. *$\mathfrak{p}$ is prime;*

2. *for all right ideals I and J such that $IJ \subset \mathfrak{p}$, either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$;*
3. *for all left ideals I and J such that $IJ \subset \mathfrak{p}$, either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$;*
4. *whenever $x, y \in A$ are such that $xAy \subset \mathfrak{p}$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.*

As you know, a commutative ring is prime if and only if it is a domain, but for non-commutative rings condition (4) in the previous lemma is the appropriate substitute—it replaces the condition that $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Matrix rings $M_n(k)$ over a field k provide the simplest examples of prime rings that are not domains—$\{0\}$ is prime because $M_n(k)$ has no two-sided ideals other than $\{0\}$ and itself. More generally, the following holds.

Lemma 1.3 *Every maximal two-sided ideal is prime.*

Lemma 1.4 *The annihilator of a simple module is prime.*

Proof. Let M be a simple left A-module. If $xAy \subset \operatorname{Ann} M$ then $xAyM = 0$. Because AyM is a submodule of M it is either zero, in which case $y \in \operatorname{Ann} M$, or equal to M, in which case $x \in \operatorname{Ann} M$. □

A two-sided ideal is **primitive** if it is the annihilator of a simple module. A maximal ideal $\mathfrak{m}$ is primitive, because any simple $A/\mathfrak{m}$-module has annihilator $\mathfrak{m}$ when viewed as an A-module. Exercise 1 shows that a primitive ideal need not be maximal.

However, for a ring that is finite over its center, the primitive ideals coincide with the maximal ideals.

Proposition 1.5 *If R is a left (or right) artinian ring, then*

$$\{\textit{prime ideals}\} = \{\textit{maximal ideals}\}.$$

Proof. We observed that every maximal ideal is prime in Lemma 1.3. Let $\mathfrak{p}$ be a prime ideal of R. We must show that $R/\mathfrak{p}$ is simple, so we may replace R by $R/\mathfrak{p}$, and it then suffices to show that a prime artinian ring has no ideals other than itself and zero.

Let R be a prime left artinian ring. Let J denote the intersection of all the maximal left ideals of R. Let M be a simple left R-module. If $0 \neq m \in M$, then $\operatorname{Ann}(m)$ is a maximal left ideal so $Jm = 0$. Hence $JM = 0$. If J is non-zero, then it has a simple left submodule because R is artinian, say I. Now $JI = 0$; but $\{0\}$ is a prime ideal of R so either $J = 0$ or $I = 0$; we conclude that $J = 0$.

Thus the intersection of all maximal left ideals of R is zero, say

$$0 = \bigcap_{\lambda \in \Lambda} I_\lambda$$

where I_λ runs through all the maximal left ideals of R. But R is artinian, so there is a *finite* collection of maximal left ideals such that $0 = I_1 \cap \cdots \cap I_n$. The diagonal map

$$R \to R/I_1 \oplus \cdots \oplus R/I_n$$

is injective; thus R is a submodule of a semisimple module, and is therefore semisimple itself.

A semisimple ring is, by the Artin-Wedderburn Theorem, isomorphic to a direct product/sum of matrix algebras over division rings, say

$$R \cong M_{d_1}(D_1) \oplus \cdots M_{d_r}(D_r).$$

But this ring is prime only if $r = 1$, in which case it is also simple. □

Lemma 1.6 *Over a left artinian ring, two simple left modules are isomorphic if and only if their annihilators coincide.*

Proof. Let R be a left artinian ring and M_1 and M_2 simple left R-modules. Isomorphic modules have the same annihilators, so we must show that if $\operatorname{Ann} M_1 = \operatorname{Ann} M_2$, then $M_1 \cong M_2$. By Lemma 1.4, $\operatorname{Ann} M_1$ is a prime ideal, and hence maximal by Proposition 1.5. A simple artinian ring is isomorphic to $M_d(D)$ for some division ring D, so M_1 and M_2 are simple modules over such a ring. However, $M_d(D)$ has a unique simple left module up to isomorphism, namely the column $D^d = (D, \dots, D)^{\mathsf{trans}}$, so $M_1 \cong M_2$. □

Lemma 1.7 *A left artinian ring has a finite number of simple left modules up to isomorphism.*

Proof. Let R be a left artinian ring. Then R has a composition series

$$R = I_0 \supset I_1 \supset \cdots \supset I_n = \{0\}$$

of left ideals such that I_j/I_{j+1} is a simple left module. If M is a simple left R-module, then $M \cong /I$ for some maximal left ideal I, so M is a composition factor of R. However, the Jordan-Holder theorem says that the composition factors are uniquely determined up to mulitplicity and re-ordering, so M is isomorphic to some I_j/I_{j+1}. Hence R has at most n simple modules up to isomorphism. □

The next theorem summarizes the preceding results.

Theorem 1.8 *A left artinian ring has a finite number of prime (equivalently, maximal) ideals and there is a bijection*

$$\{\textit{maximal ideals}\} = \{\textit{prime ideals}\} \longleftrightarrow \{\textit{simple modules up to isomorphism}\}.$$

EXERCISES

1.1 Show that the center of a simple ring is a field.

1.2 Let F be a field. For each pair of non-zero elements $a, b \in F$, define the generalized quaternion algebra

$$A := \left(\frac{a, b}{F}\right) = F \oplus Fi \oplus Fj \oplus Fk$$

to be the F-algebra with multiplication table

$$i^2 = a,\ j^2 = b;\ ij = -ji = k.$$

Show that this defines an associative F-algebra of dimension 4 that is isomorphic to $F\langle x, y\rangle/(x^2 - a, y^2 - b, xy + yx)$.

1.3 Suppose that $\operatorname{char} F \neq 2$. Show that A is a simple ring with center equal to F.

1.4 Show that A is a division ring if and only if the curve $X^2 - aY^2 - bZ^2 = 0$ in $\mathbb{P}^2_F$ has no points; i.e., if and only if the only solution to $X^2 - aY^2 - bZ^2 = 0$ in K is $X = Y = Z = 0$.

1.5 What happens if $ab = 0$? Is A a simple ring? If not describe its maximal (=prime) ideals.

3.2 Spec A

We now impose a topology on $\operatorname{Spec} A$ that reflects the containment properties of the ideals.

Notation. If I is a two-sided ideal we write

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subset \mathfrak{p}\}.$$

Proposition 2.1 *Let I, J, and I_j, $j \in \Lambda$, be ideals in the ring A. Then*

1. *$I \subset J$ implies $V(J) \subset V(I)$;*
2. *$V(0) = \operatorname{Spec} A$;*
3. *$V(A) = \phi$;*
4. *$\bigcap_{j\in\Lambda} V(I_j) = V(\sum_{j\in\Lambda} I_j)$;*
5. *$V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

Proof. The first four statements are clear, so we only prove the fifth.

Since $IJ \subset I \cap J$ and $I \cap J$ is contained in both I and J, (1) imples that

$$V(I) \cup V(J) \subset V(I \cap J) \subset V(IJ).$$

Let $\mathfrak{p}$ be a prime that is not in $V(I) \cup V(J)$; then $I \not\subset \mathfrak{p}$ and $J \not\subset \mathfrak{p}$, whence $IJ \not\subset \mathfrak{p}$; i.e., $\mathfrak{p} \notin V(IJ)$; thus $V(IJ) \subset V(I) \cup V(J)$. The equalities in (5) follow. □

Contrast parts (4) and (5) of the proposition. Part (5) extends to finite unions: if Λ is finite, then $\cup_{j\in\Lambda} V(I_j) = V(\prod_{j\in\Lambda} I_j) = V(\cap_{j\in\Lambda} I_j)$. To see that part (5) does not extend to infinite unions, consider the ideals $(x - j)$, $j \in \mathbb{Z}$, in $\mathbb{R}[x]$.

The Zariski topology. The Zariski topology on $\operatorname{Spec} A$ is defined by declaring the closed sets to be those subsets of $\operatorname{Spec} A$ of the form $V(I)$ as I ranges over all two-sided ideals of A. (This topology was first introduced by *Jacobson* in this general situation.) Proposition 2.1 shows that this is a topology. Parts (2) and (3) show that $\operatorname{Spec} A$ and the empty set are closed. Part (4) shows that the intersection of a collection of closed sets is closed. Part (5) shows that a finite union of closed sets is closed.

The closed points of $\operatorname{Spec} A$ are exactly the maximal ideals. As in the commutative case, when picturing $\operatorname{Spec} A$ we usually only consider the closed points (i.e., the maximal ideals). For the rings of interest in this course, that is rings that are finite modules over their centers, we shall see that the pictures are similar to those we have for affine varieties.

Warning. In contrast to the case for commutative rings, the rule $\mathsf{Rings} \to \mathsf{Top}$, $A \mapsto \operatorname{Spec} A$, is not a functor because if $\phi : A \to B$ is a ring homomorphism and $\mathfrak{p} \in \operatorname{Spec} B$, $\phi^{-1}(\mathfrak{p})$ need not be a prime ideal of A: consider, for example, the inclusion

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \subset \begin{pmatrix} k & k \\ k & k \end{pmatrix}.$$

There is a partial cure for this disease.

Definition 2.2 (Procesi) A ring homomorphism $\phi : A \to B$ is central if B is generated by $\phi(A)$ and elements that commute with $\phi(A)$.

The subcategory of Rings consisting of all rings but only central homomorphisms is denoted by $\mathsf{Rings_c}$. ◊

A composition of central homomorphisms is central, and identity maps are central, so $\mathsf{Rings_c}$ really is a subcategory. Important examples of central homomorphisms are:

1. the natural map $A \to A/I$ for every two-sided ideal I;
2. any homomorphism from a commutative ring $R \to A$, especially
3. the inclusion of the center $Z(A) \to A$;
4. localizations $A \to A[\mathcal{S}^{-1}]$ whenever $\mathcal{S}$ is a multiplicatively closed subset of the *center* of A.

If $\phi : A \to B$ is a central homomorphism, then $\phi(I)B = B\phi(I)$ is a two-sided ideal of B whenever I is a two-sided ideal of A.

Proposition 2.3 (Procesi) *If* $\phi : A \to B$ *is a central homomorphism then the map* $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ *is a continuous map* $\operatorname{Spec} B \to \operatorname{Spec} A$. *Thus* Spec *is a functor* $\mathsf{Rings_c} \to \mathsf{Top}$.

Proof. First, we show that $\phi^{-1}(\mathfrak{p})$ is a prime in A. Since ϕ induces an injective map $A/\phi^{-1}(\mathfrak{p}) \to B/\mathfrak{p}$, we can assume that $\mathfrak{p}$ is the zero ideal and that ϕ is simply inclusion; we must therefore show that A is prime if B is.

Suppose that $x, y \in A$ and that $xAy = 0$. If $b \in B$ commutes with A, then $xAby = 0$. However, the hypothesis ensures that $B = \sum Ab_i$ for some elements $b_i \in B$ that commute with A. Hence $xBy = x(sumAb_i)y = 0$; since B is prime we conclude that either $x = 0$ or $y = 0$. Hence A is prime.

Second, we show that the map $\mathfrak{p} \to \phi^{-1}(\mathfrak{p})$ is continuous. The preimage of a typical closed subspace $V(I) \subset \operatorname{Spec} A$, I a two-sided ideal of A, consists of those $\mathfrak{p} \in \operatorname{Spec} B$ such that $\phi^{-1}(\mathfrak{p}) \supset I$. But $\phi^{-1}(\mathfrak{p}) \supset I$ if and only if $\mathfrak{p} \supset \phi(I)$, if and only if $\mathfrak{p} \supset B\phi(I)$. Hence, the preimage of $V(I)$ is the closed set $V(B\phi(I))$. □

The following part of the previous proof warrants a separate statement.

Corollary 2.4 *If an inclusion $A \to B$ is a central homomorphism and B is prime, then A is prime.*

Running through the examples of central homomorphisms given prior to Proposition 2.3, the corresponding continuous maps between spectra are:

- if I is a two-sided ideal of A the map $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is an isomorphism onto $V(I)$, and we identify $\operatorname{Spec} A/I$ with the closed subset $V(I) \subset \operatorname{Spec} A$;
- the inclusion of the center induces a map $\operatorname{Spec} A \to \operatorname{Spec} Z(A)$ to something to which we can apply the tools of algebraic geometry—we view $\operatorname{Spec} A$ as lying over $\operatorname{Spec} Z(A)$;
- if $\mathcal{S}$ is a multiplicatively closed subset of $Z(A)$, the map $\operatorname{Spec} A[\mathcal{S}^{-1}] \to \operatorname{Spec} A$ is an isomorphism onto the complement of $V(\sum_{s \in \mathcal{S}} As)$, and we identify $\operatorname{Spec} A[\mathcal{S}^{-1}]$ with this open subset of $\operatorname{Spec} A$.

Proposition 2.5 *Let $\phi : A \to B$ be an injective central homomorphism such that B is a finitely generated A-module. Then the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Proof. [Robson-Small] Let $\mathfrak{q} \in \operatorname{Spec} A$. Choose $\mathfrak{p} \in \operatorname{Spec} B$ maximal such that $\phi^{-1}(\mathfrak{p}) \subset \mathfrak{q}$; we now check that such $\mathfrak{p}$ exists. The set of ideals J in B such that $\phi^{-1}(J) \subset \mathfrak{q}$ is non-empty—it contains 0 because ϕ is injective; by Zorn's Lemma, there is an ideal $\mathfrak{p}$ in B that is maximal subject to $\phi^{-1}(\mathfrak{p}) \subset \mathfrak{q}$; such $\mathfrak{p}$ is prime because if there were ideals I and J of B strictly larger than $\mathfrak{p}$ such that $IJ \subset \mathfrak{p}$, then

$$\mathfrak{q} \supset \phi^{-1}(\mathfrak{p}) \supset \phi^{-1}(IJ) \supset \phi^{-1}(I)\phi^{-1}(J),$$

so either $\phi^{-1}(I) \subset \mathfrak{q}$ or $\phi^{-1}(J) \subset \mathfrak{q}$, contradicting the maximality of $\mathfrak{p}$.

Write $\bar{B} = B/\mathfrak{p}$ and $\bar{A} = A/\phi^{-1}(\mathfrak{p})$; thus ϕ induces an inclusion $\bar{A} \to \bar{B}$ and, by Corollary 2.4, $\bar{A}$ is prime. The choice of $\mathfrak{p}$ implies that if I is a two-sided ideal of $\bar{B}$ such that $I \cap \bar{A} \subset \mathfrak{q}\bar{A}$, then $I = 0$.

Write $\bar{B} = \sum_{i=1}^{t} \bar{A}b_i$ where $b_1 = 1$ and each b_i commutes with the elements of $\bar{A}$. We may assume that all b_i are non-zero. Fix i; if $a \in \bar{A}$ and $ab_i = 0$, then

$$a\bar{B}b_i = \sum_{j=1}^{t} ab_j\bar{A}b_i = \sum_{j=1}^{t} b_j ab_i \bar{A} = 0;$$

but $\bar{B}$ is prime, so $a = 0$. It follows that each $\bar{A}b_i$ is isomorphic to $\bar{A}$ as both a right and left $\bar{A}$-module. Renumbering the b_is if necessary, we may pick m maximal such that $T = \bar{A}b_1 + \cdots + \bar{A}b_m$ is a direct sum, and hence a free $\bar{A}$-module.

It follows that for all i, $J_i := \operatorname{Ann}_{\bar{A}}(b_i + T/T)$ is a non-zero two-sided ideal of $\bar{A}$; since $\bar{A}$ is prime, the product of the J_is is non-zero, and hence $J := \cap_{i=1}^{t} J_i$ is a non-zero ideal of $\bar{A}$.

Because $Jb_i \subset T$, it follows that $J\bar{B} \subset T$, and hence $\mathfrak{q}J\bar{B} \subset \mathfrak{q}T \subset T$. Now $\mathfrak{q}J\bar{B}$ is a two sided ideal of $\bar{B}$ and $\mathfrak{q}J\bar{B} \cap \bar{A} \subset \mathfrak{q}T \cap \bar{A} = \mathfrak{q}\bar{A}$, the last equality being because $T = \bar{A} \oplus C$ with C an A-A-bimodule. The inclusion $\mathfrak{q}J\bar{B} \cap \bar{A} \subset \mathfrak{q}\bar{A}$ implies that $\mathfrak{q}J\bar{B} = 0$, whence $\mathfrak{q}J = 0$; but $\bar{A}$ is prime and J is non-zero, so the image of $\mathfrak{q}$ in $\bar{A}$ is zero. Thus $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. □

3.3 Spec $A \to$ Spec $Z(A)$ when A is finite over its center

The following is an immediate consequence of Proposition 2.5.

Proposition 3.1 *If A is a finite module over its center $Z(A)$, then the map* Spec $A \to$ Spec $Z(A)$ *is surjective.*

If A is a finite module over its center $Z(A)$, and $Z(A)$ is noetherian, then A is noetherian as both a right and as a left $Z(A)$-module. Since every A-submodule of A is a $Z(A)$-module, A is both right and left noetherian as a module over itself.

Let I be a two-sided ideal of A. The image of $Z(A)$ under the map $A \to A/I$ is contained in the center of A/I; however, the center of A/I may be larger than this. If A is a finitely generated module over its center, then A/I is a finitely generated module over its center.

Lemma 3.3 below uses the fact that the endomorphism ring of a noetherian module over a commutative noetherian ring R is a noetherian R-module. This is a special case of the next lemma.

Lemma 3.2 *Let R be a commutative noetherian ring. If M and N are noetherian R-modules, so is* $\operatorname{Hom}_R(M, N)$.

Proof. By hypothesis, there is a surjective map $R^n \to M$ for some integer n, and hence an injective map $\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(R^n, N)$ of R-modules. But $\operatorname{Hom}_R(R^n, N)$ is isomorphic to $\operatorname{Hom}_R(R, N)^n \cong N^n$, which is noetherian by hypothesis. So $\operatorname{Hom}_R(M, N)$ is noetherian. □

Lemma 3.3 *Suppose that A is a finite module over its center $Z(A)$, and that $Z(A)$ is noetherian. If M is a simple A-module, then*

1. $\operatorname{Ann} M$ *is a maximal ideal of A, and*
2. $Z(A) \cap \operatorname{Ann} M$ *is a maximal ideal of $Z(A)$.*

Proof. Write $\bar{A} = A/\operatorname{Ann} M$ and $\bar{Z}$ for the image of Z in $\bar{A}$. The action of Z on M gives an injective ring homomorphism

$$\bar{Z} \to \operatorname{End}_A M =: D.$$

By Schur's Lemma, D is a division ring.

(2) Since A is a finitely generated Z-module, so is M; hence $\operatorname{End}_Z M$ and its subring $\operatorname{End}_A M$ are finitely generated $\bar{Z}$-modules. Let $0 \neq z \in \bar{Z}$. Then z has an inverse z^{-1} in D, and there is an ascending chain $\bar{Z} \subset \bar{Z}z^{-1} \subset \bar{Z}z^{-2} \subset \cdots$ that must eventualy terminate because D is a noetherian $\bar{Z}$-module. It follows that $z^{-n-1} \in \bar{Z}z^{-n}$ for some $n \geq 0$, and hence $z^{-1} \in \bar{Z}$. Hence $\bar{Z}$ is a field; but $\bar{Z} = Z/Z \cap \operatorname{Ann} M$ so (2) holds.

(1) Since $\bar{A}$ is a finitely generated $\bar{Z}$-module, it is artinian. It is also prime because $\operatorname{Ann} M$ is a prime ideal, so $\bar{A}$ is a simple ring by Proposition 1.5. □

Proposition 3.4 *Suppose that A is a finite module over its center $Z(A)$, and that $Z(A)$ is noetherian. Then*

1. *the map* $\operatorname{Spec} A \to \operatorname{Spec} Z(A)$ *restricts to a surjective map* $\pi : \operatorname{Max} A \to \operatorname{Max} Z(A)$;
2. *the fibers of π are finite;*
3. *if* $\mathfrak{n} \in \operatorname{Max} Z(A)$, *then* $\pi^{-1}(\mathfrak{n})$ *is the set of primes (=maximal) ideals in the finite-dimensional algebra* $A/A\mathfrak{n}A$;
4. $\pi^{-1}(\mathfrak{n})$ *is in natural bijection with the set of simple A-modules that are annihilated by* $\mathfrak{n}$.

Proof. Write $Z = Z(A)$.

Let $\mathfrak{m}$ be a maximal ideal of A. Then $\bar{A} = A/\mathfrak{m}$ is a simple ring and there are inclusions

$$\bar{Z} = Z/Z \cap \mathfrak{m} \subset Z(\bar{A}) \subset \bar{A}.$$

The center of a simple ring is a field: if $z \in Z(\bar{A})$, then $z\bar{A}$ is a two-sided ideal of $\bar{A}$, so equals $\bar{A}$, whence z has an inverse in $\bar{A}$, and it is easily seen that this inverse must be central. Hence $Z(\bar{A})$ is a field.

Since A is a finitely generated Z-module, $\bar{A}$ is a finitely generated $\bar{Z}$-module; because Z is noetherian, it follows that $Z(\bar{A})$ is a finitely generated $\bar{Z}$-module. Hence $Z(\bar{A})$ is integral over $\bar{Z}$; a non-zero $x \in \bar{Z}$ has an inverse x^{-1} in $Z(\bar{A})$ and this satisfies a monic polynomial $x^{-n} + \alpha_1 x^{-n+1} + \cdots + \alpha_{n-1}x^{-1} + \alpha_n = 0$ with each $\alpha_i \in \bar{Z}$; multiplying through by x^{n-1}, we see that $x^{-1} \in \bar{Z}$; thus $\bar{Z}$ is also a field. In other words, if $\mathfrak{m}$ is maximal in A, then $\mathfrak{m} \cap Z$ is maximal in Z. This proves (1) (except for the surjectivity of π).

Now let $\mathfrak{n}$ be a maximal ideal of Z. Then $A/A\mathfrak{n}A$ is a finitely generated $Z/\mathfrak{n}$-module so is artinian. The statements (2)-(4) now follow from Theorem 1.8.

The map π is surjective: by Proposition 3.1, $\mathfrak{n} \in \operatorname{Max} Z(A)$ is equal to $\mathfrak{p} \cap Z(A)$ for some $\mathfrak{p} \in \operatorname{Spec} A$; but $\mathfrak{p}$ contains $A\mathfrak{n}A$, so $\mathfrak{p}/A\mathfrak{n}A$ is a prime ideal in the artinian ring $A/A\mathfrak{n}A$, and is therefore maximal. □

EXERCISES

3.1 Let $A = \mathbb{C}[x, y]$ have defining relations $yx - xy = x$. This has basis $x^i y^j$, $i, j \geq 0$. Show that $A/A(x-1)$ is a simple module and that its annihilator is 0, so $\{0\}$ is a non-maximal primitive ideal. It might be easier to view the module in question as $\mathbb{C}[t]e^t$ with the actions of x and y given by

$$y \mapsto -t\tfrac{d}{dt}, \qquad x \mapsto \tfrac{d}{dt}$$

and observe that the generator $e^t \in \mathbb{C}[t]e^t$ is annihilated by $x - 1$.

3.2 Describe the map $\operatorname{Spec} A \to \operatorname{Spec} Z(A)$ when $A = \mathbb{C}[x, y]$ with relation $xy + yx = 0$.

3.3 Fix $n \geq 2$ and $\xi = e^{2\pi i/n}$. Describe the map $\pi : \operatorname{Spec} A \to \operatorname{Spec} Z(A)$ when $A = \mathbb{C}[x, y, \sigma]$ with relations

$$\sigma^n = 1, \ xy = yx, \ \sigma x = \xi x \sigma, \ \sigma y = \xi y \sigma.$$

(a) Show that $Z(A) = \mathbb{C}[x^n, xy, y^n]$; thus $\operatorname{Spec} Z(A)$ is the singular surface $uv = t^n$ with unique singularity at $(t, u, v) = (0, 0, 0)$. The category of right A-modules is equivalent to $\mathsf{Qcoh}[\mathbb{C}^2/\mathbb{Z}_n]$, where $[\mathbb{C}^2/\mathbb{Z}_n]$ denotes the stack-theoretic quotient for the $\mathbb{Z}_n$ action on $\mathbb{C}^2$ given by $\sigma : (x, y) \mapsto (\xi^{-1}x, \xi y)$.

(b) Describe the fiber of π over the singular point.

(c) Show that away from the singular point π is bijective.

3.4 Describe the map $\operatorname{Spec} A \to \operatorname{Spec} Z(A)$ when $A = \mathbb{C}[x, y]$ with relation $yx = qxy$ and q is a primitive n^{th} root of unity.

3.4 $I(-)$, $V(-)$, and $\sqrt{-}$

If X is a subset of $\operatorname{Spec} A$, we define

$$I(X) := \bigcap_{\mathfrak{p} \in X} \mathfrak{p}.$$

It is easy to see that $J \subset I(V(J))$ and $X \subset V(I(X))$, but the maps $V(-)$ and $I(-)$

$$\{\text{ideals in } A\} \longleftrightarrow \{\text{closed subspaces of } \operatorname{Spec} A\} \qquad (4\text{-}1)$$

are not generally inverses of one another.

Lemma 4.1 *Let X be a subset of* $\operatorname{Spec} A$. *Then*

1. *$V(I(X)) = \bar{X}$, the closure of X;*

2. *if X is closed, then $V(I(X)) = X$.*

Proof. It is clear that (2) follows from (1), so we shall prove (1).

Certainly, $V(I(X))$ contains X and is closed. On the other hand, any closed set containing X is of the form $V(J)$ for some ideal J; but then every $\mathfrak{p} \in X$ is in $V(J)$ so $J \subset \mathfrak{p}$ and $J \subset I(X)$; whence $V(J) \supset V(I(X))$. Thus $V(I(X))$ is the smallest closed set containing X. □

Definition 4.2 Let J be a two-sided ideal in a ring A. The radical of J is the ideal

$$\sqrt{J} := \bigcap_{\mathfrak{p} \supset J} \mathfrak{p}.$$

If $J = \sqrt{J}$ we call J a radical, or semiprime, ideal. ◊

Obviously $J \subset \sqrt{J}$. A prime ideal is radical. Intersections of radical ideals are radical.

We could equally well define $\sqrt{J}$ as $I(V(J))$.

The notation $\sqrt{J}$ agrees with its use in commuttaive algebra (see, e.g., Proposition 6.5 below).

The next lemma explains why semiprime ideals are important.

Lemma 4.3 *If J is a two-sided ideal, then*

1. *$I(V(J)) = \sqrt{J}$;*

2. *$V(J) = V(\sqrt{J})$.*

Proof. (1) The definitions of $I(-)$ and $\sqrt{-}$ give

$$I(V(J)) = \bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p} = \bigcap_{\mathfrak{p} \supset J} \mathfrak{p} = \sqrt{J}.$$

(2) Since $J \subset \sqrt{J}$, $V(J) \supset V(\sqrt{J})$. However, if $\mathfrak{p} \in V(J)$, then $\mathfrak{p} \supset J$, so $\mathfrak{p} \supset \sqrt{J}$, whence $\mathfrak{p} \in V(\sqrt{J})$; thus $V(J) \subset V(\sqrt{J})$, and there is equality. □

3.5 Irreducible components

A topological space is irreducible if it is not the union of two proper closed subsets.

The main result in this section is that when A is right or left noetherian, every closed subset of $\operatorname{Spec} A$ is a finite union of irreducible subsets; and the closed irreducible subsets of $\operatorname{Spec} A$ are the $V(\mathfrak{p})$ as $\mathfrak{p}$ ranges over the prime ideals. For this reason prime rings are a particular focus.

Let W be a closed irreducible subspace of a topological space X. If Y and Z are closed subspaces of X such that $W \subset Y \cup Z$, then either $W \subset Y$ or $W \subset Z$ because $W = (W \cap Y) \cup (W \cap Z)$.

Lemma 5.1 *Let $f : X \to Y$ be a continuous map. If Z is an irreducible subspace of X, then $f(Z)$ is irreducible.*

Proof. We can replace X by Z, Y by $f(Z)$, and f by $f|_Z : Z \to f(Z)$. Thus, we can assume that X is irreducible, that f is surjective, and we must show that $Y = f(X)$ is irreducible.

If $Y = W \cup W'$ is the union of closed subspaces, then $X = f^{-1}(W) \cup f^{-1}(W')$ is a union of closed subspaces. Hence, either $X = f^{-1}(W)$ or $X = f^{-1}(W')$, and either $f(X) \subset W$ or $f(X) \subset W'$. □

Proposition 5.2 *A closed subspace $X \subset \operatorname{Spec} A$ is irreducible if and only if $I(X)$ is a prime ideal.*

Proof. Let $\mathfrak{p} = I(X)$.

($\Rightarrow$) Suppose X is irreducible. To show that $I(X)$ is prime, suppose that $IJ \subset I(X)$ where I and J are two-sided ideals containing $I(X)$; then $X \subset V(I(X)) \subset V(IJ) = V(I) \cap V(J)$; since X is irreducible, either $X \subset V(I)$ or $X \subset V(J)$; suppose that $X \subset V(J)$; then $I(X) \supset I(V(J)) \supset J$. Hence $I(X)$ is prime.

($\Leftarrow$) Suppose that $I(X)$ is a prime ideal.

Let $V(I)$ and $V(J)$ be closed subspaces of X such that $X = V(I) \cup V(J)$. Then

$$I(X) = I(V(I) \cup V(J)) = I(V(I)) \cap I(V(J)) \supset I \cap J \supset IJ.$$

Hence $I(X)$ contains either I or J. If $J \subset I(X)$, then $J \subset \cap_{\mathfrak{p} \in X} \mathfrak{p}$, so every $\mathfrak{p} \in X$ belongs to $V(J)$; i.e., $X = V(J)$. □

Corollary 5.3 *There is a bijection*

$$\{\textit{closed irreducible subsets of } \operatorname{Spec} A\} \longleftrightarrow \{\textit{prime ideals}\}.$$
$$X \mapsto I(X)$$
$$V(\mathfrak{p}) \leftarrow \mathfrak{p}.$$

Noetherian spaces. A topological space X is noetherian if every descending chain of closed subsets

$$X \supset Z_1 \supset Z_2 \supset \cdots$$

is eventually constant.

A closed subspace of a noetherian space is obviously noetherian.

If A is right or left noetherian, then $\operatorname{Spec} A$ is noetherian.

Proposition 5.4 *Let X be a noetherian topological space. There is a unique way of writing $X = X_1 \cup \cdots \cup X_n$ where each X_i is a closed irreducible subspace of X and $X_i \not\subset X_j$ if $i \neq j$.*

Proof. First we show that X is a finite union of irreducible subspaces. Suppose to the contrary that X is not such a union. In particular, X is not irreducible, so we can write $X = Y_1 \cup Z_1$ as a union of proper closed subspaces. If both Y_1 and Z_1 were finite unions of closed irreducible subspaces, X would be too, so one of them, say Z_1, is not such a union. In particular, Z_1 is not irreducible, so we can write $Z_1 = Z_2 \cup Y_2$ as a union of proper closed subspaces, and one of these, say Z_2, is not a finite union of irreducible subspaces.

Repeating this process leads to an infinite descending chain $Z_1 \supset Z_2 \supset \cdots$ of subspaces contradicting the hypothesis that X is noetherian. We therefore conclude that X is a finite union of irreducible subspaces.

It remains to prove the uniqueness. Suppose that $X = X_1 \cup \cdots \cup X_n = Y_1 \cup \cdots \cup Y_m$ where each X_i and each Y_s is irreducible and $X_i \not\subset X_j$ if $i \neq j$ and $Y_s \not\subset Y_t$ if $s \neq t$. Then the irreducible space $X_i = (X_i \cap Y_1) \cup \cdots \cup (X_i \cap Y_m)$ is a union of closed subspaces so must equal one of them, whence $X_i \subset SY_s$ for some s. Likewise each Y_s is contained in some X_j. But then $X_i \subset Y_s \subset X_j$, so $X_i = Y_s$. It follows that $m = n$ and $\{X_1, \ldots, X_n\} = \{Y_1, \ldots, Y_m\}$. □

Definition 5.5 The closed subspaces $X_1, \ldots, X_n$ appearing in Proposition 5.4 are called the irreducible components of X. ◊

Thus, when A is right or left noetherian, every closed subspace of Spec A is the (finite) union of its irreducible components.

Definition 5.6 The dimension of a topological space X is the largest n for which there is a chain of distinct closed irreducible subspaces

$$\phi \neq X_0 \subset X_1 \subset \cdots \subset X_n.$$

The dimension of the empty set is $-\infty$. ◊

EXERCISES

5.1 Let $f : X \to Y$ be a continuous map. Are the following true?

(a) If f is injective, then $\dim X \leq \dim Y$.

(b) If f is surjective, then $\dim X \geq \dim Y$.

5.2 If $f : X \to Y$ is a surjective continuous map with finite fibers, is $\dim X = \dim Y$?

3.6 Minimal primes

Definition 6.1 Let I be a two-sided ideal. A minimal prime over I is a prime ideal $\mathfrak{p} \supset I$ such that if $\mathfrak{p} \supset \mathfrak{p}' \supset I$ and $\mathfrak{p}'$ is prime, then $\mathfrak{p}' = \mathfrak{p}$. ◊

It is obvious that

$$\{\text{minimal primes over } J\} = \{\text{minimal primes over } \sqrt{J}\}.$$

Theorem 6.2 *Let A be left noetherian. The irreducible components of $V(J)$ are $V(\mathfrak{p}_1), \dots, V(\mathfrak{p}_n)$, where $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \{$minimal primes over $J\}$.*

Proof. Write $V(J)$ as the union of its irreducible components, say $V(J) = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_n)$. If $\mathfrak{q}$ is any prime containing J, then $\mathfrak{q}$ belongs to some $V(\mathfrak{p}_i)$, so $\mathfrak{q} \supset \mathfrak{p}_i$. Hence $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ must be the complete set of minimal primes containing J. □

Corollary 6.3 *Let J be an ideal in a left noetherian ring.*

1. *If $\mathfrak{q}$ is a prime containing J, there is a minimal prime $\mathfrak{p}$ over J such that $\mathfrak{q} \supset \mathfrak{p} \supset J$.*
2. *The set of minimal primes containing J is finite.*
3. *$\sqrt{J} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, where $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \{$the minimal primes over $J\}$.*

Proposition 6.4 *Let J be an ideal in a left noetherian ring. Then J contains a finite product of prime ideals, say $\mathfrak{q}_1 \cdots \mathfrak{q}_m \subset J$, where each $\mathfrak{q}_i$ is a minimal prime over J.*

Proof. Let A be a left noetherian ring. Let

$$\mathcal{S} := \{\text{ideals } J \text{ that do not contain a finite product of primes}\}.$$

If $\mathcal{S} = \phi$, we are done. If not, $\mathcal{S}$ has maximal members; pick one, say J. Certainly, J is not prime, so contains a product, II', of ideals each of which is strictly larger than J. By maximality of J, I and I' contain finite products of primes; but if $\mathfrak{p}_1 \cdots \mathfrak{p}_n \subset I$ and $\mathfrak{q}_1 \cdots \mathfrak{q}_m \subset I'$ are products of primes, then J contains $\mathfrak{p}_1 \cdots \mathfrak{p}_n \mathfrak{q}_1 \cdots \mathfrak{q}_m$.

Hence J contains a product of primes, each of which contains J; each such prime contains a minimal prime over J, and replacing that prime by the minimal one still gives a product, now of minimal primes, that is contained in J. □

Proposition 6.5 *Let J be an ideal in a left noetherian ring A. Then $\sqrt{J}$ is the largest ideal I such that $I^n \subset J$ for some n.*

Proof. First we check that there is a *largest* ideal I such that $I^n \subset J$ for some n. Let

$$\mathcal{S} := \{\text{ideals } I \mid I^n \subset J \text{ for some } n\}.$$

This set is non-empty because it contains $\{0\}$. Since A is left noetherian $\mathcal{S}$ has maximal members; pick one, say I_1, and choose n such that $I_1^n \subset J$. Let $I_2 \in \mathcal{S}$; choose m such that $I_2^m \subset J$; by increasing n we can assume that $n \geq m$, so that $I_2^n \subset J$ too; now $(I_1 + I_2)^{2n} \subset J$, whence $I_1 + I_2 \in \mathcal{S}$; but I_1 is a maximal member of $\mathcal{S}$ so $I_1 + I_2 \subset I_1$, and we deduce that $I_2 \subset I_1$, thus showing that $\mathcal{S}$ has a unique maximal member, say I.

Suppose that $I^n \subset J$.

If $\mathfrak{p}$ is a prime containing J, then $I^n \subset \mathfrak{p}$, so $I \subset \mathfrak{p}$; hence $I \subset \sqrt{J}$. On the other hand, $J \supset \mathfrak{p}_1 \cdots \mathfrak{p}_m$ where each $\mathfrak{p}_i$ is a minimal prime over J; but $\sqrt{J} \subset \mathfrak{p}_i$, so $(\sqrt{J})^m \subset J$, so $\sqrt{J} \subset I$. Hence $I = \sqrt{J}$. □

In particular, in a left (or right) noetherian ring, $\sqrt{0}$ is the largest nilpotent ideal.

3.7 The Artin-Tate Lemma

The following two results are fundamental to the study of rings that are finite over their centers.

Theorem 7.1 (Artin-Tate Lemma) *Let A be a finitely generated k-algebra. If A is finite over its center, then*

1. *its center is a finitely generated k-algebra, hence noetherian;*
2. *A is noetherian.*

Proof. Write Z for the center of A. Suppose that $A = k[x_1, \dots, x_m]$ and $A = Za_1 + \cdots + Za_n$. There is a finite set

$$\{\alpha_{pqr}, \beta_{st}\} \subset Z$$

such that

$$a_p a_q = \sum_{r=1}^{n} \alpha_{pqr} a_r \qquad \text{and} \qquad x_s = \sum_{t=1}^{n} \beta_{st} a_t.$$

Since $Z' := k[\alpha_{pqr}, \beta_{st}]$ is a finitely generated, commutative k-algebra, it is a noetherian ring. Since any product of the x_is is in $Z'a_1 + \ldots + Z'a_n$, A is generated as a Z'-module by $a_1, \dots, a_n$. Hence A is a noetherian Z'-module, and is therefore a noetherian ring. Since $Z' \subset Z \subset A$, Z is a finitely generated Z'-module, hence a finitely generated k-algebra, and thus a noetherian ring. □

Theorem 7.2 *Let A be a finitely generated k-algebra that is finite over its center. Let M be a simple A-module. Then*

1. $\dim_k M < \infty$*;*
2. *M is annihilated by a maximal ideal of the center of A;*
3. *if k is algebraically closed,* $\operatorname{End}_A M = k$.

Proof. Let M be a simple A-module. The hypotheses on A also apply to $A/\operatorname{Ann} M$, so it is enough to prove the result for $A/\operatorname{Ann} M$. Therefore we can, and do, assume that $\operatorname{Ann} M = 0$.

Write Z for the center of A. We already proved (2) in Lemma 3.3. This implies that Z is a field.

By the Artin-Tate Lemma, Z is a finitely generated k-algebra hence of finite dimension over k. It follows that A, M, and $\operatorname{End}_A M$, are finite dimensional over k. This proves (1). Finally, if k is algebraically closed, then the division algebra $\operatorname{End}_A M$ must equal k. □

Corollary 7.3 *Let A be a finitely generated k-algebra that is finite over its center. Then the closed points in* $\operatorname{Spec} A$ *are in bijection with the simple A-modules, and in bijection with the maximal ideals of A.*

EXERCISES

7.1 Let $\phi : A \to B$ be a surjective ring homomorphism. Show that the induced map $Z(A) \to Z(B)$ need not be surjective. If you need help, consider $A = \mathbb{C}[x, y, z]$ with relations $zx - xz = zy - yz = 0$ and $xy - yx = z$. Also give an example in which A is a finite module over its center.

3.8 The Azumaya locus

We continue to suppose that A is a finitely generated algebra over an algebraically closed field k, and that A is finite over its center Z. We consider the map

$$\pi : \operatorname{Spec} A \to \operatorname{Spec} Z.$$

for each $\mathfrak{n} \in \operatorname{Max} Z$, $\pi^{-1}(\mathfrak{n}) = \operatorname{Spec} A/A\mathfrak{n}$. Now $A/A\mathfrak{n}$ is a finite dimensional algebra over the field $Z/\mathfrak{n}$. Thus A provides a family of finite dimensional algebras parametrized by $\operatorname{Spec} Z$.

The algebras $A/A\mathfrak{n}$ along one irreducible component of $\operatorname{Spec} Z$ can have little to do with their behavior along another component, so we now suppose add the hypothesis that A is prime because this then forces Z to be a domain and $\operatorname{Spec} Z$ therefore is a single irreducible component. We will show that over a dense open set $U \subset \operatorname{Spec} Z$ there is a uniformity to the behavior of the finite dimensional algebras: we will show that all are isomorphic to $M_n(k)$ for a fixed n. Thus $\pi : \pi^{-1}(U) \to U$ is bijective and for each $\mathfrak{n} \in U$ there is a unique simple A-module annihilated by $\mathfrak{n}$ and that simple has dimension n.

The first step towards this uniformity result is a criterion for deciding when a finite dimensional algebra is a matrix algebra. Suppose that A is an algebra over a commutative ring k. Then A is an A-A-bimodule, hence a left module over $A \otimes_k A^{\text{op}}$, via the action $(a \otimes b).c = acb$. The module action yields a k-algebra homomorphism $\varphi : A \otimes_k A^{\text{op}} \to \operatorname{End}_k A$.

Lemma 8.1 *Let A be a finite dimensional algebra over an algebraically closed field k. Then A is isomorphic to $M_n(k)$ for some n if and only if the map $\varphi : A \otimes_k A^{\text{op}} \to \operatorname{End}_k A$ is an isomorphism.*

Proof. ($\Rightarrow$) In this case, A^{op} is also isomorphic to $M_n(k)$, so $A \otimes_k A^{\text{op}} \cong M_{n^2}(k)$. This is a simple ring, so φ is injective. But $\dim_k \operatorname{End}_k A = (\dim_k A)^2 = \dim_k A \otimes_k A^{\text{op}}$, so φ is also surjective.

($\Leftarrow$) Certainly A is a simple module over $\operatorname{End}_k A$ so, since φ is an isomorphism, A is a simple $A \otimes_k A^{\text{op}}$-module. But the $A \otimes_k A^{\text{op}}$-submodules of A are its two-sided ideals, so A is a simple ring. By the Artin-Wedderburn Theorem it is therefore a matrix algebra over a division ring. That division ring has finite dimension over k, so is isomorphic to k. □

Proposition 8.2 *Let A be a prime noetherian ring that is finite over its center Z. Then*

1. *Z is a domain and non-zero elements of Z are not zero-divisors in A;*
2. *if F is the field of fractions of Z, then the center of $R := A \otimes_Z F$ is F;*
3. *non-zero ideals of R have non-zero intersection with A;*
4. *$R \cong M_n(D)$ for some division algebra D, and integer n;*
5. *the induced F-algebra homomorphism $\varphi : R \otimes_F R^{\mathrm{op}} \to \operatorname{End}_F R$ is an isomorphism.*

Proof. (1) If $x \in Z$ and $y \in A$ are such that $xy = 0$, then $(AxA)(AyA) = 0$, so either x or y is zero. Hence non-zero elements of Z are regular in A. In particular, Z is a domain.

(2) Because A is a torsion-free Z-module the map $A \to R := A \otimes_Z F$, $a \mapsto a \otimes 1$ is an injective ring homomorphism. Elements of R are of the form ax^{-1} with $a \in A$ and $x \in Z$, so if ax^{-1} commutes with all elements of R, so does $ax^{-1}x = a$.

(3) Let I be a non-zero ideal of R, and Let ax^{-1} be a non-zero element of an ideal I in R, where $a \in A$ and $0 \neq x \in Z(A)$. Then $0 \neq a \in I \cap A$.

(4) Clearly R is a finite module over F, so is artinian. And R is prime, because A is: if I and J are non-zero ideals of R such that $IJ = 0$, then $I \cap A$ and $J \cap A$ are non-zero ideals of A whose product is zero. A prime artinian ring is simple, so the Artin-Wedderburn Theorem gives the result.

(5) The theory of central simple algebras shows that $R \otimes_F R^{\mathrm{op}}$ is a simple ring, so φ is injective. However, the F-vector-space dimension of both $R \otimes R^{\mathrm{op}}$ and $\operatorname{End}_F R$ is $(\dim_F R)^2$, so φ must be an isomorphism. □

Proposition 8.3 *Let A be finite over its center Z. Suppose that elements of Z are regular in A (this implies Z is a domain) and let $F = \operatorname{Fract} Z$. If $A \otimes_Z F$ is a simple artinian ring, then A is prime.*

Proof. The map $A \to R := A \otimes_Z F$, $r \mapsto r \otimes 1$, is an injective ring homomorphism. Non-zero ideals I and J of A generate non-zero ideals of R, namely $I \otimes_Z F$ and $J \otimes_Z F$. Because R is simple artinian the product $(I \otimes_Z F)(J \otimes_Z F)$ is non-zero; but that product is $IJ \otimes_Z F$, so $IJ \neq 0$. Hence A is prime. □

Lemma 8.4 *Let C be a commutative noetherian domain with field of fractions F. Let $f : M \to N$ be a homomorphism of finitely generated C-modules. If the induced map $M \otimes_C F \to N \otimes_C F$ is an isomorphism, then there is a non-zero element $x \in C$ such that the induced map $M[x^{-1}] \to N[x^{-1}]$ is an isomorphism.*

Proof. Let K and L be the kernel and cokernel of f. Both are finitely generated, and $K \otimes_C F \cong L \otimes_C F \cong 0$. If $a_1, \ldots, a_m$ is a set of generators for K, then there are non-zero elements $x_1, \ldots, x_m$ in C such that $a_i x_i = 0$ for all i. Similarly, we can choose elements $b_1, \ldots, b_n$ generating L, and non-zero elements $y_1, \ldots, y_n \in C$ such that $b_i y_i = 0$ for all i. If $x = x_1 \ldots x_m y_1 \ldots y_n$, then $Kx = Lx = 0$, whence $K \otimes_C C[x^{-1}] \cong L \otimes_C C[x^{-1}] \cong 0$. □

Lemma 8.4 and its proof can be phrased more geometrically. If $Z = \operatorname{Spec} C$ and $Y = \operatorname{Supp} K \cup \operatorname{Supp} L$, then $f : M \to N$ is an isomorphism on $Z \backslash Y$.

Furthermore, $M \otimes_C F$ is free, of rank d say, so there is a proper closed subscheme $Y' \subset Z$ such that M and N are both free of rank d on $Z \backslash Y'$. To see this, choose any set of generators $m_1, \ldots, m_n$ for M; choose a subset of the generators that is maximal subject to being linearly independent over C; then the quotient $\bar{M}$ of M by the submodule these elements generate is a torsion C-module so $\bar{M}[x^{-1}] = 0$ for a suitable non-zero $x \in C$, whence $M[x^{-1}]$ is free.

Theorem 8.5 *Let A be a prime ring that is finite over its center Z. Suppose further that A is a finitely generated algebra over an algebraically closed field k. Let $\pi : \operatorname{Max} A \to \operatorname{Max} Z$ be the map $\mathfrak{m} \mapsto \mathfrak{m} \cap Z$. Then there is a dense open set $U \subset \operatorname{Max} Z$, and an integer n, such that*

1. *$\pi : \pi^{-1}(U) \to U$ is bijective, and*
2. *for each $\mathfrak{n} \in U$ the (unique) simple A-module annihilated by $\mathfrak{n}$ has dimension n, and*
3. *if $\mathcal{O}_q$ is the simple A-module annihilated by $\mathfrak{n} \in U$, then $A/\operatorname{Ann}\mathcal{O}_q = A/A\mathfrak{n} \cong M_n(k)$.*

Proof. Let $\varphi : A \otimes_Z A^{\mathrm{op}} \to \operatorname{End}_Z A$ be the natural map. Let F denote the field of fractions of Z, and set $R := A \otimes_Z F$. By Proposition 8.2, $\varphi \otimes F$ is an isomorphism, so by Lemma 8.4, $\varphi \otimes Z[x^{-1}]$ is an isomorphism for some non-zero x in Z. By the remarks after that lemma we can also assume, perhaps after replacing x by some xx', that $A[x^{-1}]$ is a free $Z[x^{-1}]$-module of rank d. Let $\mathfrak{n}$ be a maximal ideal of Z that does not contain x. Then $\varphi \otimes Z[x^{-1}] \otimes_Z Z/\mathfrak{n}$ is an isomorphism. But $Z[x^{-1}] \otimes_Z Z/\mathfrak{n} \cong Z/\mathfrak{n} \cong k$, so this is the map

$$\varphi \otimes Z/\mathfrak{n} : A/A\mathfrak{n} \otimes_k A/A\mathfrak{n} \to \operatorname{End}_k A/A\mathfrak{n}.$$

Therefore by Lemma 8.1, $A/A\mathfrak{n}$ is a matrix algebra over k. Since $A[x^{-1}]$ is free of rank d, $\dim_k A/A\mathfrak{n} = d$. Hence for all $\mathfrak{n}$ not containing x, $A/A\mathfrak{n} \cong M_n(k)$ where $n^2 = d$. □

The largest open set U in the theorem is called the Azumaya locus of A.

Example 8.6 Let k be an algebraically closed field of characteristic two, and let $A = k[x, y, z]$ be the k-algebra with defining relations

$$xy - yx = z, \quad xz = zx, \quad yz = zy.$$

Thus A is isomorphic to the enveloping algebra of the Heisenberg Lie algebra, and to the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 matrices. The elements $x^i y^j z^k$, $(i,j,k) \in \mathbb{N}^3$ form a basis for A. The center of A is $k[x^2, y^2, z]$, the coordinate ring of $\mathbb{A}^3$. It is clear that A is a finite module over its center. Since $A/(z)$ is commutative, if p is a closed point of $\operatorname{Max} A$ lying on the hypersurface $z = 0$, $\dim \mathcal{O}_p = 1$. On the other hand, if p is a closed point of $\operatorname{Max} A$ such that $\mathcal{O}_p$ is annihilated by $z - \lambda$ for some non-zero scalar λ, then $\dim \mathcal{O}_p = 2$. To see this, we argue as follows.

Since $\mathcal{O}_p$ is a finite dimensional simple it is annihilated by a maximal ideal of $k[x^2, y^2, z]$, so $A/\operatorname{Ann}\mathcal{O}_p$ is spanned by the images of 1, x, y, and xy, so has dimension at most four. Thus the dimension of $\mathcal{O}_p$ is at most two; if $\dim_k \mathcal{O}_p$ were one, then both x and y would act on $\mathcal{O}_p$ as scalars so $z = xy - yx$ would act as zero, contrary to our hypothesis.

Thus the Azumaya locus of A is the complement to the coordinate plane $z = 0$. $\Diamond$