Math 436 Spring 2003

ITERATED FUNCTION SYSTEMS AND THE "CHAOS GAME"

Continuation

3. Proof of Hutchinson's Theorem.

Let's recall the theorem (reworded slightly). Below $\mathcal{H}(\mathbb{R}^n)$ denotes the space of NONEMPTY compact subsets of $\mathbb{R}^n$, equipped with the Hausdorff metric d.

THEOREM 2 (Hutchinson). *Suppose that* $\{f_1, \ldots, f_m\}$ *is an IFS in* $\mathbb{R}^n$. *Define* $\mathcal{F} : \mathcal{H}(\mathbb{R}^n) \to \mathcal{H}(\mathbb{R}^n)$ *by*

$$\mathcal{F}(E) = \bigcup_{i=1}^{m} f_i(E).$$

Then $\mathcal{F}$ *has a unique fixed point* K *and all orbits under* $\mathcal{F}$ *converge to* K.

Partial proof. The proof consists of three steps: (i) $(\mathcal{H}(\mathbb{R}^n), d)$ is a complete metric space; (ii) $\mathcal{F}$ is a contraction on $(\mathcal{H}(\mathbb{R}^n), d)$; (iii) every contraction on a complete metric space has a unique fixed point to which all orbits converge.

Let $(\mathcal{X}, \rho)$ be a metric space. A sequence $\{x_n\}$ in $\mathcal{X}$ is a *Cauchy sequence* if for any $\epsilon > 0$, there is an integer N such that

$$n, m \geq N \;\Rightarrow\; \rho(x_n, x_m) < \epsilon.$$

It is easy to see that any sequence converging to a limit x in $\mathcal{X}$ is a Cauchy sequence. The metric space is said to be *complete* if the converse is true: that is, every Cauchy sequence in $\mathcal{X}$ has a limit $x \in \mathcal{X}$.

Proof of (i) is omitted. It is a good exercise on the level of Math 424. You can read it e.g. in [Barnsley, *Fractals Everywhere*, p. 35–39].

Proof of (ii). Recall that f_i are contractions, that is, for some $\beta \in (0, 1)$,

$$|f_i(x) - f_i(y)| \leq \beta |x - y|, \quad i = 1, \ldots, m; \quad \forall\, x, y \in \mathbb{R}^n. \tag{1}$$

We claim that $\mathcal{F}$ is a contraction on $(\mathcal{H}(\mathbb{R}^n), d)$ with the same contractivity factor β. Let E and E' be two nonempty compact sets and $\delta = d(E, E')$. This implies that $E' \subseteq E_\delta$. We claim that

$$f_i(E') \subseteq (f_i(E))_{\beta\delta}, \quad i = 1, \ldots, m.$$

Indeed, for any $x \in E'$ there exists $y \in E$ such that $|x - y| \leq \delta$. Then $|f_i(x) - f_i(y)| \leq \beta\delta$, and $f_i(y) \in f_i(E)$, verifying the statement. Now it follows that

$$\mathcal{F}(E') = \bigcup_{i=1}^{m} f_i(E') \subseteq \bigcup_{i=1}^{m} (f_i(E))_{\beta\delta} = \left(\bigcup_{i=1}^{m} f_i(E) \right)_{\beta\delta} = (\mathcal{F}(E))_{\beta\delta}.$$

Similarly, $\mathcal{F}(E) \subseteq (\mathcal{F}(E'))_{\beta\delta}$, hence

$$d(\mathcal{F}(E), \mathcal{F}(E')) \leq \beta d(E, E')$$

proving (ii).

Proof of (iii). Basically, we are proving

CONTRACTION FIXED POINT THEOREM. *Let $(\mathcal{X}, \rho)$ be a complete metric space and let $f : \mathcal{X} \to \mathcal{X}$ be a contraction. Then f has a unique fixed point x, and all f-orbits converge to x.*

Proof. Pick any $x_0 \in \mathcal{X}$ and consider $x_n = f^n(x_0)$. Let us prove that this is a Cauchy sequence. We have (denoting by β the contractivity factor),

$$\rho(x_n, x_{n+1}) = \rho(f^n(x_0), f^{n+1}(x_0)) \leq \beta\rho(f^{n-1}(x_0), f^n(x_0)) \leq \ldots \leq \beta^n \rho(x_0, x_1).$$

Therefore, we can write for $n, k > 0$:

$$\begin{aligned} \rho(x_n, x_{n+k}) &\leq \rho(x_n, x_{n+1}) + \cdots + \rho(x_{n+k-1}, x_{n+k}) \\ &\leq \beta^n \rho(x_0, x_1) + \cdots + \beta^{n+k-1} \rho(x_0, x_1) \\ &= (\beta^n + \cdots + \beta^{n+k-1})\, \rho(x_0, x_1) \\ &\leq \frac{\beta^n}{1-\beta}\, \rho(x_0, x_1) \end{aligned}$$

It follows that

$$\frac{\beta^N}{1-\beta}\, \rho(x_0, x_1) \leq \epsilon \implies \rho(x_n, x_m) \leq \epsilon \text{ for } n, m \geq N.$$

Since $\beta \in (0, 1)$, this proves that the sequence $\{x_n\}$ is Cauchy. By completeness, there is a limit $x = \lim x_n \in \mathcal{X}$. We have $x = \lim_{n\to\infty} f^n(x_0)$, and since f is continuous (being a contraction), we conclude that $f(x) = x$. Observe that f cannot have two distinct fixed points: if $f(y) = y$ and $x \neq y$, then

$$\rho(x, y) = \rho(f(x), f(y)) \leq \beta\rho(x, y) < \rho(x, y),$$

which is a contradiction. Finally, since the point x_0 at the beginning of the proof was arbitrary, we conclude that every orbit converges to a fixed point. But since the fixed point is unique, every orbit converges to x. □

This concludes the proof of Hutchinson's Theorem as well! □

4. Dimension of self-similar sets

Recall that a set S is *self-similar* if $S = \bigcup_{i=1}^m S_i$ where S_i is congruent to $r_i S$ for some $r_i \in (0,1)$. Every self-similar set S is the attractor of an IFS $\{f_1, \ldots, f_m\}$ where f_i are linear affine mappings with the property that

$$|f_i(x) - f_i(y)| = r_i |x - y| \tag{2}$$

(note that we have equality here, rather than inequality for a general contraction in (1); functions satisfying (2) are called *similitudes*). The "fractal dimensions" (box-counting, Hausdorff, etc.) of a self-similar set can be computed exactly when the sets S_i are "non-overlapping" or "just-touching." This is made precise by the following definition.

Definition. Let $\{f_1, \ldots, f_m\}$ be an IFS. We say that it satisfies the *Open Set Condition* if there exists an open set $V \subseteq \mathbb{R}^n$ such that

$$f_i(V) \subseteq V, \quad i \leq m, \quad \text{and} \quad f_i(V) \cap f_j(V) = \emptyset, \quad i \neq j.$$

Theorem 3. *Let $\{f_1, \ldots, f_m\}$ be an IFS where f_i is a similitude satisfying (2), and suppose that the Open Set Condition is satisfied. Then the box-counting dimension (and also the Hausdorff dimension) of the attractor exists and equals the unique positive number s such that*

$$\sum_{i=1}^m r_i^s = 1.$$

If all the contraction rates are equal to r, then this dimension is equal to

$$s = \frac{\log m}{\log(1/r)}$$

(which we called the similarity dimension of the attractor).

For the proof of this theorem, see e.g. [Falconer, *Fractal Geometry*, 1990, p. 117–120].

Examples. 1. If the sets S_i that the self-similar set is composed of, are disjoint, then the Open Set Condition holds (this is an easy exercise.) Thus, the Cantor set and its variants (when the limit set is totally disconnected) are covered by Theorem 3.

2. The standard fractals, such as the Sierpinski gasket and carpet, and the von Koch curve, all satisfy the Open Set Condition. For instance, for the Sierpinski gasket, we need to take the OPEN triangle as our open set V; its images are disjoint since the only intersection of the triangles in the construction is along the boundary. In other cases, we take V to be the open square, etc.