1. Prove that all of the zeros of the polynomial $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ lie in the disk centered at 0 with radius $R = \sqrt{1 + |c_{n-1}|^2 + \cdots + |c_1|^2 + |c_0|^2}$.

2. Let $f : \mathbb{D} \to G$ be a conformal map with $f(0) = 0$, and define $g(z) = z\sqrt{f(z^2)/z^2}$. Prove that g is a well-defined analytic function and a conformal map.

3. Prove that if a sequence of analytic polynomials converges uniformly on a region Ω thenthe sequence converges uniformly on a simply connected region containing Ω.

4. Does there exist a function f analytic on $\{z : |z| \leq 300\}$, with $f(0) = 1$, with 10 zeros in $\{z : |z| \leq 100\}$ and satisfying $|f(z)| < 1024$ when $|z| = 300$? Produce such a function f or prove it does not exist.

5. Suppose f is an entire function with $|f(z)| = 1$ for $|z| = 1$. Show that $f(z) = e^{i\theta} z^n$ for some θ and $n \in \mathbb{N}$.

6. Prove that there do not exist entire functions f and g such that $e^{f(z)} + e^{g(z)} = 1$ for all $z \in \mathbb{C}$.

7. Let $\{f_n\}$ be a collection of analytic functions on a region Ω with $|f_n| \leq 1$ on Ω. Let K be compact and contained in Ω. Supose $\{f_n\}$ converges at infinitely many points in K. Then is it true or false that $\{f_n\}$ necessarily converges at every point of Ω?

8. Let F_M be the set of functions analytic on the open unit disk $\mathbb{D}$ and continuous on the closed unit disk which satisfy

$$\int_0^{2\pi} |f(e^{i\theta}| \, d\theta \leq M.$$

Show F_M is a normal family on $\mathbb{D}$ with respect to the Euclidean metric.

9. Construct the conformal map of the following regions Ω onto $\mathbb{D}$, with $f(z_0) = 0$ and $f'(z_0) > 0$. You may leave your answer as an explicit sequence of maps, but you must show that each map does what you claim it does.

(i) $\Omega = \{z : 0 < \text{Im}(z) < 1\},\ z_0 = i/2$; (ii) $\Omega = \mathbb{D} \cap \{z : \text{Im}(z) > 1/2\},\ z_0 = 3i/4$;

$$\text{(iii) } \Omega = \{z : 0 < \text{Im}(z) < 1,\ \text{Re}(z) > 0\},\ z_0 = (1+i)/2.$$

10. Find any conformal map of the following region onto $\mathbb{D}$:

$$\Omega = \{z : |z| < 1\} \cap \{z : |z-1| < 1\} \cap \{z : \text{Im}(z) > 0\}.$$

In addition, look over the following problems from Gamelin and make sure you know how to solve them:

8.1: 5, 6; 8.2: 1, 3, 4; 8.5: 1, 3; 8.6: 3, 6; 9.2: 5-9; 9.3: 1-4; 10.1: 2; 10.2: 1; 10.3: 9