The Volatility-stabilized Market Model

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Models for equity markets

- Market with n stock capitals $(X_1, \dots, X_n)$.
- Standard model of time-dynamics - geometric BM.

$$d \log X_i(t) = \gamma_i(t)dt + \sigma_i(t)dW_i(t), \qquad i = 1, 2, \dots$$

- $(W_1, \dots, W_n)$ - correlated BM.
- Too general.

Market weights

- Market weights:

$$\mu_i = \frac{X_i}{X_1 + \ldots + X_n}, \quad i = 1, 2, \ldots$$

- Measure the influence of individuals.
- Empirically observed: lower μ_i = higher growth & volatility.
- Models where γ_i and σ_i depend on μ_i ?

Volatility-stabilized markets

- (Fernholz-Karatzas '05) Volatility-stabilized market (VSM).
- (P. '09) Parameters $(\delta_1, \ldots, \delta_n) \geq 0$.

$$d \log X_i(t) = \frac{\delta_i - 1}{2\mu_i(t)} dt + \frac{1}{\sqrt{\mu_i(t)}} dW_i(t), \quad i = 1, 2, \ldots.$$

- $(W_1, \ldots, W_n)$ are iid BMs.
- Existence and weak uniqueness.

Wild fluctuation; sedate market

- $S = X_1 + \ldots X_n \quad ; \delta_0 = \delta_1 + \ldots + \delta_n$.
- Then
$$dS(t) = \frac{\delta_0}{2} S(t)dt + S(t)d\beta(t), \qquad \text{GBM.}$$
- The market is sedate.
- Individuals fluctuate wildly. No diversity.
- Describe process of market weights.

Market weights

- Peculiar characteristics of market weights.

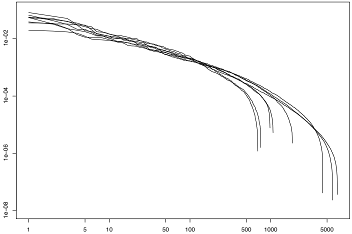

Figure 1: Capital distribution curves: 1929–1999

- $\log \mu_i \approx -\gamma \log i$, or $\mu_i \approx i^{-\gamma}$.
- Does VSM lead to Power-law decay?

Some answers

(P. '09)

- Market weights process - diffusion !
- Independent of S.
- The famous Wright-Fisher model.
- Genetic transfer to non-overlapping generations.

Wright-Fisher models

- Markov chain - urn model.
- M Balls of n - colors.
- Pick M randomly w.r.
- Fill another urn with copies.
- Ultimately monochromatic.
- No absorption.

Mutations

- Introduce mutation probabilities q_{ij}, $1 \leq i, j \leq n$.
- Suppose we pick color i from Urn $(k-1)$.
- But replace color i in Urn k w.p.

$$1 - \sum_{j \neq i} q_{ij}$$

- Or else, replace by color j w.p. q_{ij}.

Simplifications

- Popular simplification:
$$q_{ij} = q_j.$$
- Diffusion approximation to proportion of colors.
- Rate of mutation δ_j:
$$q_j = \delta_j \Delta t.$$
- Diffusion on the unit simplex
$$\{x_i \geq 0 : \quad x_1 + \ldots + x_n = 1\}$$
- WF diffusion - parameters $(\delta_1, \ldots, \delta_n)$.

So what ?

- So what ? Can one do any math ?
- MC is difficult.
- Even invariant distributions are not known explicitly.
- The diffusion is better.
- Invariant distribution Dirichlet$(\delta_1, \ldots, \delta_n)$.
- But still ...

BESQ connection

- Z - Bessel-square - Q_z^δ

$$Z(t) = z + \delta t + 2\int_0^t \sqrt{|Z(s)|}d\beta(s), \qquad z \geq 0, \quad \delta \geq 0.$$

- $\delta = 1, 2, \ldots$ - norm-square of δ dimensional BM.
- $\delta = 0$ - Feller branching diffusion.

A time-change relation

F-K '05, P. '09

- X - VSM$(\delta_1, \ldots, \delta_n)$.
- $(Z_1, \ldots, Z_n)$ - BESQ $(2\delta_1, \ldots, 2\delta_n)$.
- $\zeta = Z_1 + \ldots + Z_n$.

$$X_i = Z_i\left(4C_t\right), \quad C_t = \int_0^t \zeta^{-1}(u)du.$$

- $S(t) = \zeta\left(4C_t\right)$, ζ-BESQ$(2\delta_0)$

Corollaries

- Assume each $\delta_i > 0$. Individuals can hit zero. No absorption.
- $\delta_0 > 1$. The entire market does not hit zero.
- Market weights:
$$\mu_i(t) = \frac{X_i}{S}(t) = \frac{Z_i}{\zeta}(C_t).$$
- Invariant distribution $\mathrm{Dir}(\delta_1, \ldots, \delta_n)$.
- $\varsigma_a = \inf\{t : \; S(t) \geq a\} \qquad \sigma_a = \inf\{t : \; \zeta(t) \geq a\}.$
- $$\mu_i(\varsigma_a) = \frac{Z_i}{\zeta}(\sigma_a).$$

BESQ exit density

- $Z_1, \ldots, Z_n$ - independent BESQ - $(2\delta_1, \ldots, 2\delta_n)$.
- Time zero - $Z(0)$ in

$$U = \{y: \quad y_i \geq 0, \quad y_1 + \ldots y_n \leq 1\}.$$

- Z - transient. Exit density on boundary ?
- $p(\varsigma_1, z, y)$.

Stopped density

- $X(0) = x; s = x_1 + \ldots + x_n < a$.
- $y_i \geq 0, y_1 + \ldots + y_n = 1$.
- $p(\varsigma_a, x, y)$ - stopped density of market weights.

Theorem.(P. '09)

$$\begin{aligned}\varphi(x,y) = (1 - s/a) \sum_{m=0}^{\infty} \frac{\Gamma(2m+\delta_0)}{m!\Gamma(m+\delta_0)} (1+s/a)^{-2m-\delta_0} \\ \times \sum_{k:\ k_1+\ldots+k_n=m} \binom{m}{k_1 \cdots k_n} \prod_{i=1}^{n} (x_i/a)^{k_i} \operatorname{Dir}(y; k+\delta).\end{aligned} \tag{1}$$

$$\operatorname{Dir}(y;\gamma) = \frac{\prod_{i=1}^{n} \Gamma(\gamma_i)}{\Gamma\left(\sum_i \gamma_i\right)} \prod_{i=1}^{n} y_i^{\gamma_i - 1}. \tag{2}$$

Market weights at fixed times

- $\mu = (\mu_1, \ldots, \mu_n)$ - diffusion independent of S.
- (P. '09) Transition density $p(t, \xi, y)$

$$p(t,\xi,y) = \sum_{m=0}^{\infty} \frac{\Gamma(2m+d)}{m!\Gamma(m+d)} b_m(t) \times \sum_{k_1+\ldots+k_n=m} \binom{m}{k_1 \cdots k_n} \prod_{i=1}^{n} (\xi_i)^{k_i} \operatorname{Dir}(y; k+\delta).$$

$$\operatorname{Dir}(y;\gamma) = \frac{\prod_{i=1}^{n} \Gamma(\gamma_i)}{\Gamma\left(\sum_i \gamma_i\right)} \prod_{i=1}^{n} y_i^{\gamma_i - 1}.$$

- Coefficients are explicit.

Idea of the proof

- $Z_1, \ldots, Z_n$ - BESQ$(\theta_1, \ldots, \theta_n)$. $\quad \theta = \theta_1 + \ldots + \theta_n$.

$$L = \sum_{i=1}^{n} \theta_i \partial_i + 2 \sum_{i=1}^{n} x_i \partial_i^2 .$$

- Inversion wrt simplex

$$I(x) = \frac{x}{\left(\sum_{i=1}^{n} x_i\right)^2}, \quad K[u](x) = \left(\sum_{i=1}^{n} x_i\right)^{1-\theta/2} u\left(I(x)\right).$$

- Analogous to Kelvin transform for the laplacian.

The BESQ potential

- $u : D \to \mathbb{R}$.

$$Lu = 0, \quad x \in D \quad \Leftrightarrow \quad LK[u] = 0, \quad x \in I(D).$$

- Potential kernel

$$u_y(x) = u(x, y) = \int_0^\infty \prod_{i=1}^n p(Z_i(t) = y_i \mid Z_i(0) = x_i)\, dt.$$

- Explicit description - non-central χ^2.

$$Lu_y(x) = \delta_y(x).$$

The Green kernel inside the simplex

- y inside simplex.

$$
\begin{aligned}
v_y(x) &= v(x,y) = u_y - K[u_y] \\
Lv_y &= \delta_y, \qquad v_y|_{\mathbb{S}} = 0.
\end{aligned}
$$

- $\mathbb{S} = \{x \geq 0, \quad \sum_i x_i = 1\}$.
- Explicit Green kernel $v(x,y)$.

The exit density as a derivative

- BESQ exit density $\varphi(x,y)$: $\sum_i x_i < 1, \sum_i y_i = 1.$

$$\varphi(x,y) = -2\frac{\omega(y)}{\omega(x)} \sum_{i=1}^{n} y_i \ \ \partial_i v(y,x), \qquad y \in S,$$

$$\omega(x) = \prod_{i=1}^{n} x_i^{\theta_i/2-1}.$$

- Proof - integration by parts.

Poisson-Dirichlet

- Random point process.
- $0 < \alpha < 1, \theta > -\alpha$.
- Stick-breaking scheme.
- $Y_1, Y_2, \dots$ - Y_n - $\mathrm{Beta}(1-\alpha, \theta + n\alpha)$ - independent.

$$V_1 = Y_1, \quad V_n = (1-Y_1)\cdots(1-Y_{n-1})Y_n.$$

- Order $V_{(1)} \geq V_{(2)} \geq \cdots$ - $\mathrm{PD}(\alpha, \theta)$.

In equilibrium

- Invariant distribution of market weights Dirichlet$(\delta_1, \ldots, \delta_n)$.
- Power law decay ?
- Suppose
$$n \to \infty, \quad \max_i \theta_i \to 0, \quad \sum_i \delta_i \to \theta.$$
- Ordered $\mu_{(1)} \geq \mu_{(2)} \geq \ldots$ converges to PD$(0, \theta)$.
- Exponential decay.
- Contrast Rank-based models.

Thank you