Spectral properties of random regular graphs

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Definition

- d-Regular graph: every vertex - degree d.
- Simple graphs - no self-loops, no multiple edges.
- Random d-regular graph on n vertices

  - Uniform distribution.
- $G(n, d)$ - random of degree d on n vertices.
- Vertices randomly labeled $\{1, 2, \ldots, n\}$.

Adjacency matrices

- A - $n \times n$ matrix.

$$A(i,j) = \begin{cases} 1, & \textit{if} \quad i \sim j \\ 0, & \textit{otherwise} \end{cases}$$

- Random symmetric matrices.
- Eigenvalues ? Eigenvectors ?
- Combinatorial optimization, clustering algorithms, connectivity properties, principal component analysis.
- Fan Chung - *Spectral Graph Theory*.

Two parameters

- $G(n, d_n)$ - note the growth of d_n.
- E.g., the complete graph $d_n = n - 1$.

$$A = \mathbf{1}\mathbf{1}' - I.$$

- Eigenvalues $(n-1), -1$.
- Empirical Spectral Distribution, ESD

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}.$$

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ - eigenvalues of A.

What we know for fixed d

- When $d_n \equiv d$ - (McKay '81)

$$\text{ESD of} \quad G(n,d) \xrightarrow{p} \int f_d(x)dx.$$

  f_d - density supported on $[-2\sqrt{d-1},\ 2\sqrt{d-1}]$.
- Kesten ('59) - d regular tree.
- Spectral gap -
  -Füredi and Komlós ('81)
  -Broder and Shamir ('87)
  -Friedman ('91, '08)
- Markov chain mixing - Lubetzky and Sly ('09)

Alon Conjecture

- $G(n,d)$ - $\lambda_1 = d$ vector $n^{-1/2}\mathbf{1}$.
- $\delta_2 = \max(|\lambda_2|, |\lambda_n|)$

$$\delta_2 \geq 2\sqrt{d-1} + O\left(\frac{\log d}{\log n}\right).$$

- Alon Conjecture: w.h.p.

$$\delta_2 \leq 2\sqrt{d-1} + \epsilon, \quad n \to \infty, \ d - \text{fixed}.$$

- Freidman '08 - w.p. $1 - O\left(n^{-\tau}\right)$.

The case of slowly growing *d*

- What if d_n grows “slowly” with n ?
- Few rigorous results.
- Simulations (Jacobson et. al ’03, Elon ’08)
- A Gaussian limit ?
  Histograms of eigenvectors appear Gaussian.
- Confusion on the domain
  Linial - personal communication - not true for d=2,3.

Universality in Random Matrix Theory

- GOE - A random real symmetric matrix
  IID $N(0,1)$ entries on upper diagonal.
- Wigner - random real symmetric matrix
  IID entries $\sim F$, $\int x dF = 0$, $\int x^2 dF = 1$.
- "Wigner matrices have similar spectral behavior as GOE".
- Enormous literature -
  Bai ('93), Bai and Yao ('05), Soshnikov ('99), Guionnet and Zeitouni ('00), Erdös Schlein and Yau ('09), Tao and Vu ('09).
- Focus on weakening F - "at least 3 atoms, moment assumptions" etc.

Comparison with Wigner

- Adjacency matrices are not Wigner.
-
  1. No independence.
  2. Atomic distribution on $\{0,1\}$.
  3. Sparse.
- To make mean zero

$$\tilde{A} = A - \frac{d}{n}\mathbf{1}\mathbf{1}'.$$

- Sparsity affects concentration.

  Wigner $\quad \lambda_1 = 2\sqrt{n} \pm O\left(n^{-1/6}\right)$.

  $\lambda_1(\tilde{A}) \gg 2\sqrt{d-1} \quad$ when $d \gg \log n$.

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  $\lambda_1(\tilde{A}) \gg 2\sqrt{d-1} \quad$ when $d \gg \log n$.
- However, not all is lost ...

Convergence to semicircular law

Theorem (Dumitriu-P.)

Assume

$$\lim_{n\to\infty} d_n = \infty, \qquad d_n - 1 = n^{\epsilon_n}, \quad \epsilon_n = o(1).$$

Then

$$\textit{ESD of} \quad (d_n - 1)^{-1/2} A_n \xrightarrow{p} \textit{semicircular law on } [-2, 2],$$

density

$$f_{sc}(x) := \frac{1}{2\pi}\sqrt{4 - x^2}, \quad -2 < x < 2.$$

E.g., $d_n = (\log n)^\gamma$ for any positive γ.

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Semicircle law is the limiting ESD for $n^{-1/2}$ Wigner matrices.

Rate of convergence

Let

$$W_n = \frac{1}{\sqrt{d_n - 1}} \left(A_n - \frac{d_n}{n} \mathbf{1}\mathbf{1}' \right), \quad \mathcal{N}_I = \#\{\lambda_i(W_n) \in I\}.$$

Theorem (Dumitriu-P.)

Fix $\delta > 0$. Assume $d_n = (\log n)^{\gamma}, \quad 0 < \gamma < 1$.
Let

$$\eta_n = \frac{6(\log d_n)^{1+\sigma}}{\sqrt{\log n}}, \ \textit{for some } \sigma > 0 \,. \tag{1}$$

Take $|I| \geq \max\{2\eta_n, \eta_n/(-\delta \log \delta)\}$,

$$P\left(\sup_I \left| \frac{\mathcal{N}_I}{n} - \int_I f_{sc}(x)dx \right| > \delta\,|I| \right) = o\left(d_n^2/n \right).$$

Semicircular law and recursion

- Stieltjes transform: $B - n \times n$ symmetric.

$$s_n(z; B) = \frac{1}{n} \mathrm{tr}\,(B - z)^{-1}, \quad z \in \mathbb{C},\ \Im(z) > 0.$$

- Recursion

$$s_n(z; B) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{B_{kk} - z - b_k'(B_k - z)^{-1} b_k}.$$

- B_k - row k and col k removed.
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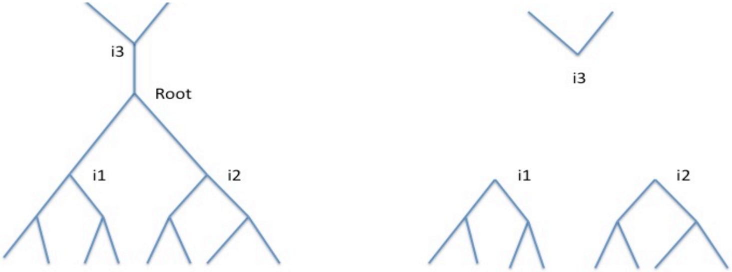
- When B adj matrix: B_k adj matrix of a subgraph

The local tree picture

$d \ll n$,   graph - locally d-regular tree for most vertices.



The local picture

- For d-regular graphs

$$s_n(z;A) = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{-z - \sum_{i\sim k}(A_k - z)^{-1}(i,i) + \epsilon_k}$$

$$(A_k - z)^{-1}(i,i) \approx (A - z)^{-1}(k,k) \approx s_n(z;A)$$

- Thus

$$s_n(z;A) \approx \frac{1}{-z - d\, s_n(z;A)}.$$

$$s_n(z;d^{-1/2}A) \approx \frac{1}{-z - s_n(z;d^{-1/2}A)}.$$

- $d_n \to \infty$ - errors go to zero.

Fixed point equation

- Fixed point equation

$$\phi(z) = \frac{1}{-z - \phi(z)}, \quad z \in \mathbb{C}.$$

- Unique solution $\phi(z) = \frac{1}{2}\left(-z + \sqrt{z^2 - 4}\right)$

$$\phi(z) = \int_{-2}^{2} \frac{1}{x - z} f_{sc}(x) dx.$$

- $\lim_{n\to\infty} s_n(z; d_n^{-1/2} A) = \phi(z) \quad \Rightarrow \quad$ ESD convergence.

Previous literature

- Local weak convergence -
  Aldous ('01 ?), Benjamini and Schramm ('01)
- In similar contexts - Bordenave and Lelarge ('08)
  Bhamidi, Evans, and Sen ('09)
- All for fixed d.
- In our case:
  - tree approximation for growing d_n.
  - semicircular law.
  - rate of convergence.

Eigenvectors

- $G(n, d_n)$ - adjacency matrix A_n.

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$
$$v_1 \geq v_2 \geq \ldots \geq v_n.$$

- In general, ill-defined

$$P\left(\lambda_i > \lambda_{i+1}, \text{ for all } i\right) = ?$$

- $v_1 = n^{-1/2}\mathbf{1}$. What can we say about v_i, $i > 1$?

The GOE picture

- Suppose $A - d/n\mathbf{1}\mathbf{1}' \quad \approx$ GOE.
- All eigenvectors of GOE is uniform over sphere.
- $X \sim \quad$ Uniform over sphere
  - Exchangeable coordinates -

$$\sqrt{n}X_i \sim N(0,1).$$

  - Lack of directional bias: H -subspace

$$P\left(\mathrm{dist}\left(X,H\right) < \delta\right) < \epsilon.$$

  - Delocalization: $J \subset \{1,2,\ldots,n\}$,

$$\sum_{i\in J} X_i^2 \approx \frac{|J|}{n}.$$

- But ... sparsity ...

Lack of bias

Every $v_i \perp \mathbf{1}$ for $i > 1$.

Theorem (Dumitriu-P.)

Fix $\delta > 0$. $d_n = (\log n)^{\gamma}, \quad 0 < \gamma < 1$.

$$\eta_n = \frac{6(\log d_n)^{1+\sigma}}{\sqrt{\log n}} \ , \ \textit{for some } \sigma > 0 \ . \tag{2}$$

$H \perp \mathbf{1}$*- dimension* $o(1/\eta_n)$. *Then*

$$P\left(\exists \ \ v_i \ s.t. \ \ \|v_i\| = 1 \ and \ \ \|P_H(v_i)\|^2 \geq 1 - \delta\right) \leq o\left(d_n^2/n\right) .$$

$P_H(v_i)$*-projection of* v *onto* H.

Delocalization

Theorem (Dumitriu-P.)

Same set-up. Fix $\delta > 0$.

Let $T_n \subseteq \{1, 2, \ldots, n\}$*; size* $L_n = o(\eta_n^{-1})$.

$$E_n = \left\{ \exists\, v_i \quad s.t. \quad \sum_{i \in T_n} v_i^2(j) \geq 1 - \delta \right\}.$$

Then,

$$P\left(E_n^c\right) \geq e^{-L_n \eta_n / d_n} \left(1 - o\left(d_n^2/n\right)\right).$$

Compare with Wigner, dense graphs

- Erdös, Schlein, Yau ('09), Tao and Vu ('09)
  Delocalization of eigenvectors

  $$\mathbb{L}^{\infty} - \text{norm of every eigenvector } \approx n^{-1/2}\text{log order terms.}$$

- Erdös-Renýi: $p = 1/2$, Dekel, Lee, and Linial ('07)
- General idea: concentration of measure applied to IID row entries.
- We don't have such luxury.

Open problems

- Prove that (even in the Wigner case) $\sqrt{n} v_i(j) \to N(0,1)$.
- A process version

$$Y_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} \left(v_i^2(j) - \frac{1}{n} \right) \to \text{Brownian Bridge.}$$

  Bai, Miao, and Pan ('07) - Wishart matrices.
- Conjecture (Elon). $d(j_1, j_2) = k$. $\quad Cov\left(v_i(j_1), v_i(j_2)\right)$

$$\approx \frac{1}{d(d-1)^{k/2}} \left[(d-1) U_k \left(\frac{\lambda_i}{2\sqrt{d-1}} \right) - U_{k-2} \left(\frac{\lambda_i}{2\sqrt{d-1}} \right) \right].$$

  U_k - Chebyshev polynomial.

Idea of proofs

- Central idea - estimate Stieltjes transform near $\mathbb{R}$.

$$W_n = \frac{1}{\sqrt{d-1}}\left(A - \frac{d}{n}\mathbf{1}\mathbf{1}'\right)$$

- Compare $s_n(z) = s_n(z; W_n)$ with $s(z)$:

$$\sup_{z:\Im(z)\geq\eta_n} |s_n(z) - s(z)|$$

- Smaller η_n, better control.

Idea of proofs

- Inversion formula: G continuous distribution

$$G[a,b] = \lim_{\epsilon \longrightarrow 0} \frac{1}{\pi} \int_a^b \Im\left(s_G(x+i\epsilon)\right) dx.$$

- Estimating $s_n(z) \approx s(z)$ for $\epsilon \approx 0$.
- Rate of convergence.
- Main idea - estimating tree approximation.

Idea of proofs

- Eigenvectors - estimate resolvents

$$a'(W_n - z)^{-1}a = \sum_{i=1}^{n} \frac{\langle a, v_i \rangle^2}{\lambda_i - z}.$$

- Estimate left-side by tree approximation.
- For $z = \lambda_i + i\eta_n$, leading term on right

$$\frac{\langle a, v_i \rangle^2}{\lambda_i - z} = \frac{\langle a, v_i \rangle^2}{i\eta_n}.$$

- Use $\eta_n \approx 0$.
- H= span$\{a_1, \ldots, a_m\}$

$$P_H(v_i) = \sum_{j=1}^{m} \langle a_j, v_i \rangle^2.$$

Idea of proofs

- Stiletjes transform and the Moment method.

$$(A-z)^{-1} = \sum_{s=0}^{2r_n} (z-\alpha)^{-s-1} (A-\alpha)^s + E(r_n).$$

- A - adjacency

$$A^s(i,j) = \#\text{ paths from } i \text{ to } j.$$

- Estimate the depth of tree approximation. $r_n \to \infty$.
- Compare with d_n-regular infinite tree.

Idea of proofs

Theorem (Mckay, Wormald, Wysocka '04)

$G(n, d_n)$. Let g_n be s.t.

$$(d_n - 1)^{2g_n - 1} = o(n).$$

Number of cycles of length r ($r \leq g_n$) is

$$\approx \textit{Poisson}\left(\frac{(d_n - 1)^r}{2r}\right).$$

We want $g_n \to \infty$. Hence $d_n = n^{\epsilon_n}$, $\epsilon_n = o(1)$.

Odds and Ends

- Monotonicity.
  Compare $G(n, d_n)$ with infinite d_n-regular tree.
  Monotonicity as d_n increases.

- **Theorem (Friedman '91)**
  Let $\delta_2 = \max(|\lambda_2|, |\lambda_n|)$. *Any* $\beta > 1$,

$$|\delta_2| \geq \left[2\sqrt{d-1}\left(1 + \frac{\log d}{\sqrt{d}} + O\left(\frac{1}{\sqrt{d}}\right)\right) + O\left(\frac{d^{3/2}\log\log n}{\log n}\right)\right]\beta$$

w.p.

$$\frac{\beta^2}{n^{2\lfloor\sqrt{d-1}/2\rfloor \log\beta/\log d}}.$$

Thank you