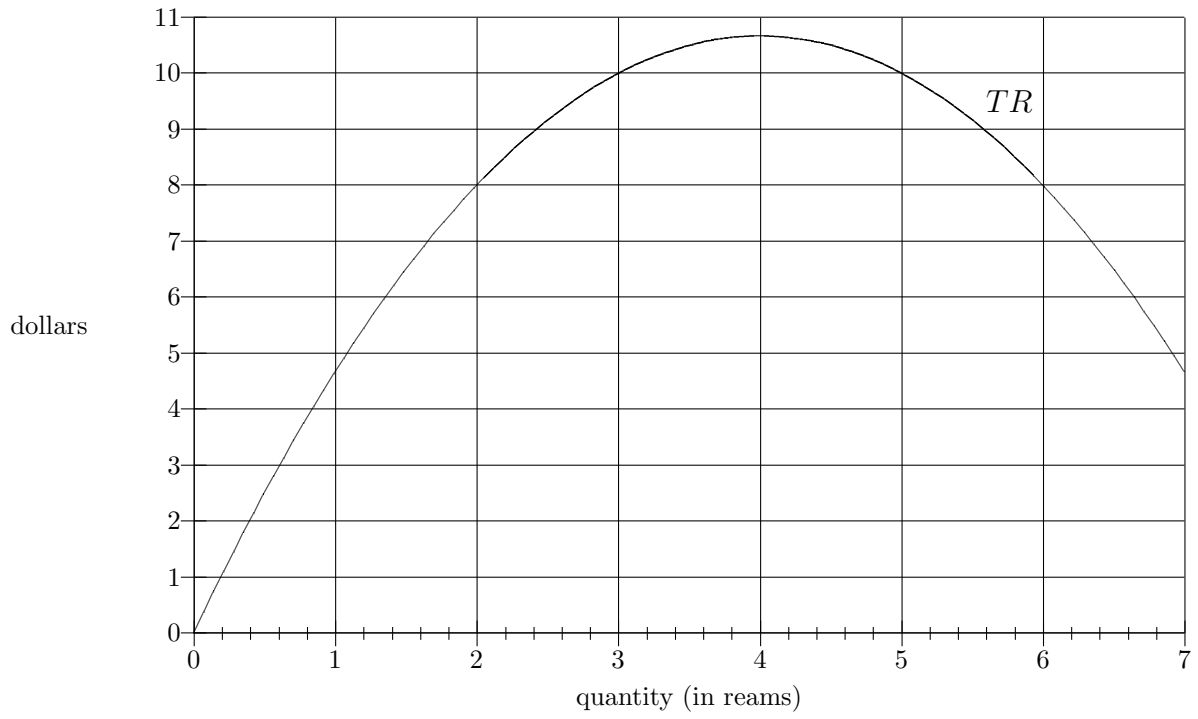
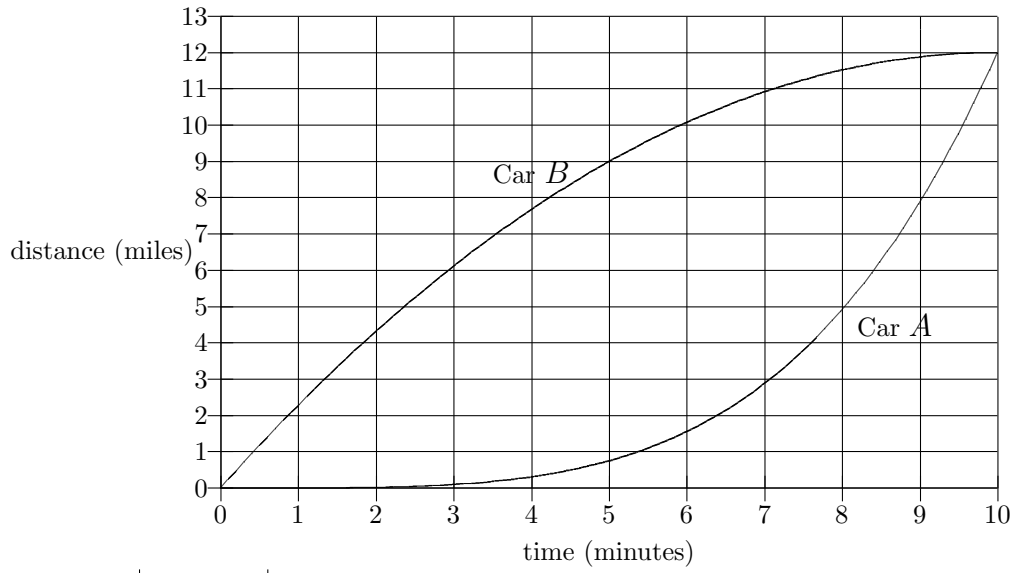


Math 112 – Spring 2005
Lecture Materials

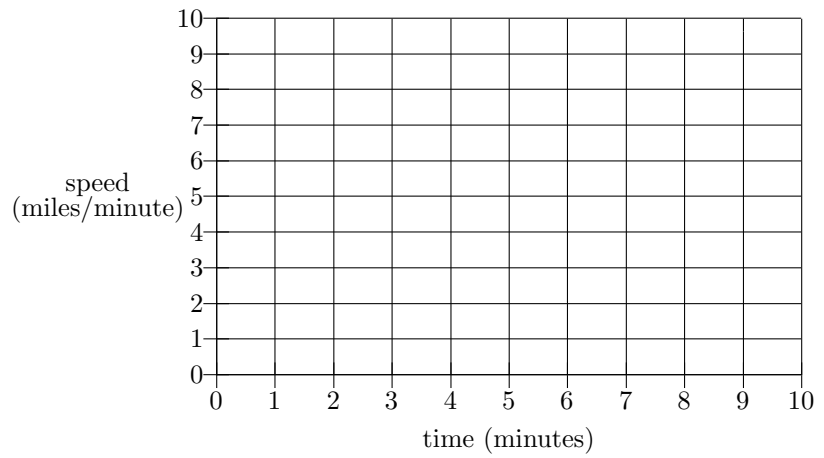
MATH 112
Lecture to Accompany Worksheet 2



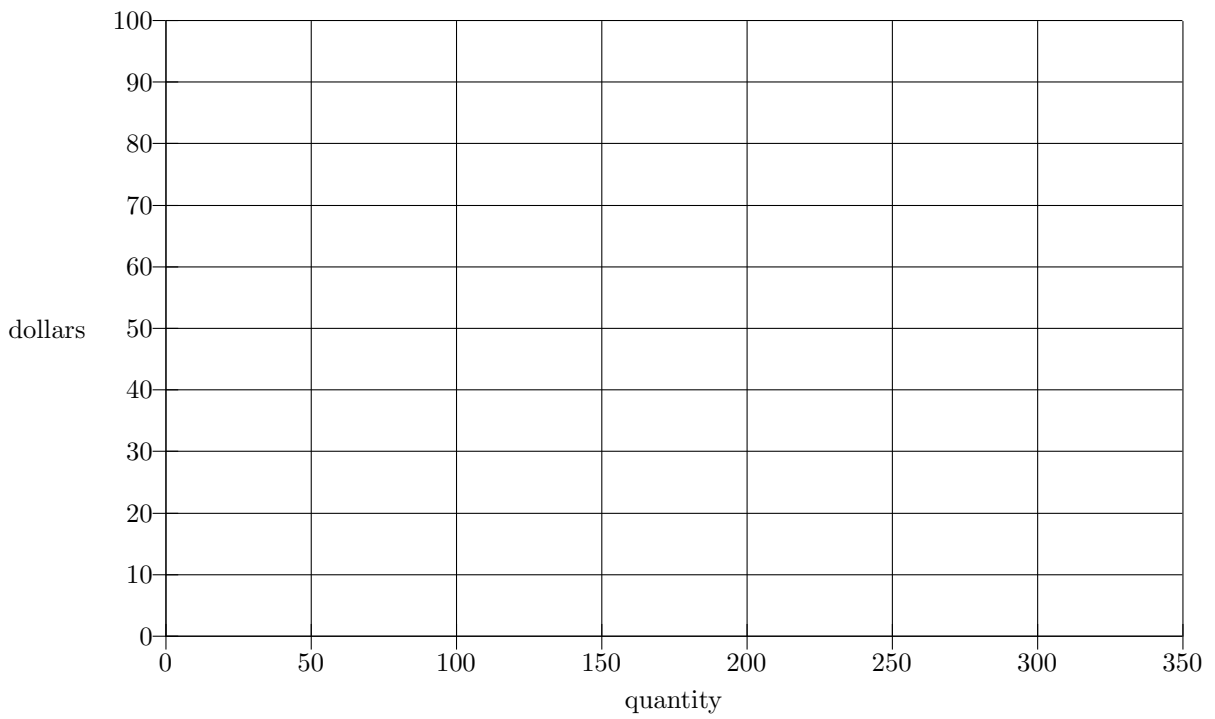
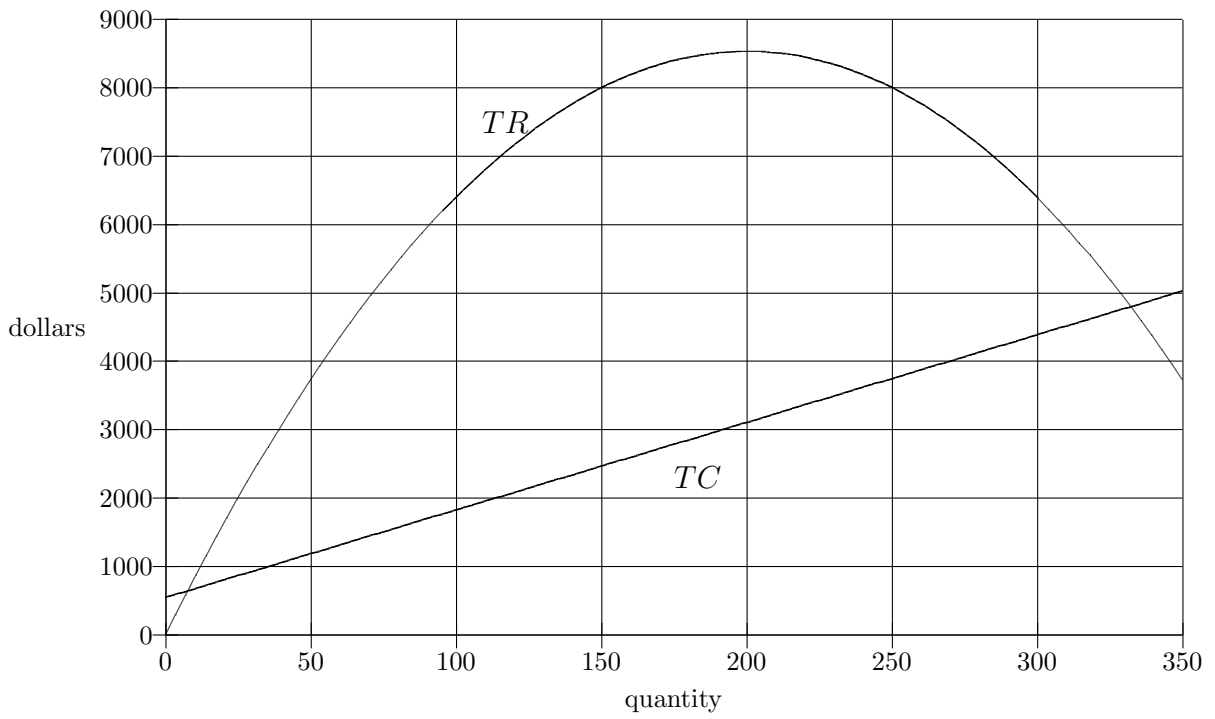
MATH 112
Lecture to Accompany Worksheet 4



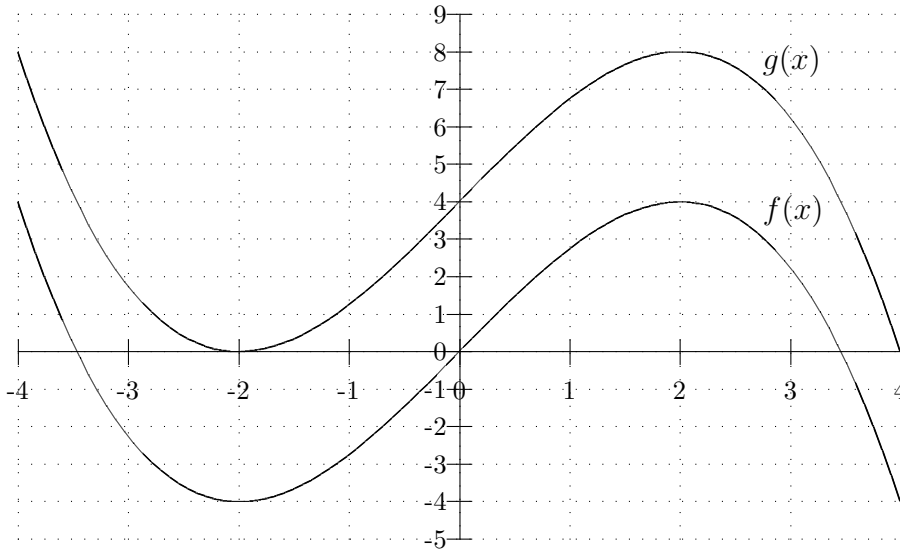
t	car A's speed	car B's speed
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		



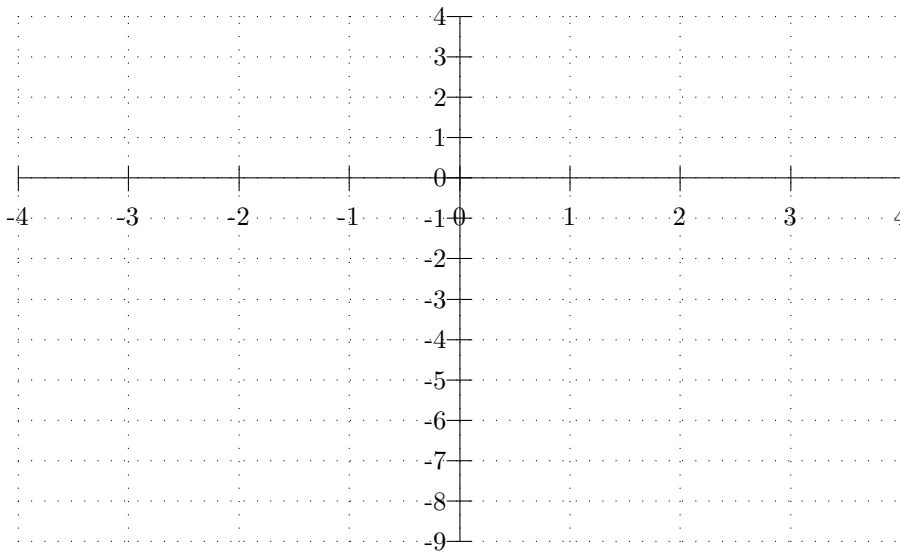
MATH 112
Lecture to Accompany Worksheet 5

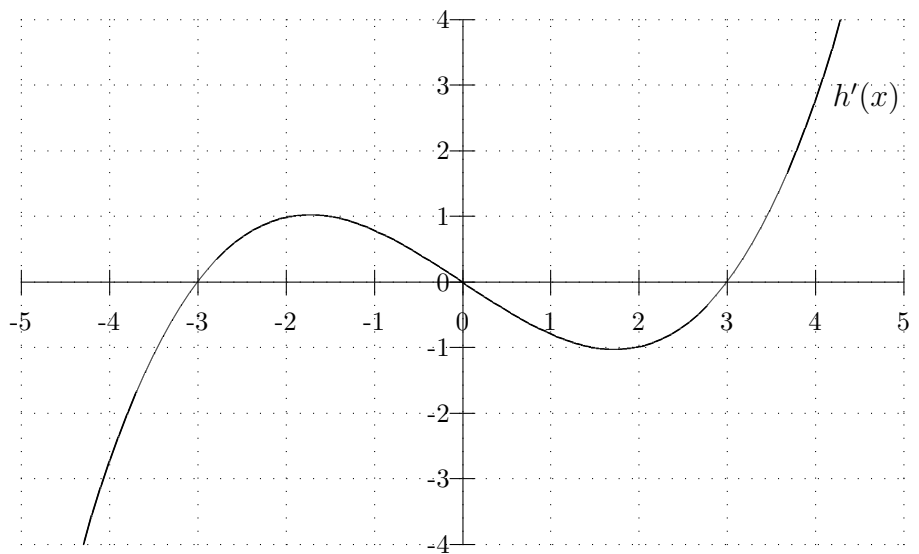
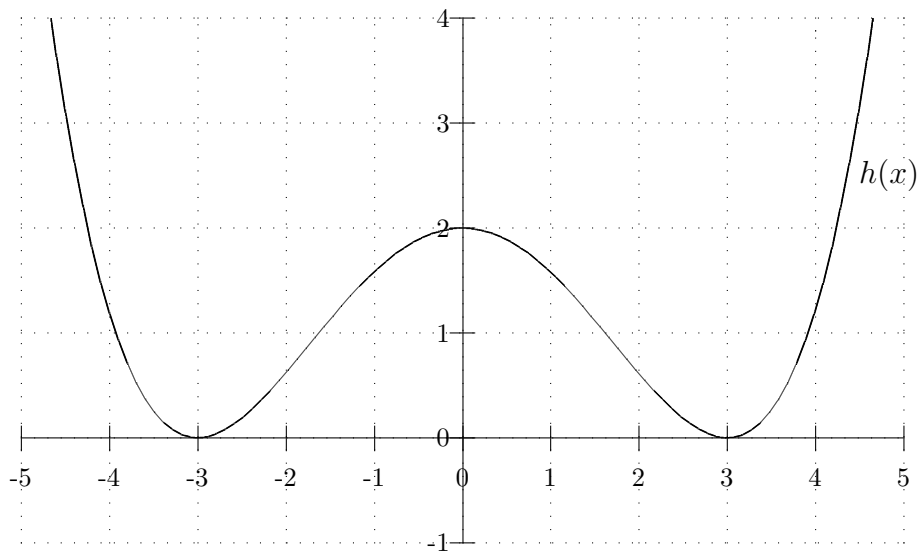


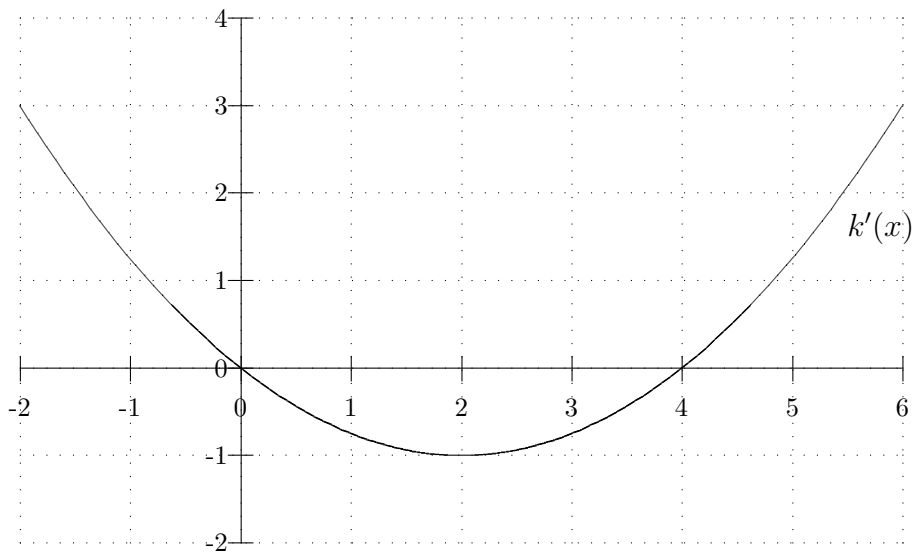
MATH 112
Lecture to Accompany Worksheet 6



x		-4		-3		-2		-1		0		1		2		3		4	
$f'(x)$																			
$g'(x)$																			







MATH 112

Lecture Questions to Accompany Worksheet 6

1. What does $f'(2)$ represent?
2. What is the value of $f'(2)$?
3. Where does the graph of $f'(x)$ cross the x -axis?
4. What happens to the graph of $f(x)$ at those places?
5. How are the graphs of $g(x)$ and $f(x)$ related?
6. What does $g'(1)$ represent?
7. What is the value of $g'(1)$?
8. Where does the graph of $h'(x)$ cross the x -axis?
9. What's happening to the graph of $h(x)$ at those values of x ?
10. Find an interval on which $h'(x)$ is negative.

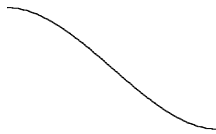
11. What does that mean about the graph of $h(x)$?

12. Find an interval on which $h'(x)$ is positive.

13. What does that mean about the graph of $h(x)$?

14. Find an interval on which $k'(x)$ is decreasing and positive.

15. Which of these two graphs could be the graph of $k(x)$ for $x = 0$ to $x = 4$?



16. Where does $k'(x)$ cross the x -axis?

17. Where does $k'(x)$ have horizontal tangents?

18. Where does $k(x)$ have horizontal tangents?

19. Find the intervals on which $k(x)$ is increasing.

MATH 112

Lecture to Accompany Worksheet 10

Recall the derivative rules:

$$f(x) = 3x^6 - 4x^2 + x + 7 \Rightarrow f'(x) =$$

Note that the variable in the function need not be named x :

$$g(q) = 10q^9 - 5q^2 + 9q - 3 \Rightarrow g'(q) =$$

PROFIT ANALYSIS USING DERIVATIVES (WS 10)

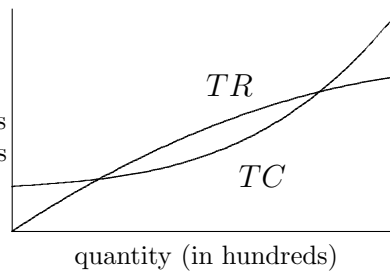
You manufacture hats.

The TR and TC functions are

$$TR : R(q) = -3.75q^2 + 28.5q$$

$$TC : C(q) = 2q^3 - .4q^2 + 3q + 15$$

where q is given in hundreds of hats of dollars and TR and TC are in hundreds of dollars. The graphs give TR and TC for 0 to about 300 hats.



1. Write out a formula for profit, $P(q)$.
2. **If** the formula for profit was a quadratic and its graph was a parabola that opens downward, describe how you would find the maximum profit. Explain why that method is inappropriate in this case.
3. **If** you had nice graphs of TR and TC , give at least two ways to find the quantity that maximizes profit.
4. Since profit is *not* a quadratic and we *don't* have nice graphs of TR and TC , we'll need to get formulas for MR and MC to find where profit is maximized. Recall that, even though the MR is by definition the slope of a secant line, we often approximate MR using the slope of a tangent line to TR . The slope of the tangent line is given by the derivative. Find formulas for MR and MC .

5. Give a verbal description of the graphs of MR and MC .

6. Sketch rough graphs of MR and MC .

7. Find the quantity that maximizes profit.

8. What is the maximum profit?

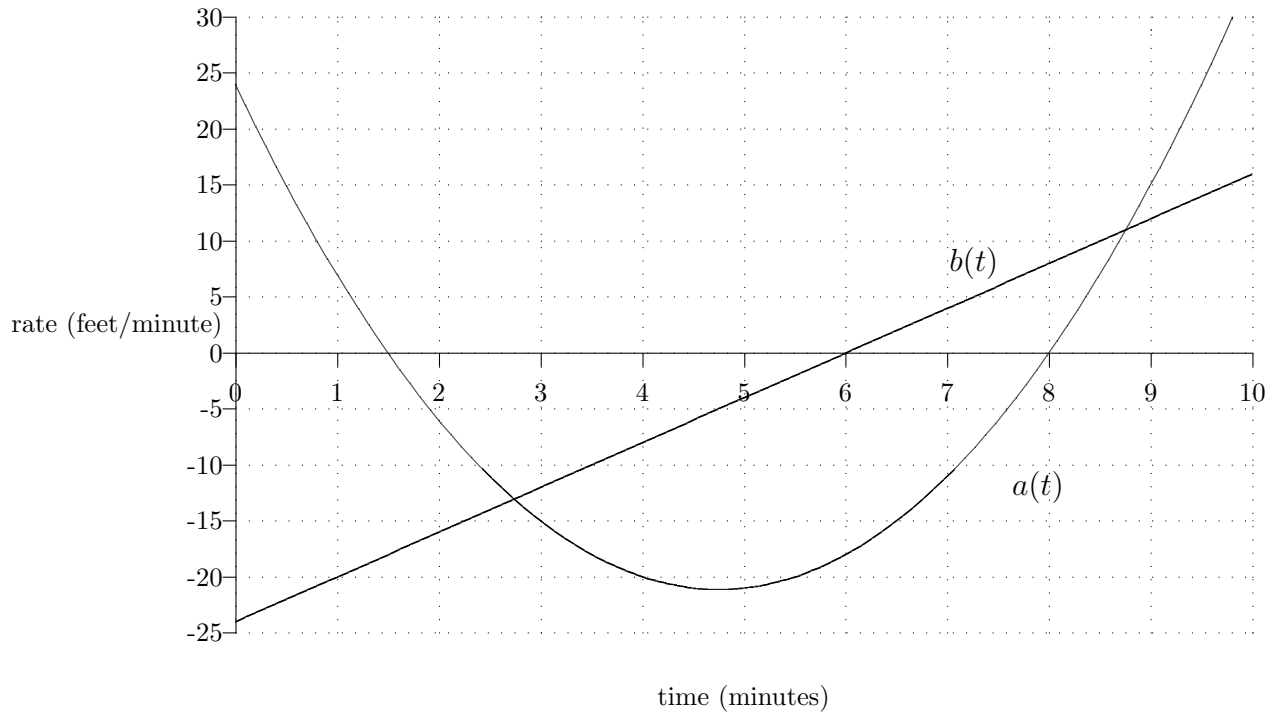
9. What quantity maximizes total revenue?

10. What is maximum total revenue?

11. What quantity gives the smallest marginal cost?

12. What is the cost of the 121st hat?

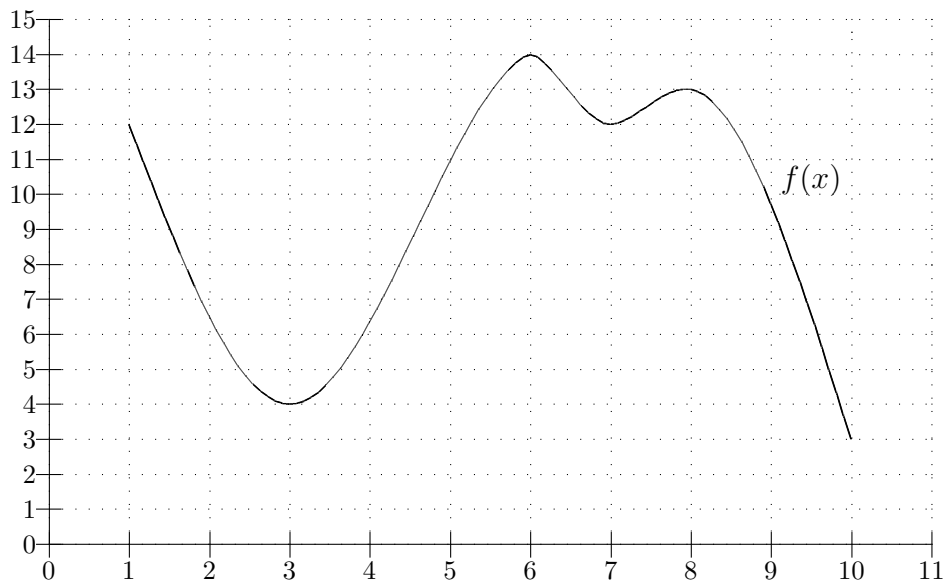
MATH 112
Lecture to Accompany Worksheet 11



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Lecture to Accompany Worksheets 15 and 16

The following is the graph of a function $f(x)$ on the interval from $x = 1$ to $x = 10$.



Two things are happening at $x = 3$:

1.

2.

Q: Name another value of x for which properties (1) and (2) are both true.

Q: What's happening at $x = 6$ and $x = 8$?

Q: What's happening at $x = 1$?

Q: at $x = 10$?

So, $f(x)$ has minimum values at $x = \underline{\hspace{2cm}}$, $x = \underline{\hspace{2cm}}$, and $x = \underline{\hspace{2cm}}$. On the interval from $x = 1$ to $x = 10$, $\underline{\hspace{2cm}}$ OR $\underline{\hspace{2cm}}$ is the absolute smallest value of $f(x)$. We say that, on this interval:

- $f(x)$ has $\underline{\hspace{2cm}}$ minimum values at $x = \underline{\hspace{2cm}}$, $x = \underline{\hspace{2cm}}$, and $x = \underline{\hspace{2cm}}$.
- $f(x)$ has a $\underline{\hspace{2cm}}$ minimum value at $x = \underline{\hspace{2cm}}$.

Q: Where does $f(x)$ have local maximum values on this interval?

Q: Where does $f(x)$ have a global maximum value on this interval?

Q: What is the global maximum value of $f(x)$ on this interval?

Def: We use the collective term *optimum values* to refer to maximum and minimum values.

FACT: The optimum values of a function $g(x)$ on the interval from $x = a$ to $x = b$ occur:

-
-

TWO APPLICATIONS:

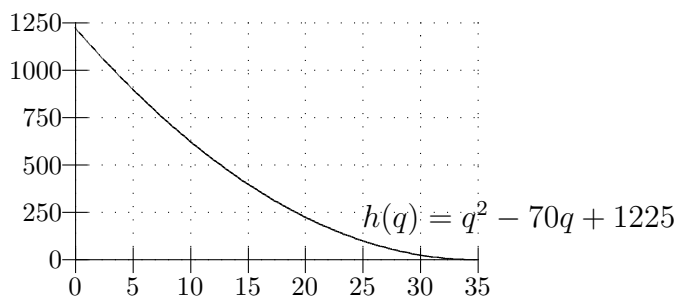
I. You sell *things*. You set the price per thing depending on the size of the order. If a customer orders q things, then the *price per thing* is:

$$p = h(q) = q^2 - 70q + 1225.$$

For example, if a customer orders 10 things, then $p = h(10) = 10^2 - 70(10) + 1225 = 625$ and the customer pays \$625 per thing for their order. Your revenue for that order is

$$R = \text{price} \cdot \text{quantity} = \$625 \cdot 10 = \$6250.$$

The function $h(q)$ that gives the price per thing is called a *demand function*. Its graph is a *demand curve*.



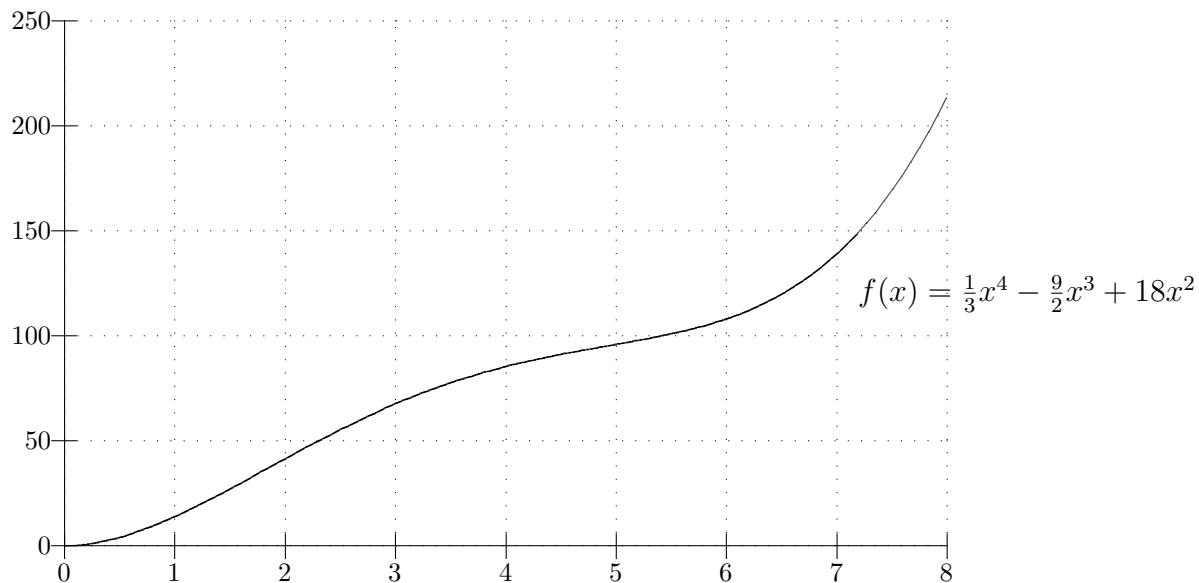
Goal: Find maximum possible revenue on the interval from $q = 0$ to $q = 35$.

Step 1: Compute $A'(q)$ and find where it equals 0.

Step 2: Compute $A(q)$ at the endpoints of your interval and at the points you found in Step 1.

Step 3: Draw a rough sketch of $A(q)$ and pick out the optima.

II. We have a function $f(x) = \frac{1}{3}x^4 - \frac{9}{2}x^3 + 18x^2$.



We define a function $S(x)$ as the slope of the diagonal line from the origin to the point $(x, f(x))$. That is,

$$S(x) = \frac{f(x)}{x}.$$

Goal: Find the maximum and minimum values of $S(x)$ on the interval from $x = 1$ to $x = 7$.

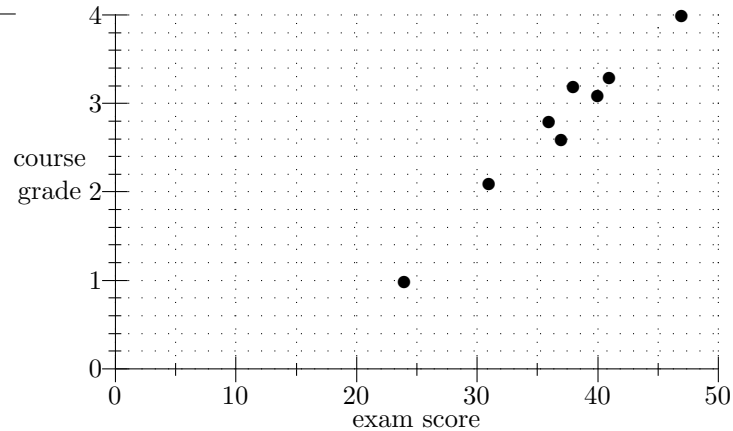
MATH 112

Lecture to Accompany Worksheet 18

LINE FITTING

Situation: The following is a chart containing the first midterm exam scores and final course grades for eight students in Math 112. Next to the chart is a graph of these data.

Student	Exam	Course Grade
1	31	2.1
2	40	3.1
3	38	3.2
4	47	4.0
5	24	1.0
6	37	2.6
7	36	2.8
8	41	3.3



Goal: We want to find a formula that will “predict” a student’s approximate course grade from his/her first exam score. This formula will be a *linear function* $g(t) = mt + b$ where m and b are chosen to give the line that fits the data best. The variable t represents the first midterm exam score and the value of $g(t)$ will be our prediction of the course grade.

Phase 1: Given a specific line, find a number that indicates *how well* the line fits the data.

Let $g(t) = 0.1t - 1$. Sketch the graph of $g(t)$ above. How well do you think it fits the data (not well, fairly well, very well)? To quantify this, complete the chart below:

i	1	2	3	4	5	6	7	8
t_i	31	40	38	47	24	37	36	41
s_i	2.1	3.1	3.2	4.0	1.0	2.6	2.8	3.3
$g(t_i)$								
$s_i - g(t_i)$								
$(s_i - g(t_i))^2$								

The number $g(t_i)$ is what our formula would predict for the course grade for student number i . The number $s_i - g(t_i)$ is the *error* in using the formula to make the prediction. We want to add up all of the errors. Since some of the errors might be negative and some might be positive, we square the errors to avoid the cancelling out that can occur.

We take the average of the squared errors:

$$\frac{(s_1 - g(t_1))^2 + (s_2 - g(t_2))^2 + \dots + (s_8 - g(t_8))^2}{8} = \frac{1}{8} \sum (s_i - g(t_i))^2 =$$

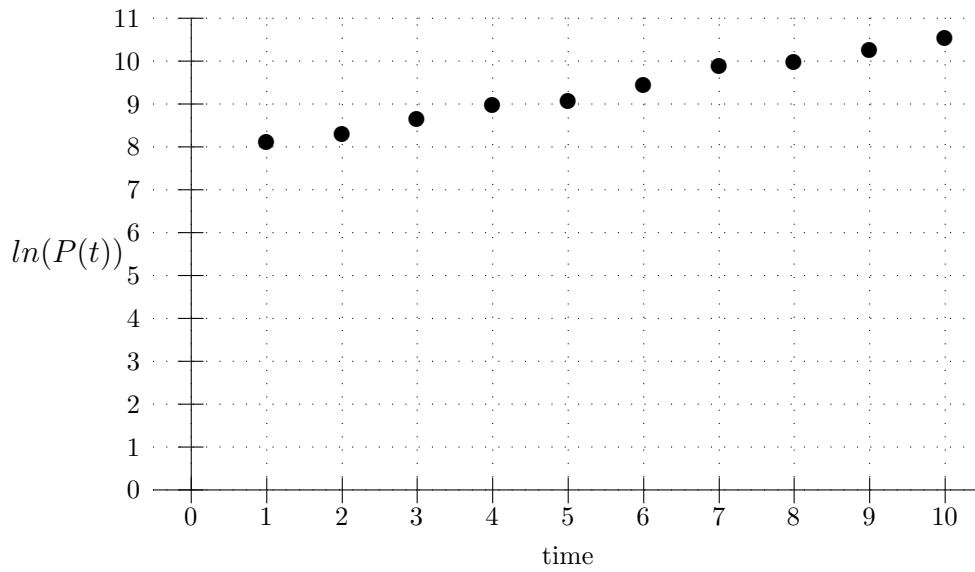
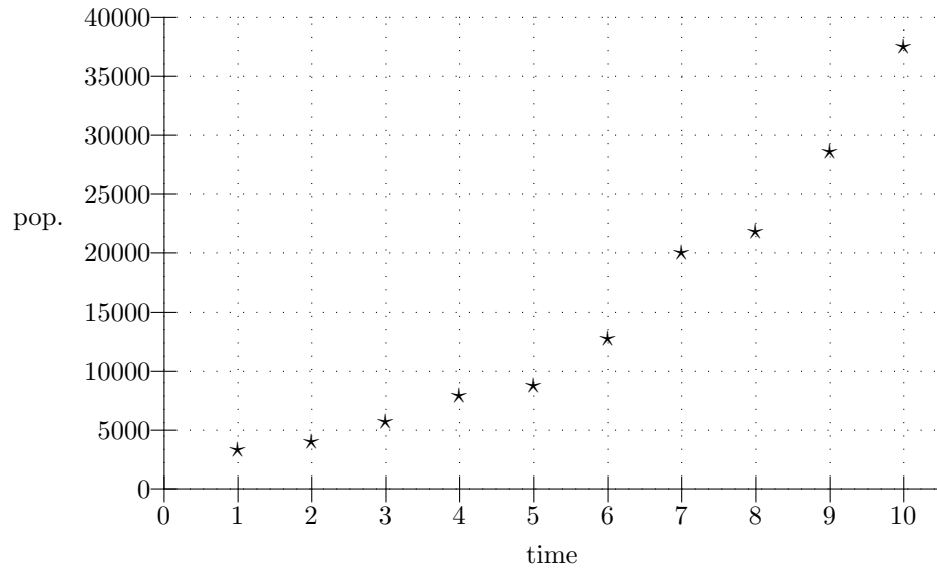
This number is the *mean squared error* for the line $g(t) = 0.1t - 1$. The smaller the MSE, the better the fit.

Phase 2: Find the line with the *smallest* mean squared error.

Let $g(t) = mt + b$ be a candidate for the line with the smallest mean squared error. The MSE for this line depends on the values of m and b . We want to find the values of m and b that *minimize* MSE.

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Lecture to Accompany Worksheet 14

t	$P(t)$
1	3310
2	4030
3	5680
4	7940
5	8740
6	12760
7	20000
8	21890
9	28680
10	37570



i	t_i	z_i	t_i^2	z_i^2	$t_i z_i$
1	1	8.10	1	65.61	8.10
2	2	8.30	4	68.89	16.6
3	3	8.64	9	74.6496	25.92
4	4	8.98	16	80.6404	35.92
5	5	9.08	25	82.4464	45.4
6	6	9.45	36	89.3025	56.7
7	7	9.90	49	98.01	69.3
8	8	9.99	64	99.8001	79.92
9	9	10.26	81	105.2676	92.34
10	10	10.53	100	110.8809	105.3
$\sum t_i = 55$		$\sum z_i = 93.23$	$\sum t_i^2 = 385$	$\sum z_i^2 = 875.4975$	$\sum t_i z_i = 535.5$

$$\begin{aligned}
E(b, m) &= \frac{1}{10} \{ 10b^2 + 385m^2 + 2(55)bm - 2(93.23)b - 2(535.5)m + 875.4975 \} \\
&= b^2 + 38.5m^2 + 11bm - 18.646b - 107.1m + 87.54975
\end{aligned}$$

$$\frac{\partial E}{\partial b} = 2b + 11m - 18.646 \quad \frac{\partial E}{\partial m} = 77m + 11b - 107.1$$

The smallest possible Mean Squared Error occurs when

$$\frac{\partial E}{\partial b} = 0 \text{ and } \frac{\partial E}{\partial m} = 0.$$

This occurs when $b = 7.807$ and $m = 0.2756$. So, the best-fitting line is

$$z = 0.2756t + 7.807.$$

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Lecture to Accompany Worksheet 19

Example: To earn a bit of extra cash, you start selling dried fruit and nut mixtures to hungry shoppers at the Fremont Sunday Flea Market. You sell two varieties:

- Mostly Nuts!, which contains 25% fruit and 75% nuts
- Fruitilicious!, which contains 55% fruit and 45% nuts

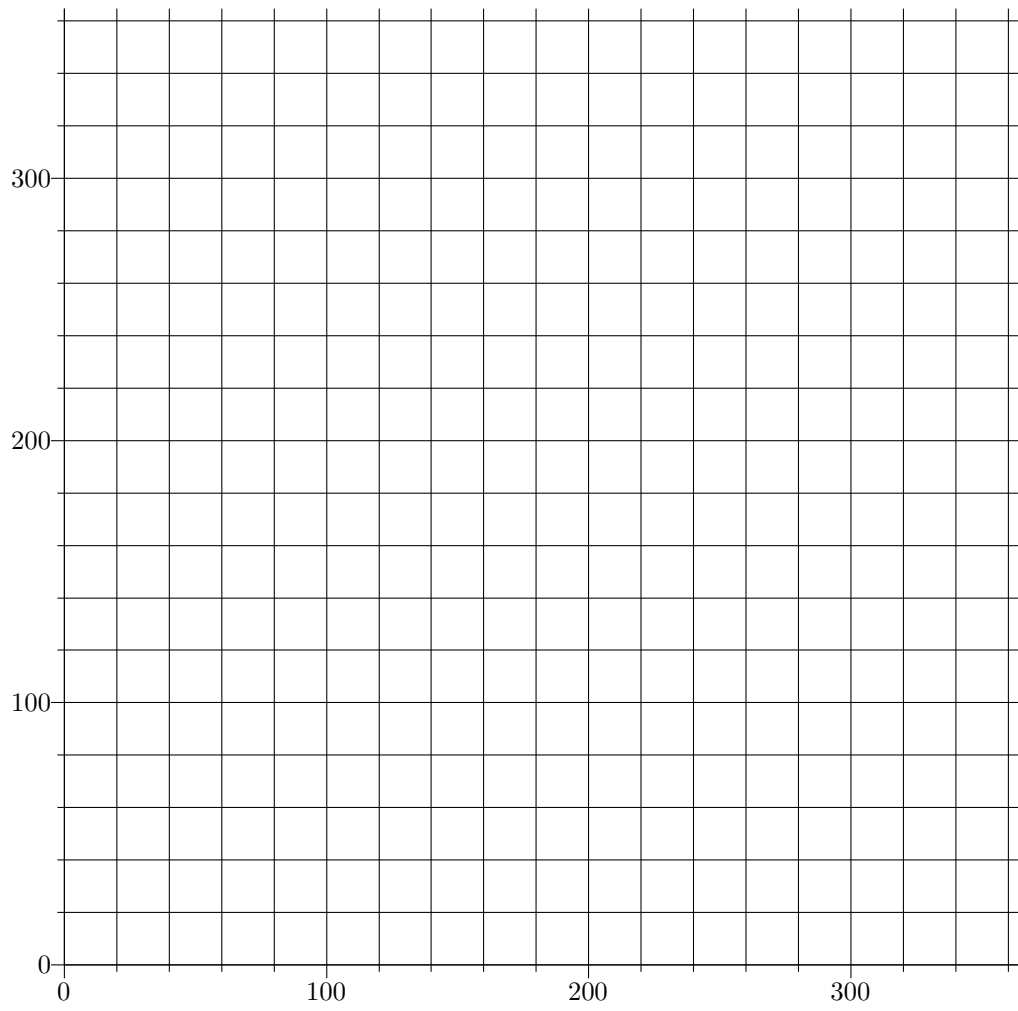
For every pound of Mostly Nuts! you sell, you make \$.35 profit; and for every pound of Fruitilicious!, you make \$.60 profit. Your supply of fruit is limited to 90 pounds a day; your supply of nuts is limited to 120 pounds a day.

Let x be the amount of Mostly Nuts! that you make (in pounds) and y be the amount of Fruitilicious! that you make (in pounds).

Key Question: How much of each mixture should you make in order to maximize profit?

Another Example: Gina inherits a large sum of money and a bunch of pet cages from an animal-loving aunt. She decides to rescue some unwanted pets from a shelter. She has 20 cages that can each house either a bunny or a ferret. She does some research and finds that, on average, it costs \$0.60 a day to feed one ferret and \$0.80 a day to feed one bunny. Gina can budget no more than \$14.40 a day for pet food. But cuddliness is an issue for Gina. She figures that bunnies are twice as cuddly as ferrets. That is, ferrets are each worth one cuddle-unit, while bunnies are each worth two.

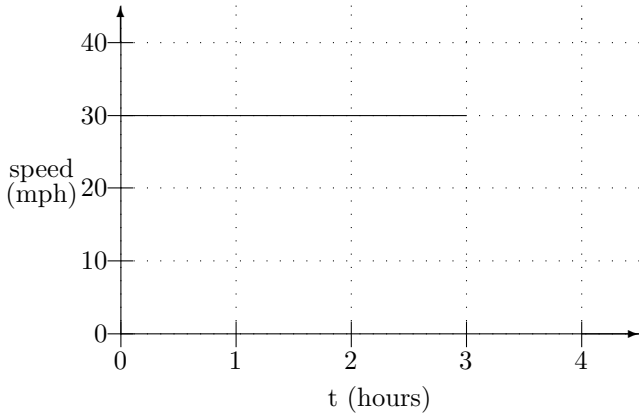
Determine how many of each pet Gina should adopt to maximize cuddliness while staying within her budget and without buying more cages.



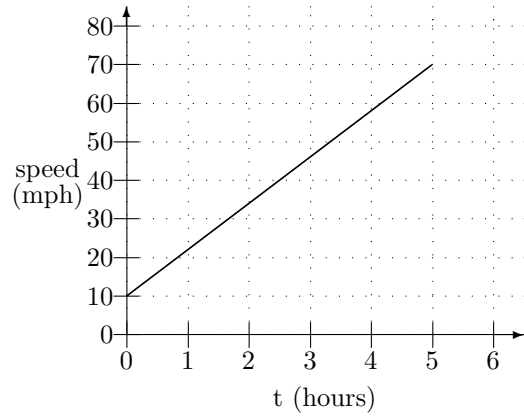
MATH 112

Lecture to Accompany Worksheet #20

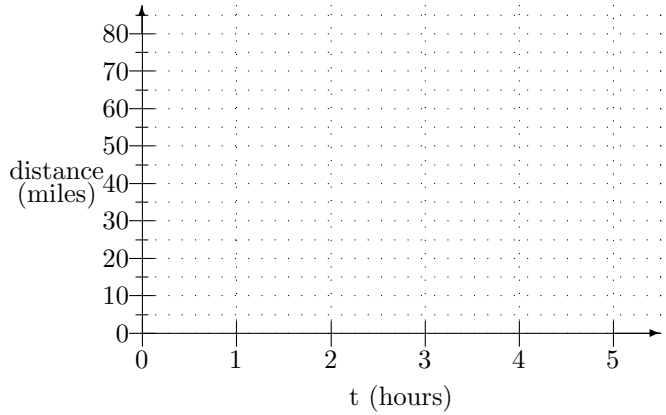
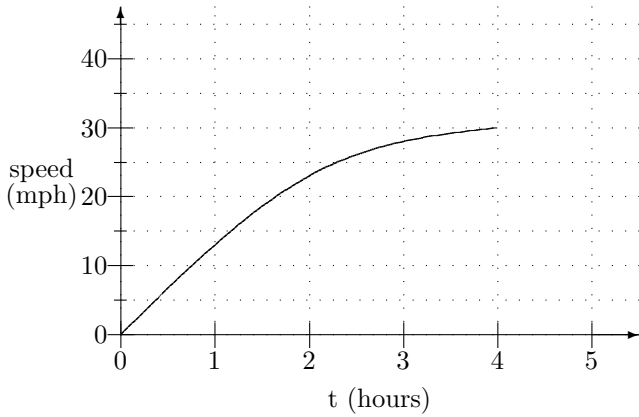
Example 1:



Example 2:



Example 3:

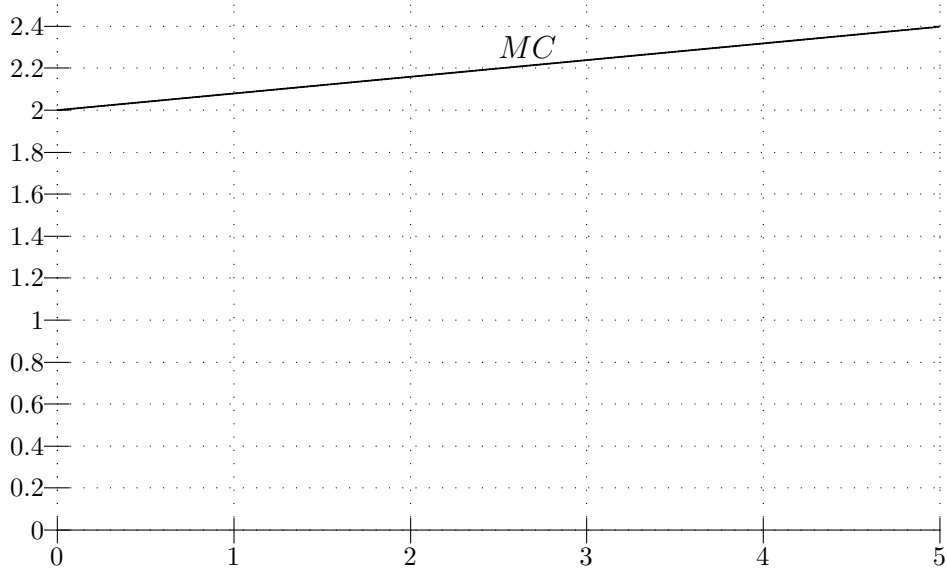
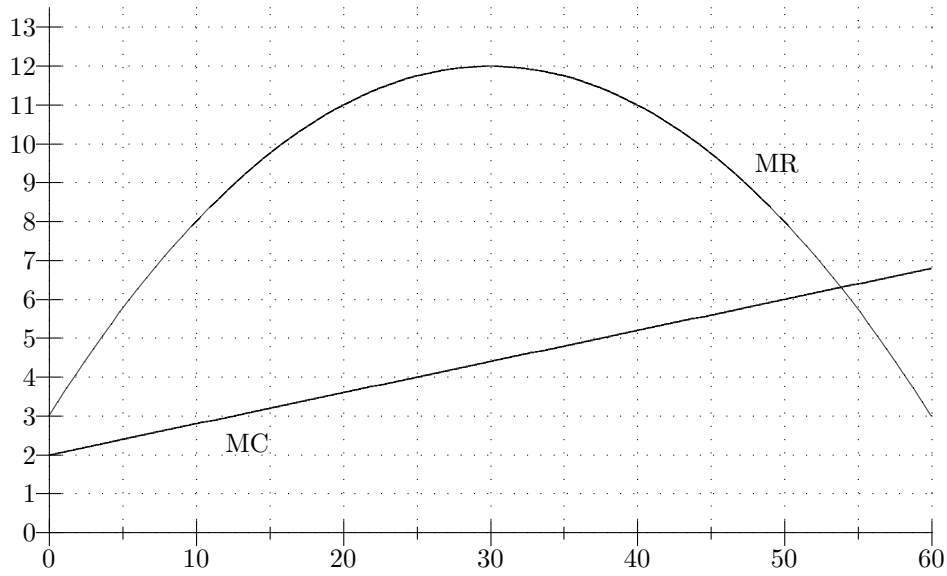


Interval	0-1	1-2	2-3	3-4
Distance covered in that interval				

time	0	1	2	3	4
Distance covered by that time					

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Lecture to Accompany Worksheet 21

TR/TC From MR/MC



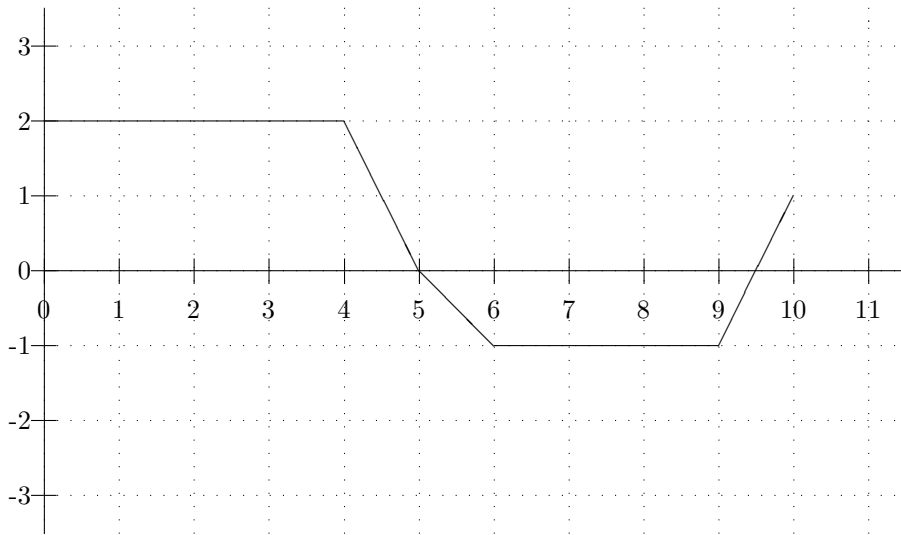
Interval	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Area under <i>MC</i> on that interval								

<i>q</i>	0	5	10	15	20	25	30	35	40
Area under <i>MC</i> from 0 to <i>q</i>									

Interval	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Area under <i>MR</i> on that interval								

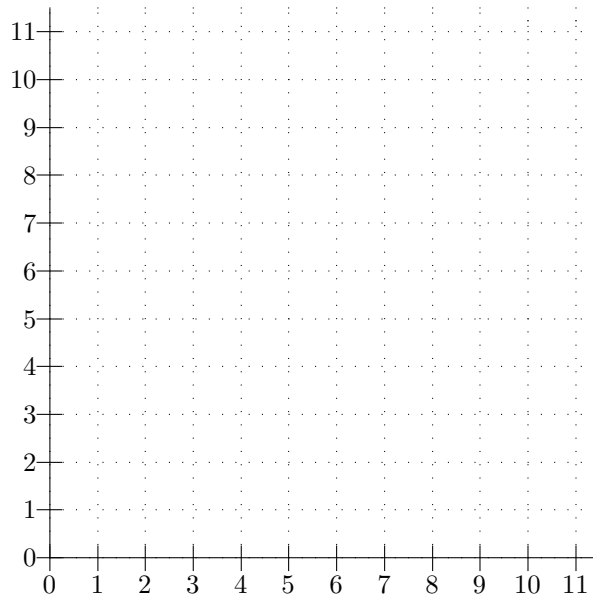
<i>q</i>	0	5	10	15	20	25	30	35	40
Area under <i>MR</i> from 0 to <i>q</i>									

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Lecture to Accompany Worksheet 22



Interval	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10
Integral of $f(x)$ on that interval										

m	0	1	2	3	4	5	6	7	8	9	10
$A(m)$											



Math 112 – Winter 2005
Additions to the Text

Math 112

Worksheet 18A - *Multivariable Functions*

In the previous sections of the course, you worked with functions that consisted of only one variable. For example, $R(q) = 25q - 0.5q^2$ is a formula for Total Revenue. Its only input is quantity q . In this section, you will explore functions that have multiple inputs. Since each input is represented by a different variable, these functions are called multivariable functions.

Introduction to Multivariable Functions

Example: During 2001, Puget Sound Energy utilized two methods for calculating the electrical bill for residential users: flat rate and time-of-use.

With the first method, a customer pays a flat rate for each kilowatt-hour of electricity used, no matter what time of day the usage occurs. Puget Sound Energy charged \$0.05 per kilowatt-hour for flat rate usage.

1. Customer 1 chooses the flat rate method.
 - (a) Determine the charge for 2000 kilowatt-hours to Customer 1.
 - (b) Develop a formula $C_1(x)$ that gives the charge to Customer 1 for x kilowatt-hours.

On the other hand, the time-of-use method charges different amounts per kilowatt-hour depending on when the electricity is used. During peak times, when the demand is highest, the charge per kilowatt-hour is higher than that at off-peak times. The time-of-use system has three rates:

peak hours \$0.062 per kilowatt-hour for usage between 6 a.m. and 10 a.m. and between 5 p.m. and 9 p.m.

daytime hours \$0.0536 per kilowatt-hour for usage between 10 a.m. and 5 p.m.

off-peak hours \$0.047 per kilowatt-hour for usage between 9 p.m. and 6 a.m.

2. Determine the charge for Customer 2, who chose the time-of-use method and who had the following usage during a one-month period.

Time of Usage	Kilowatt-hours	Rate (\$)	Charge
Peak Hours	1000	0.062	
Daytime Hours	700	0.0536	
Off-Peak Hours	300	0.047	

Total Charge:

In order to make a formula for the charge to Customer 2, one variable will not be enough. We'll need three variables, one for each time category.

- Let x be the electricity usage during peak hours, y be the usage during daytime hours, and z be the usage during off-peak hours, all for a one-month period. Develop a formula for $C_2(x, y, z)$, the charge to Customer 2.

Note: Just like formulas with only one input, the variables are listed in parentheses. The stuff in the parentheses is like the marquee at a movie theater: “Now showing the formula *Charge for Customer 2*, starring the variables x , y , and z .”

- Suppose Customer 2 purchases smart appliances for a total of \$3000. These smart appliances (dishwasher, washer/dryer, water heater, etc.) can be programmed to operate during off-peak hours. Customer 2 also adjusts patterns of electrical usage so that the meter reading per month is now given by the following.

Time of Usage	Kilowatt-hours	Rate (\$)	Charge
Peak Hours	500	0.062	
Daytime Hours	300	0.0536	
Off-Peak Hours	1200	0.047	
Total Charge:			

How many months will it take for Customer 2 to recoup the investment in smart appliances? That is, how long will it take before Customer 2 has saved a total of \$3000 on electric bills?

Example: Another example of a function that has multiple inputs is the formula used to compute the balance of a certificate of deposit compounded continuously, $A = Pe^{rt}$. This function has three inputs: the principal P , the interest rate r , and the time of maturation t . However, many lending institutions set P equal to \$1. The balance can then be found easily by multiplying by the principal. Thus, the formula becomes $A(r, t) = e^{rt}$ and the two inputs are interest rate r (expressed as a decimal, of course) and time t (in years). The following chart, showing the balance of an account under different interest rates and at different times, is similar to charts found in bank pamphlets.

		Time of Maturation (years)				
		1	2	3	4	5
Interest Rate (%)	4.5	1.0460	1.0942	1.1445	1.1972	1.2523
	5	1.0513	1.1052	1.1618	1.2214	1.2840
	5.5	1.0565	1.1163	1.1794	1.2461	1.3165
	6	1.0618	1.1275	1.1972	1.2712	1.3499
	6.5	1.0672	1.1388	1.2153	1.2969	1.3840

5. Use the chart to determine the values of $A(0.06, 4)$ and $A(0.055, 3)$. (Remember that r is expressed as a decimal.)
6. Use the chart to estimate the values of $A(0.05625, 3)$ and $A(0.045, 3.25)$.

Rates of Change of Multivariable Functions

Rates of change (not to be confused with interest rates) can be found in multivariable functions in a manner similar to that for functions with only one variable. The technique is to look at the change in only one variable at a time. That means you have to “fix” the other variables and think about what happens to the function as the remaining variable changes.

As an example, think about what the fourth row of the chart represents. Each entry in that row is for an account paying 6% interest. If you cover up the remaining rows and look only at the fourth row, you have different values of a function $A(0.06, t)$. You can think of this as a function of a single variable t . The interest rate is fixed at 0.06. In this way, a multivariable function can be reduced to a function of only one variable.

7. Describe what $A(r, 2)$ represents.

When you consider rates of change for a multivariable function, think of it as moving across a row or down a column. For example, $\frac{A(0.05, 3) - A(0.05, 1)}{2}$ is the rate of change in the balance at 5% interest as time changes from 1 to 3 years. The balance changes as time changes, while the interest rate stays fixed at 5%. Evaluating this expression gives

$$\frac{A(0.05, 3) - A(0.05, 1)}{2} = \frac{1.1618 - 1.0513}{2} = 0.05525.$$

Thus, the incremental rate of change of the balance at 5% between year 1 and year 3 is 0.05525 dollars per year.

8. (a) Determine the incremental rate of change in the balance at 4.5% between year 2 and year 5.
- (b) Determine the incremental rate of change in the balance for rates between 4.5% and 6% for year 3.
- (c) Determine the *overall* rate of change in the balance at 5% over 4 years.
9. (a) Use the result of #8(a) and the value of $A(0.045, 3)$ to estimate $A(0.045, 3.25)$. How close is your answer to that in #6?
- (b) Use the result in #8(b) and the value of $A(0.055, 3)$ to estimate $A(0.05625, 3)$. How close is your answer to that in #6?

Note: The rates of change in #8 (a) and (b) are not quite the same thing. Both are rates of change in the balance. But in part (a), time is changing, and in part (b), the interest rate is changing. The rate of change across a row is different than down a column. This is evident in the units of the rates of change.

10. Use the units to explain the difference between the two rates of change.

Derivatives of Multivariable Functions

In the examples above, we were able to treat multivariable functions just like regular functions when evaluating them and determining rates of change. We just had to take the precaution of dealing with only one variable at a time. A similar approach is used when taking the derivative of a multivariable function. The derivative rules for single-variable functions work here as well, but the rules are applied to one of the variables at a time, while the others are considered fixed.

For example, let's consider the balance function $A(r, t) = e^{rt}$. We'll start by thinking of the time as fixed and find the rate of change of A , where r is the only variable.

- First, we need to indicate which of our multiple variables we're allowing to change. Here, we are thinking of time as fixed and we are allowing the interest rate to change. We take the derivative of A *with respect to* r . The derivative of A with respect to the interest rate r is denoted $\frac{\partial A}{\partial r}$. The delta ∂ is used to indicate that the function is a multivariable function. The notation $\frac{dA}{dr}$ would indicate that the function A involves only one variable.
- Next, think of the other variable, t as being fixed. Although we will not be assigning the variable t a numerical value, we treat the variable t as if it were a number. Some people like to indicate that a variable is fixed by marking the variable with an asterisk. We could write the formula as $A(r, t^*) = e^{rt^*}$. The asterisk is a reminder that time is fixed.
- Finally, we take the *partial derivative* using rules from previous worksheets. Remember we treat r as the variable and t as a fixed number. Applying the derivative rule for exponential functions, we get

$$\frac{\partial A}{\partial r} = e^{rt^*} \cdot (t^*).$$

The last part is the derivative of rt^* . (Recall, for example, that the derivative of $x \cdot 3 = 3x$ is simply 3. Since we are treating t^* just like a number, the derivative of $rt^* = t^* \cdot r$ is simply t^* .)

11. Compute $\frac{\partial A}{\partial t}$.

As another example, let's look at the multivariable function we computed for the time-of-use electricity charges:

$$C_2(x, y, z) = 0.062x + 0.0536y + 0.047z.$$

To compute $\frac{\partial C_2}{\partial x}$, think of everything without an x in it as a number. Even though the terms $0.0536y$ and $0.047z$ have a variable in them, they don't have an x in them. So, we treat those two terms like constants. Thus,

$$\frac{\partial C_2}{\partial x} = 0.062.$$

12. Compute $\frac{\partial C_2}{\partial y}$ and $\frac{\partial C_2}{\partial z}$.

For practice, consider the function:

$$E(b, m) = b^2 + 3.4m^2 + 2.833bm - 5.6b - 15.4m + 102.3.$$

Then, the partial derivative of E with respect to m is computed by treating all of the b 's as numbers:

$$\frac{\partial E}{\partial m} = 6.8m + 2.833b - 15.4.$$

13. Compute $\frac{\partial E}{\partial b}$.

Notation: You may also see partial derivatives denoted with subscripts. For example, the partial derivative of E with respect to m might be written $E_m(b, m) = 6.8m + 2.833b - 15.4$.

14. Let $f(x, y) = x^3 + xy + y^2 - x - y + 5$. Compute $f_x(x, y)$ and $f_y(x, y)$.

MATH 112
Homework for Worksheet 14

- Do #1, 2, and 3 as written.
- Most of the table for #4 is done below. Fill in the missing values in the table and complete the plot of the data in #4. (You are plotting *time* vs. $\ln(\text{value})$.)

t	value	$\ln(\text{value})$	t	value	$\ln(\text{value})$	t	value	$\ln(\text{value})$
0.0833	\$25.50	3.24	1.0833	\$37.30	3.62	2.0833	\$87.50	
0.167	\$24.38	3.19	1.167	\$42.38		2.167	\$75.06	4.32
0.250	\$22.92	3.13	1.250	\$44.75	3.80	2.250	\$75.00	4.32
	\$30.38			\$45.06	3.81		\$81.31	4.40
	\$31.00		1.417	\$42.41	3.75	2.417	\$80.69	4.39
0.500	\$31.59	3.45	1.500	\$54.19	3.99		\$90.19	4.50
0.583	\$35.34	3.57		\$54.97		2.583	\$85.81	4.45
0.667	\$33.05	3.50	1.667	\$47.97	3.87	2.667	\$92.56	4.53
0.750	\$33.08	3.50	1.750	\$55.03	4.01	2.750	\$90.56	
	\$32.50	3.48		\$52.94	3.97		\$92.56	4.53
0.917	\$35.38	3.57	1.917	\$61.00		2.917	\$91.05	4.51
1.000	\$32.31	3.48	2.000	\$69.34	4.24	3.000	\$116.75	

- Let t_i be the time and s_i be the $\ln(\text{value})$ in the i^{th} row of the table. Then,

$$\sum t_i = 55.501 \quad \sum s_i = 141.57$$

$$\sum t_i^2 = 112.54 \quad \sum s_i^2 = 564.22 \quad \sum t_i s_i = 232.21.$$

- Use these values and the process of Worksheet 18 to find the best fitting line for these data.
 - On the grid from #4, sketch the line you found.
 - If we let $f(t)$ be the value of the stock at time t , then you have just found a formula for $\ln(f(t))$. In subsequent questions, use this equation when it tells you to “use the equation for $\ln(f(t))$, which you found in Problem 6.”
- Do #8, 9, 10, 11, and 12 as written.