MATH 300 D — Spring 2011
Midterm Study Problems

1. Give a useful denial of each statement. You may use symbols like $\forall$ and $\exists$, et cetera.
Please put a box around your final answer.

   (a) For every real number x, there exists a real number y such that $x < y$.
   (b) There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for every real number x, $f(2x) = 4f(x)$.
   (c) If x is a positive real number, then $\ln x \geq 0$ or $\frac{1}{x} > 1$.
   (d) For all real x, $x \neq 0 \Rightarrow x^2 > 0$.
   (e) For every real M, there exists a real number N such that, for every $n > N$, $|f(n)| > M$.
   (f) For all $\epsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

2. Let P be the statement: "Every prime number is odd." Which of the following are logically equivalent to P? (Check all that apply.)

_______ n is odd only if n is prime.
_______ n is odd if n is prime.
_______ If n is odd, then n is prime.
_______ If n is prime, then n is odd.
_______ There exist numbers that are both prime and odd.
_______ No even number is prime.
_______ n is odd if and only if n is prime.

3. Let A, B, and C be sets.

   (a) Prove that $(A \cap B) - C \subseteq (B \cup C) - (B \cap C)$.
   (b) Give a counterexample that demonstrates that $(B \cup C) - (B \cap C)$ is not necessarily a subset of $(A \cap B) - C$.

4. (a) Prove that, for all $x \in \mathbb{Z}$, if $x^2 - 1$ is divisible by 8, then x is odd.
   (b) Let P be the statement: "For all $x \in \mathbb{Z}$, if x is odd, then $x^2 - 1$ is divisible by 8."
      i. What is the negation of the statement P?
      ii. Which is true: P or its negation? Prove your claim.

5. Prove or give a counterexample for each of the following statements.

   (a) For all real numbers x and y, $|x + y| = |x| + |y|$.
   (b) For all real numbers x and y, $|xy| = |x||y|$.
   (c) For all positive integers, $x^2 + x + 23$ is prime.

(d) There is a natural number M such that, for every natural number $n > M$, $\frac{1}{n} < 0.002$.

(e) For all integers a and b, if $a|b$ and $b|a$, then $|a| = |b|$.

(f) For all integers m and n, if $n + m$ is odd, then $n \neq m$.

6. (a) Let x be an integer. Prove that if $\sqrt{2x}$ is an integer, then x is even.

   (b) Is the converse of the statement you proved in (a) true? Prove it or give a counterexample.

   (c) What can you conclude about $\sqrt{2x}$ if x is odd?

7. We proved in class that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}.$$

   Use this fact and induction to prove that $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$.

8. Prove that $3|(7^n - 4)$ for every $n \in \mathbb{N}$.