MATH 300 D — Spring 2011
Midterm Study Problem Answers

1. (a) There exists a real number x such that, for every real y, $x \geq y$.
   (b) For every function $f : \mathbb{R} \to \mathbb{R}$, there exists a real number x such that $f(2x) \neq 4f(x)$.
   (c) There is a positive real number x such that $\ln x < 0$ and $\frac{1}{x} \leq 1$.
   (d) There is a real x such that $x \neq 0$ and $x^2 \leq 0$.
   (e) There exists a real number M such that, for all real N, there is an $n > N$ such that $|f(n)| \leq M$.
   (f) There exists an $\epsilon \in P$ such that, for all $\delta \in P$, there exist $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

2. "Every prime number is odd" is equivalent to:
   - n is odd if n is prime.
   - If n is prime, then n is odd.
   - No even number is prime.

3. (a) Let A, B, and C be sets. Choose an arbitrary $x \in (A \cap B) - C$. Then $x \in A \cap B$ and $x \notin C$. So, $x \in A$ and $x \in B$ but $x \notin C$. Since $x \in B$, $x \in B \cup C$. Since $x \notin C$, $x \notin B \cap C$. So, $x \in (B \cup C) - (B \cap C)$. Thus, $(A \cap B) - C \subseteq (B \cup C) - (B \cap C)$. □
   (b) Many answers are possible. Here's one: Let $A = \{1, 3\}$, $B = \{1, 2, 3\}$, and $C = \{3\}$. Then $A \cap B = \{1, 3\}$ and $(A \cap B) - C = \{1\}$. On the other hand, $B \cup C = \{1, 2, 3\}$ and $B \cap C = \{3\}$, which means $(B \cup C) - (B \cap C) = \{1, 2\}$. So $(B \cup C) - (B \cap C) \not\subseteq (A \cap B) - C$.

4. (a) Choose an arbitrary $x \in \mathbb{Z}$ and suppose that $x^2 - 1$ is divisible by 8. Then, $x^2 - 1 = 8k$ for some $k \in \mathbb{Z}$. So, $x^2 = 8k + 1 = 2(4k) + 1$, which is odd. We need to show that x is odd. Suppose not. Then x is even and $x = 2n$ for some $n \in \mathbb{Z}$. This means $x^2 = (2n)^2 = 2(2n^2)$, which is even. Thus, since x^2 is odd, x must be odd. □
   (b) i. $\sim P$: There exists an integer x such that x is odd and $x^2 - 1$ is not divisible by 8.
   ii. The statement P is true: Choose $x \in \mathbb{Z}$ and suppose that x is odd. Then $x = 2k+1$ for some $k \in \mathbb{Z}$. This means $x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4k(k + 1)$. If k is even, then $k = 2n$ for some $n \in \mathbb{Z}$. Then $x^2 - 1 = 8n(k + 1)$, which is divisible by 8. On the other hand, if k is odd, then $k = 2n + 1$ for some $n \in \mathbb{Z}$ and $x^2 - 1 = 4k(2n + 2) = 8k(n + 1)$, which is divisible by 8. Therefore, if x is odd, then $x^2 - 1$ is divisible by 8. □

5. (a) This statement is false: Let $x = 10$ and $y = -5$. Then $|x + y| = |10 + (-5)| = |5| = 5$ but $|x| + |y| = |10| + |-5| = 10 + 5 = 15$ and $5 \neq 15$.
   (b) This is true: Choose $x, y \in \mathbb{R}$. There are three cases to consider.
   Case 1: Suppose $x \geq 0$ and $y \geq 0$. Then $xy \geq 0$. This means $|xy| = xy$, $|x| = x$ and $|y| = y$. So, $|xy| = xy = |x||y|$.
   Case 2: Suppose $x < 0$ and $y < 0$. Then $xy > 0$. This means $|xy| = xy$, $|x| = -x$, and $|y| = -y$. So, $|xy| = xy = (-x)(-y) = |x||y|$.

Case 3: Suppose that exactly one of x and y is non-negative. Without loss of generality, we can assume $x \geq 0$ and $y < 0$. Then $xy \leq 0$. This means $|xy| = -xy$, $|x| = x$, and $|y| = -y$. So, $|xy| = -xy = x(-y) = |x||y|$. □

(c) This is false: Let $x = 23$. Then $x^2 + x + 23 = 23^2 + 23 + 23 = 23(25)$, which is not prime.

(d) This is true: Let M=500 and choose a natural number $n > 500$. Then $\frac{1}{n} < \frac{1}{500} = 0.002$. □

(e) This is true: Choose integers a and b and suppose $a|b$ and $b|a$. Then there exist integers k_1 and k_2 such that $b = ak_1$ and $a = bk_2$. So, $b = (bk_2)k_1 = b(k_2k_1)$. This implies that $k_2k_1 = 1$, which means both k_1 and k_2 divide 1. By one of our elementary properties, the only numbers that divide 1 are ± 1. So, $k_1 = \pm 1$, which means $a = \pm b$ and, by part (b) of this problem, $|a| = |\pm 1||b| = |b|$. □

(f) This is true: Choose m and n in $\mathbb{Z}$. We'll prove the contrapositive. If $n = m$, then $n + m = 2n$, which is even. So, if $n + m$ is odd, then it must be the case that $n \neq m$. □

6. (a) Proof: Let x be an integer. Suppose $\sqrt{2x} = k$ for some integer k. Then $k^2 = 2x$. That is, k^2 is even. If k is odd, then $k = 2n + 1$ for some $n \in \mathbb{Z}$, which implies that $k^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$, which is odd. So, since k^2 is even, it must be the case that k is even. That is, $k = 2n$ for some $n \in \mathbb{Z}$ and thus $2x = k^2 = 4n^2$. This means $x = 2n^2$, which is even. □

(b) No, the converse is not true: Let $x = 4$. Then x is even but $\sqrt{2x} = \sqrt{8}$ is not an integer.

(c) Since the statement in part (a) is true, so is its contrapositive. So, if x is odd, then $\sqrt{2x}$ is not an integer.

7. Proof: By induction on n.

Basis Step: $\sum_{i=1}^{1} i^3 = 1$ and $\left(\sum_{i=1}^{1} i\right)^2 = 1^2 = 1$. ✓

Induction Step: Suppose $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for some $n \in \mathbb{N}$. Then:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 \\
&= \left(\sum_{i=1}^{n} i\right)^2 + (n+1)^3 \text{ (by the induction hypothesis)} \\
&= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \text{ (by the result proved in class)} \\
&= (n+1)^2\left[\frac{n^2}{4} + (n+1)\right] \\
&= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\
&= \frac{(n+1)^2(n+2)^2}{4}
\end{aligned}$$

$$= \left[\frac{(n+1)(n+1)}{2}\right]^2$$

$$= \left[\sum_{i=1}^{n+1} i\right]^2 \quad \text{(by the result proved in class)}$$

Therefore, by the principle of mathematical induction, $\displaystyle\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$. $\square$

8. Proof: By induction on n.

   Basis Step: If $n = 1$, then $7^n - 4 = 7 - 4 = 3$ and $3|3$. ✓

   Induction Step: Suppose $3|(7^n - 4)$ for some $n \in \mathbb{N}$. Then $7^n - 4 = 3k$ for some $k \in \mathbb{Z}$, which implies $7^n = 3k + 4$. We then have $7^{n+1} = 7(3k + 4) = 21k + 28$, which implies that $7^{n+1} - 4 = 21k + 24 = 3(7k + 8)$, which is divisible by 3.

   Therefore, by the principle of mathematical induction, $3|(7^n - 4)$ for all $n \in \mathbb{Z}$.