MATH 300 B — Spring 2011
Final Exam Practice Problems

1. For each of the following, determine whether the statement is TRUE or FALSE. You do not need to provide any justification of your answer.
   (a) Every infinite subset of $\mathbb{R}$ is uncountable.
   (b) There is an uncountable subset of $\mathbb{N} \times \mathbb{N}$.
   (c) There exists a bijection $f : \mathbb{Q} \to \mathbb{R}$.
   (d) There exists a bijection $f : \mathbb{Q} \to \mathbb{Z}$.
   (e) If $f : \mathbb{R} \to \mathbb{R}$ is surjective, then f must be bijective.
   (f) If $f : \mathbb{R} \to \mathbb{R}$ is injective, then f must be surjective.
   (g) Let A be a finite set. If $f : A \to A$ is surjective, then f must be injective.
   (h) Let A be a finite set. If $f : A \to A$ is injective, then f must be bijective.
   (i) Suppose the relation R on $\mathbb{Z}$ is defined by $(a, b) \in R \Leftrightarrow a < b$. R is an equivalence relation on $\mathbb{Z}$.
   (j) If a, b, and c are integers, $c \neq 0$, and $c|ab$, then it must be the case that $c|a$ or $c|b$.
   (k) If $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is defined by $f([x]) = [x^2]$, then f is injective.

2. Use induction to show that, if x is a real number such that $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

3. Use induction to prove that $2^n < n!$ for every integer $n \geq 4$.

4. We proved in class that
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}.$$
   Use this fact and induction to prove that $\displaystyle\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$.

5. Prove that $6|(n^3 - n)$ for every $n \in \mathbb{N}$.

6. Prove that $3|(7^n - 4)$ for every $n \in \mathbb{N}$.

7. Define a relation T on the set $\mathbb{R}$ of real numbers by
$$T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < 1\}.$$
   Is T an equivalence relation? (Justify your answer, of course.)

8. (a) Define a relation R on $\mathbb{N}$ by
$$(x, y) \in R \Leftrightarrow x - y \text{ is even.}$$
   Prove that R is an equivalence relation.

(b) Define a relation R on $\mathbb{Z}$ by

$$(x, y) \in R \Leftrightarrow xy \equiv 0 \pmod 4.$$

Give a counterexample to demonstrate that R is not transitive.

9. Let $D = \{1, 2, 3, 4\}$ and $T = \{u, x, y, z\}$. Define a function $f : D \to T$ defined by $f = \{(1, u), (2, x), (3, y), (4, u)\}$. Let

$$A = \{1, 2, 3\},\ B = \{3, 4\},\ X = \{u, x\},\ \text{and } Y = \{x, z\}.$$

Indicate whether each of the following is **TRUE or FALSE** and justify your answer.

(a) If $x \in D$ and $x \notin A$, then $f(x) \in f(A) \cap f(B)$.
(b) $f(A - B) = f(A) - f(B)$
(c) $f^{-1}(X - Y) = f^{-1}(X) - f^{-1}(Y)$

10. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$, and $f : A \to B$ be the function $f = \{(1, 1), (2, 1), (3, 5), (4, 2), (5, 3), (6, 3)\}$. (That is, $f(1) = 1$, $f(2) = 1$, $f(3) = 5$, etc.)

(a) Let $C = \{x \in A : x \text{ is an even integer}\}$. Lis the elements in the set $B - f(C)$.
(b) Let $D = \{y \in B : y \text{ is an odd integer}\}$. List the elements in the set $f^{-1}(D)$.
(c) Prove that, for all subsets Y_1 and Y_2 of B,

$$f^{-1}(Y_1) \cap f^{-1}(Y_2) = f^{-1}(Y_1 \cap Y_2).$$

11. Let A, B, and C be sets and consider functions $f : A \to B$ and $g : B \to A$. State whether each of the following is true or false. If the statement is true, prove it; if it is false, give a counterexample.

(a) If $g \circ f : A \to C$ is injective, then f must be injective.
(b) If $g \circ f : A \to C$ is surjective, then f must be surjective.

12. Let $A = \{x \in \mathbb{R} : x \neq 1\}$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \frac{x + 1}{x - 1}.$$

Is $f(x)$ injective? surjective? bijective? Justify each of your responses with a proof or counterexample.

13. Define a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f(x, y) = (x - y, 2x + y)$. Is f one-to-one? onto? Justify each of your responses with a proof or counterexample.

14. Let $A = \{a \in \mathbb{N} : a \text{ is even}\}$ and $B = \{b \in \mathbb{N} : b \text{ is odd}\}$.

(a) Define a function $f : A \times B \to \mathbb{N}$ by $f(a, b) = \dfrac{ab}{2}$. Is f surjective? Justify your answer.

(b) Define a function $h : A \times B \to \mathbb{N}$ by $h(a, b) = \dfrac{a + 2b}{2}$. Is h surjective? Justify your answer.

(c) Define a function $g : B \to \mathbb{N}$ by $g(b) = \frac{b+1}{2}$. Prove that g is bijective.

15. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq 2\}$ and define $f : S \to S$ by

$$f(x, y) = \left(\frac{y + 2}{x - 2}, \frac{1}{x - 2} \right).$$

(a) Prove that f is injective.

(b) Is f bijective? Prove it or give a counterexample.

16. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = f(x) + g(x)$. For each of the following, if the statement is true, prove it; otherwise, give a counterexample to show that the statement is false.

(a) If f and g are bijections, then h is a bijection.

(b) If f and g are both increasing, then h is increasing.

(c) If f is increasing and g is decreasing, then $g \circ f$ is decreasing.

17. Let f be an injective function from a non-empty set A to a set B.

(a) Define a function $g : A \to f(A)$ by $g(x) = f(x)$ for all $x \in A$. Explain why g is a bijection.

(b) Give a formal proof that the cardinality of $f(A)$ is the same as the cardinality of A in each of the following cases.

i. A is finite;

ii. A is denumerable;

iii. A is uncountable.