Hilbert's 17th Problem

to

Semidefinite Programming

&

Convex Algebraic Geometry

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References

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Polynomial Optimization: $p^* := \inf\{p(\mathbf{x}) : \mathbf{x} \in K\}$ $(*)$

- $K = \{\mathbf{x} \in \mathbb{R}^n : g_1 \geq 0, \ldots, g_m \geq 0, h_1 = 0, \ldots, h_s = 0\}$

  (basic semialgebraic set), $p, g_i, h_j \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \ldots, x_n]$
- $K = \mathbb{R}^n \longrightarrow$ unconstrained polynomial optimization
- $m = 0 \longrightarrow K = \mathcal{V}_{\mathbb{R}}(I)$ real variety of $I = \langle h_1, \ldots, h_m \rangle \subseteq \mathbb{R}[\mathbf{x}]$
- non-convex optimization problem
- NP-hard even when $\deg(p) = 4$ and $K = \mathbb{R}^n$.

GOAL: Find a sequence of relaxations of $(*)$ by convex optimization problems that converges to $(*)$.

Examples (all NP complete)

Partition problem: Given $a_1, \ldots, a_n \in \mathbb{Z}_+$, does there exist $\mathbf{x} \in \{\pm 1\}^n$ s.t. $\sum a_i x_i = 0$? Yes $\Leftrightarrow p^* = 0$ for $p := \left(\sum_{i=1}^n a_i x_i\right)^2 + \sum_{i=1}^n (x_i^2 - 1)^2$.

Distance realization: Given distances $\mathbf{d} := (d_{ij}) \in \mathbb{R}^E$, $\mathbf{d}$ is realizable in $\mathbb{R}^k$ if there exists $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^k$ such that $d_{ij} = ||\mathbf{v}_i - \mathbf{v}_j||$ for all $ij \in E$.
$\mathbf{d}$ realizable in $\mathbb{R}^k \Leftrightarrow p^* = 0$ for $p := \sum_{ij \in E} \left(d_{ij}^2 - \sum_{h=1}^k (x_{ih} - x_{jh})^2\right)^2$.

$0/1$ Integer programming: $\min\{\mathbf{c} \cdot \mathbf{x} : A\mathbf{x} \leq \mathbf{b},\ x_i^2 = x_i \ \forall\ i \in [n]\}$

Semidefinite programming

- vector space: Symm$(\mathbb{R}^{n\times n})$
- trace product: $A, B \in$ Symm$(\mathbb{R}^{n\times n})$, $\quad A \cdot B :=$ trace(AB)
- positive semidefinite: $\quad A \succeq 0$

$$\begin{aligned} &\Leftrightarrow \quad \text{all eigenvalues of } A \text{ are non-negative} \\ &\Leftrightarrow \quad \mathbf{v}^t A \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathbb{R}^n \\ &\Leftrightarrow \quad A = BB^t \quad \text{for some} \quad B \in \mathbb{R}^{n\times m} \end{aligned}$$

semidefinite program (sdp):

$$\sup\{C \cdot X \,:\, A_j \cdot X = b_j,\ j = 1, \ldots, m,\ X \succeq 0\}$$

- convex optimization, polynomial time algorithms
- Linear programming is SDP over diagonal matrices

Hilbert's 17th problem

$\mathcal{P}_n := \{p \in \mathbb{R}[\mathbf{x}] : p(\mathbf{x}) \geq 0, \forall\, \mathbf{x} \in \mathbb{R}^n\}$ non-negative polynomials

$\Sigma_n := \{\sum h_j^2 : h_j \in \mathbb{R}[\mathbf{x}]\}$ sum of squares (sos) of polynomials

$\mathcal{P}_{n,d}, \Sigma_{n,d}$ polynomials of degree at most d in $\mathcal{P}_n$ and Σ_n

$\Sigma_n \subseteq \mathcal{P}_n$ and $\Sigma_{n,d} \subseteq \mathcal{P}_{n,d}$ cones in $\mathbb{R}[\mathbf{x}]$

Hilbert 1888: $\Sigma_{n,d} = \mathcal{P}_{n,d} \Leftrightarrow n = 1$ or $d = 2$ or $(n,d) = (2,4)$.

Hilbert's 17th problem: Is every $p \in \mathcal{P}_n$ a sos of rational functions?

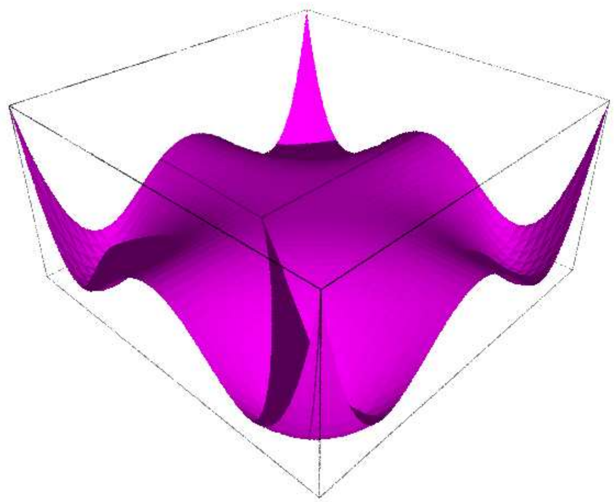
Yes! Artin 1927 using Tarski's transfer principle and orderings on fields

Motzkin (1967):

$s(x,y) := 1 - 3x^2y^2 + x^2y^4 + x^4y^2$

- $s(x,y) \in \mathcal{P}_{2,6} \setminus \Sigma_{2,6}$
- $\frac{a+b+c}{3} \geq \sqrt[3]{abc} \ \forall \ a,b,c \geq 0,$

  $a := 1,\ b := x^2y^4,\ c := x^4y^2$



Blekherman (2006): for d fixed, many more nonnegative forms than sos forms as $n \to \infty$

Lasserre (2006): for n fixed, any nonnegative polynomial can be approximated arbitrarily closely by sos polynomials.

Testing for sos representations via SDP

$p = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 2y^4$ sos

$$\Leftrightarrow \quad \exists\, A \succeq 0 \text{ s.t. } p = (x^2, xy, y^2) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$

$\Leftrightarrow$ the sdp $\{A \succeq 0 : a_{11} = 1, a_{12} = 1, a_{22} + 2a_{13} = 3, a_{23} = 1, a_{33} = 2\}$ is feasible

ex: $A = B^t B$ for $B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$ or $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \sqrt{3/2} & \sqrt{3/2} \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \end{pmatrix}$

$$p = (x^2 + xy - y^2)^2 + (y^2 + 2xy)^2 = (x^2 + xy)^2 + \frac{3}{2}(xy + y^2)^2 + \frac{1}{2}(xy - y^2)^2$$

Sos relaxations for $p^* := \inf\{p(\mathbf{x}) : \mathbf{x} \in K\}$

$$p^* := \sup\{\rho : p - \rho \geq 0 \text{ on } K\} = \sup\{\rho : p - \rho > 0 \text{ on } K\}$$

- $K = \mathbb{R}^n$: $p^{\text{sos}} := \sup\{\rho : p - \rho \text{ is sos}\}$

CAUTION Motzkin polynomial: $p^{\text{sos}} = -\infty < p^* = 0$

- $K = \{\mathbf{x} \in \mathbb{R}^n : g_1 \geq 0, \ldots, g_m \geq 0\}$:

$$p^{\text{sos}} := \sup\{\rho : p - \rho = s_0 + \sum_{i=1}^m s_i g_i, \ s_i \text{ sos}\}$$

to get sdps we look at successive truncations:

$$p_t^{\text{sos}} := \sup\{\rho : p - \rho = s_0 + \sum_{i=1}^m s_i g_i, \ s_i \text{ sos}, \ \deg(s_0), \deg(s_i g_i) \leq 2t\}$$

$$p_t^{\text{sos}} \leq p_{t+1}^{\text{sos}} \leq p^{\text{sos}} \leq p^*, \quad \lim_{t\to\infty} p_t^{\text{sos}} = p^{\text{sos}}$$

Positivstellensatz (Krivine 1964, Stengle 1974)

$T := \{\sum s_i (g_1^{i_1} g_2^{i_2} \cdots g_m^{i_m}) : (i_1, \ldots, i_m) \in \{0,1\}^m, \ s_i \text{ sos}\}$ preorder

non-negativity set of $K = \{\mathbf{x} \in \mathbb{R}^n : g_1 \geq 0, \ldots, g_m \geq 0\}$.

Positivstellensatz: Given $p \in \mathbb{R}[\mathbf{x}]$,

(i) $p > 0$ on $K \Leftrightarrow pf = 1 + g, \ f, g \in T$

(ii) $p \geq 0$ on $K \Leftrightarrow pf = p^{2k} + g, \ f, g \in T$ (solves Hilbert's 17th prob)

(iii) $p = 0$ on $K \Leftrightarrow -p^{2k} \in T$

(iv) $K = \emptyset \Leftrightarrow -1 \in T$ (compare: Hilbert's weak Nullstellensatz)

Real Nullstellensatz: p vanishes on $\{\mathbf{x} \in \mathbb{R}^n : h_1 = \cdots = h_s = 0\} \Leftrightarrow$
$p^{2k} + s = \sum_{j=1}^{s} u_j h_j$ for $u_j \in \mathbb{R}[\mathbf{x}], s$ sos. (Hilbert's strong NS)

Schmüdgen's & Putinar's refinements

$p^* := \sup\{\rho : p - \rho > 0 \text{ on } K\} = \sup\{\rho : (p-\rho) f = 1 + g,\ \ f, g \in T\}.$

Schmüdgen 1991: If K is compact then $p > 0$ on $K \Rightarrow p \in T$.

$$p_t^{\text{Schm}} = \sup\{\rho : (p-\rho) = \sum s_i (g_1^{i_1} g_2^{i_2} \cdots g_m^{i_m}),\ s_i \text{ sos},\ \deg(**) \leq t\}$$

(sdp, $\lim_{t\to\infty} p_t^{\text{Schm}} = p^*$, involves 2^m terms !)

Putinar 1993: If K is Archimedean then $p > 0$ on $K \Rightarrow p \in M$.

$M := \{s_0 + \sum_{i=1}^m s_i g_i : s_0, s_i \text{ sos}\}$ quadratic module

$$p_t^{\text{Put}} = \sup\{\rho : (p-\rho) = s_0 + \sum s_i g_i,\ s_0, s_i \text{ sos},\ \deg(**) \leq t\}$$

(sdp, $\lim_{t\to\infty} p_t^{\text{Put}} = p^*$, only $m+1$ terms !)

Moments of probability measures

μ probability measure on $\mathbb{R}^n$:

- $y_\alpha := \int \mathbf{x}^\alpha d\mu$ moment of order α
- $y := (y_\alpha : \alpha \in \mathbb{N}^n)$ moment sequence of μ

K-moment problem: Characterize moment sequences of measures supported on $K \subseteq \mathbb{R}^n$.

$y \in \mathbb{R}^{\mathbb{N}^n}$ moment sequence:

- moment matrix: $M(y) \in \mathbb{R}^{\mathbb{N}^n \times \mathbb{N}^n}$: $M(y)_{(\alpha,\beta)} := y_{\alpha+\beta}$
- truncated moment matrix: $M_t(y)$ – principal submatrix of $M(y)$ indexed by $\mathbb{N}^n_t$

Moment relaxations for $p^* := \inf\{p(\mathbf{x}) : \mathbf{x} \in K\}$

$y \in \mathbb{R}^{\mathbb{N}^n_{2t}}$ (truncated) moment sequence of μ on $K \Rightarrow \forall t \geq d_K$,

$$M_t(y) \succeq 0, \;\; M_{t-d_K}(g_i y) \succeq 0, \; i = 1, \ldots, m$$

$$p^* \;=\; \inf\{\textstyle\int_K p(\mathbf{x}) d\mu : \mu \text{ prob measure on } K\}$$

$$= \;\inf\{p^t y : y_0 = 1, y \text{ mom seq of } \mu \text{ on } K\}$$

$$p^{\text{mom}} := \inf\{p^t y : y_0 = 1, \; M(y) \succeq 0, \; M(g_i y) \succeq 0, \; i = 1, \ldots, m\}$$

to get sdps we look at successive truncations:

$$p_t^{\text{mom}} := \inf\{p^t y : y_0 = 1, \; M_t(y) \succeq 0, \; M_{t-d_K}(g_i y) \succeq 0, \; \forall\, i, \; y \in \mathbb{R}^{\mathbb{N}^n_{2t}}\}$$

$$p_t^{\text{mom}} \leq p_{t+1}^{\text{mom}} \leq p^{\text{mom}} \leq p^*, \quad \lim_{t\to\infty} p_t^{\text{mom}} = p^{\text{mom}}$$

Duality

Recall: $\mathcal{P}$ = $\{p : p(\mathbf{x}) \geq 0, \forall\, \mathbf{x} \in \mathbb{R}^n\}$, Σ = $\{\sum h_j^2 : h_j \in \mathbb{R}[\mathbf{x}]\}$.

Define $\mathcal{M} := \{y \in \mathbb{R}^{\mathbb{N}^n}$: y mom seq of μ on $\mathbb{R}^n\} \subseteq \mathbb{R}[\mathbf{x}]^*$

$\mathcal{M}_{\succeq} := \{y \in \mathbb{R}^{\mathbb{N}^n} : M(y) \succeq 0\} \subseteq \mathbb{R}[\mathbf{x}]^*$

Haviland 1935 $\mathcal{P} = \mathcal{M}^*$, $\mathcal{P}^* = \mathcal{M}$

Berg,Christensen,Jensen 1979: $\Sigma = \mathcal{M}_{\succeq}^*$, $\Sigma^* = \mathcal{M}_{\succeq}$

Corollary: $(n = 1) \Rightarrow \mathcal{M} = \mathcal{M}_{\succeq}$ Hamburger's theorem

$\exists\ K$-versions of these dual cones

sdp duality $\Rightarrow p_t^{\text{sos}} \leq p_t^{\text{mom}} \leq \cdots \leq p^{\text{sos}} \leq p^{\text{mom}} \leq p^*$

Theta Body of a Graph

$G = ([n], E)$ undirected graph

$$\text{STAB}(G) := \text{conv}\{\chi^S : S \text{ stable set in } G\}$$

max stable set problem: $\max\{\sum x_i : \mathbf{x} \in \text{STAB}(G)\}$ (NP-hard)

$$I_G := \langle x_i : i \in [n],\ x_i x_j : ij \in E\rangle \Rightarrow \mathcal{V}(I_G) = \{\chi^S : S \text{ stable set in } G\}$$

Lovász 1980

- $\text{TH}(G) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq 0\ f \text{ linear},\ f \text{ 1-sos mod } I_G\} \supseteq \text{STAB}(G)$
- G perfect $\Leftrightarrow \text{STAB}(G) = \text{TH}(G)$
- stable set problem solved in poly time if G perfect:

  $\max\{\sum x_i : \mathbf{x} \in \text{TH}(G)\}$ is an sdp

Lovász 1994: Which ideals $I \subseteq \mathbb{R}[\mathbf{x}]$ are perfect?

$$(\forall\ f \text{ linear, } f \geq 0 \text{ on } \mathcal{V}_{\mathbb{R}}(I) \Rightarrow f \text{ is 1-sos mod } I) \qquad .$$

- $f \in \mathbb{R}[\mathbf{x}]$ k-sos mod I if $f \equiv \sum h_j^2$ mod I, $\deg(h_j) \leq k$
- I is k-perfect if every linear $f \geq 0$ mod I is k-sos mod I
- $TH_k(I) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq 0, \forall\ f \text{ linear } f\ k\text{-sos mod } I\}$
- $\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \cdots \supseteq \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$ theta bodies for ideals

Gouveia, Parrilo, T. 2008

- $TH_k(I)$ is the closure of the projection of a spectrahedron
- $I = I(S)$, $S \subset \mathbb{R}^n \Rightarrow I$ k-perfect $\Leftrightarrow \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))} = \mathrm{TH}_k(I)$
- $I = I(S)$, S finite $\Rightarrow I$ perfect $\Leftrightarrow$ for every facet inequality $g(\mathbf{x}) \geq 0$ of conv(S), $S \subseteq \{g(\mathbf{x}) = 0\} \cup \{g(\mathbf{x}) = c_g\}$
- In this case, # facets, # vertices $\leq 2^n$ (both sharp)

Convex Algebraic Geometry

$$\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \cdots \supseteq \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$$

Theta bodies approximate convex hulls of real varieties

Compute theta bodies via sdp, convergence in many cases

Gouveia, Parrilo, T. 2008

For $I \subseteq \mathbb{R}[\mathbf{x}]$, $\mathrm{TH}_1(I) = \cap\{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(F)) : F$ convex quadric in $I\}$

Ex. $S = \{(0,0),(1,0),(0,1),(2,2)\}$