Note that the first property is not shared by problems whose constraints may include strict linear inequalities \( \sum a_j x_j < b \). To take a trivial example, the problem

\[
\text{maximize } x \quad \text{subject to } x < 0
\]

is neither infeasible nor unbounded and yet it has no optimal solution. The remaining two properties (ii) and (iii) tell us that, when looking for feasible or optimal solutions of an LP problem in the standard form, we may confine our search to a finite set. These two properties, easy to establish from scratch, are often used to motivate the simplex method. Our exposition has followed the reverse pattern, with an emphasis placed on actually solving the problem—and the fundamental theorem of linear programming obtained as an effortless afterthought.

**PROBLEMS**

3.1 Maximize

\[
x_1 + 3x_2 - x_3
\]

subject to

\[
2x_1 + 2x_2 - x_3 \leq 10
\]

\[
3x_1 - 2x_2 + x_3 \leq 10
\]

\[
x_1 - 3x_2 + x_3 \leq 10
\]

\[
x_1, x_2, x_3 \geq 0.
\]

3.2 In the tableau format, a natural tie-breaking rule for the choice of the pivot row favors the rows that appear higher up in the tableau. Show that in the following example (constructed by H. W. Kuhn), this tie-breaking rule leads to cycling:

\[
\text{maximize } 2x_1 + 3x_2 - x_3 - 12x_4
\]

subject to

\[
-2x_1 - 9x_2 + x_3 + 9x_4 \leq 0
\]

\[
\frac{1}{3} x_1 + x_2 - \frac{1}{3} x_3 - 2x_4 \leq 0
\]

\[
x_1, x_2, x_3, x_4 \geq 0.
\]

3.3 Solve problem 3.2 by the perturbation technique.

3.4 Arrange the following expressions in a sequence from lexicographically smallest to lexicographically largest:

\[
3 - \varepsilon_1
\]

\[
3
\]

\[
2 + 10\varepsilon_1
\]

\[
3 - 4\varepsilon_1 + \varepsilon_2
\]

\[
\varepsilon_2 + 3\varepsilon_3
\]

\[
3 + 4\varepsilon_1 + \varepsilon_3
\]

\[
3 - 4\varepsilon_1 + \varepsilon_2 + \varepsilon_3.
\]

3.5 Prove: If \( r = r_0 + r_1 \varepsilon_1 + \cdots + r_m \varepsilon_m \) is lexicographically smaller than \( s = s_0 + s_1 \varepsilon_1 + \cdots + s_m \varepsilon_m \) and if \( s \) is lexicographically smaller than \( t = t_0 + t_1 \varepsilon_1 + \cdots + t_m \varepsilon_m \), then \( r \) is lexicographically smaller than \( t \).

3.6 Use the result of problem 3.5 to prove that, in every finite set of distinct expressions, such as \( r \) and \( s \) in (3.5), there is an expression that is lexicographically smaller than all the others.
3.7 Prove that for every pair of expressions in (3.5) there is a positive number \( \delta \) such that
following two statements are equivalent: (i) \( r \) is lexicographically smaller than \( s \); (ii)
every choice of numbers \( \epsilon_1, \epsilon_2, \ldots, \epsilon_m \) such that
\[ 0 < \epsilon_i < \delta \quad \text{and} \quad 0 < \epsilon_i < \delta \epsilon_{i-1} \quad \text{for all} \ i = 2, 3, \ldots, m \]
\( r \) is numerically smaller than \( s \).

3.8 Use Theorem 3.2 and the result of problem 3.7 to prove the following. For every LP probl
\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m) \\
& \quad x_j \geq 0 \quad (j = 1, 2, \ldots, n)
\end{align*}
\]
there is a positive number \( \delta \) such that the simplex method used to
\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i + \epsilon^j \quad (i = 1, 2, \ldots, m) \\
& \quad x_j \geq 0 \quad (j = 1, 2, \ldots, n)
\end{align*}
\]
terminates whenever \( 0 < \epsilon < \delta \).

\[ \Delta \ 3.9 \]
Solve the following problems by the two-phase simplex method:

a. maximize \( 3x_1 + x_2 \)
subject to \( x_1 - x_2 \leq -1 \)
\( -x_1 - x_2 \leq -3 \)
\( 2x_1 + x_2 \leq 4 \)
\( x_1, x_2 \geq 0 \)

b. maximize \( 3x_1 + x_2 \)
subject to \( x_1 - x_2 \leq -1 \)
\( -x_1 - x_2 \leq -3 \)
\( 2x_1 + x_2 \leq 2 \)
\( x_1, x_2 \geq 0 \)

c. maximize \( 3x_1 + x_2 \)
subject to \( x_1 - x_2 \leq -1 \)
\( -x_1 - x_2 \leq -3 \)
\( 2x_1 - x_2 \leq 2 \)
\( x_1, x_2 \geq 0 \).

\[ \Delta \ 3.10 \]
Prove or disprove: A feasible dictionary whose last row reads \( z = z^* + \sum \epsilon_j x_j \) descri
an optimal solution if and only if \( \epsilon_j \leq 0 \) for all \( j \).