Math 524

Homework due 10/11/00

Problem 1. Let (X, d) be a compact metric space. We denote by $\mathcal{C}(X)$ the family of closed subsets of X. For A, $B \in \mathcal{C}(X)$ define the Hausdorff distance between A and B by,

$$D[A, B] = \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A),$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ denotes the distance from the point a to the set B, and $d(b, A) = \inf\{d(b, a) : a \in A\}$ the distance of the point b to the set A.

1.1 Show that D defines a metric on $\mathcal{C}(X)$.

1.2 Prove that $(\mathcal{C}(X), D)$ is a complete metric space.

1.3 Prove that $(\mathcal{C}(X), D)$ is totally bounded; that is, for every $\epsilon > 0$ there exists a finite cover of $\mathcal{C}(X)$ by ϵ-balls for the metric D.

Together, these imply that $(\mathcal{C}(X), D)$ is a compact metric space.

Let (X, ρ) be a metric space. Let $E \subset X$.

- A point $x \in X$ is a *limit point* or an *accumulation point* of E if $\forall r > 0$, $E \cap B(x, r) \setminus \{x\} \neq \emptyset$.
- E is said to be *perfect* if E is closed and if every point of E is an accumulation point of E.

Problem 2. Prove that a nonempty perfect set in $\mathbb{R}^n$ is uncountable.

Problems from Royden: Chapter 7, Section 4: problems 14, 15.

Chapter 7, Section 5: problem 21.
Chapter 7, Section 7: problem 27.