Math 524

Homework due 11/1/00

El Dia de los Muertos

Problem 1. (Prelim) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuously differentiable functions (i.e. $f_n \in C^1([0,1])$) which satisfy $f_n(0) = 0$, and

$$\int_0^1 |f_n'(x)|^2 \, dx \le 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove that there is a subsequence of f_n which converges uniformly on $[0,1]$.
Hint: recall that if $a, b \ge 0$ and $\epsilon > 0$ then $2ab \le \frac{a^2}{\epsilon} + \epsilon b^2$.

Problem 2. Let $X = \{f : [0,1] \to \mathbb{R}; f \text{ piecewise continuous}\}$. We say that f is piecewise continuous on $[0,1]$ if f has only finitely many discontinuities in $[0,1]$. For $f, g \in X$ we say that $f \sim g$ is $f - g \equiv 0$ on $[0,1]$, except for finitely many points. This defines an equivalence relation on X. Let $Y = \{[f] : f \in X\}$ where $[f]$ denotes the equivalence class of $f \in X$. For $[f], [g] \in Y$ define

$$d([f],[g]) = \int_0^1 |f - g| \, dx.$$

Show that d defines a metric on Y. Show that the closed ball of radius 1 and center $[0]$, i.e.

$$\{[f] \in Y : d([f],[0]) \le 1\}$$

is not compact.

$*$ **Problem:** *Caratheodory's criterion*
Let μ be a measure on $\mathbb{R}^n$. If

$$\mu(A \cup B) = \mu(A) + \mu(B), \quad \forall\ A, B \subset \mathbb{R}^n \ \text{ with } \ d(A,B) > 0,$$

then μ is a Borel measure (i.e. Borel sets are μ-measurable). Here

$$d(A,B) = \inf\{|a - b| : a \in A, b \in B\}.$$

Problems from Folland:

Chapter 2, Section 1: problems 1, 2, 9.