Math 426/576

Midterm

Instructions:

1. This midterm is due on May 9, 2008 at the beginning of class (11:30 am). NO late exams will be accepted.
2. Do problems 1 to 5. Problem 6 is an extra credit problem.
3. You are allowed to consult the following materials: your class notes, the textbook *The elements of integration and Lebesgue measure* by Bartle and the book *Real Analysis* by Royden, which is on reserve for this class at the Mathematics library. NO other materials are allowed.
4. You are not allowed to discuss this midterm with anybody other than the TA and the instructor. Failure to comply will be considered a violation of academic honesty.

Problem 1. Let m denote the Lebesgue measure in $\mathbb{R}$. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set such that $m(E) > 0$. Prove that for any $\alpha \in (0, 1)$ there is an open interval I such that

$$m(E \cap I) > \alpha m(I).$$

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \to \overline{\mathbb{R}}$ be a non-negative measurable function such that $\int f \, d\mu < \infty$. Prove that given $\epsilon > 0$ there is $\delta > 0$ such that

$$\int_E f \, d\mu < \epsilon \quad \text{whenever} \quad \mu(E) < \delta$$

Problem 3. If $f \in L^1(m)$ where m denotes the Lebesgue measure in $\mathbb{R}$ prove that the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_{-\infty}^{x} f(t) \, dt$$

is continuous on $\mathbb{R}$.

Problem 4. Generalized Dominated Convergence Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ be a sequences of measurable functions in $L^1(\mu)$. Assume that:

1. $f_n \to f$ μ-a.e. and $f \in L^1(\mu)$.
2. $g_n \to g$ μ-a.e. and $g \in L^1(\mu)$.
3. $|f_n| \leq g_n$ $\forall n \in \mathbb{N}$.
4. $\int g_n \, d\mu \to \int g \, d\mu$.

Prove that

$$\int f_n \, d\mu \to \int f \, d\mu.$$

Problem 5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions. Assume that $f_n \to f$ μ-a.e. and $f_n,\ f \in L^1(\mu)$. Show that

$$\lim_{n\to\infty} \int |f_n| \, d\mu = \int |f| \, d\mu \quad \text{iff} \quad \lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

$*$ **Problem 6.** Let $f_n(x) = ae^{-nax} - be^{-bnx}$ where $0 < a < b$. Show that:

$$\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)| \, dx = +\infty \tag{1}$$

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x) \, dx = 0 \tag{2}$$

$$\sum_{n=1}^{\infty} f_n \in L^1([0, \infty, m)) \tag{3}$$

$$\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x) \, dx = \log\left(\frac{b}{a}\right) \tag{4}$$