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Geometry of measures in $\mathbf{R}^n$: Distribution, rectifiability, and densities

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Introduction

We investigate to what extent the tangential and rectifiability properties of measures are determined by the regular behaviour of the measure of balls. One of the results of our investigation is the proof of the following conjecture. (See for example, [11, 3.3.22].)

If $0 \leq m \leq n$ are integers, Φ measures $\mathbf{R}^n$, Borel sets are Φ measurable, and

(BP) $$0 < \lim_{r \searrow 0} \Phi(B(x, r))/r^m < \infty$$

at Φ almost every x, the Φ is m rectifiable, i.e., Φ almost all of $\mathbf{R}^n$ can be covered by countably many m dimensional smooth submanifolds of class one of $\mathbf{R}^n$.

Most of the arguments used in the paper are based upon the fundamental work of A. S. Besicovitch [2], [3], [4], upon the techniques discovered by H. Federer [10] in the proof of his famous projection theorem, as well as upon the work of other mathematicians on different problems of geometric measure theory. Since these ideas may be found in the excellent book [11] (see also [12]), we restrict ourselves to a brief description of those results connected with the above conjecture and of those arguments used in their proofs which mainly influenced the present development and which might help to explain the results and arguments used here.

The case $m = 1$ and $n = 2$ was proved in a different but equivalent formulation by A. S. Besicovitch [3, Theorem 18]. (Because of this we use the notation (BP) meaning "Besicovitch Property".) In the above formulation this case was proved in [22] and the case $m = 1$, n arbitrary in [21]. If $m = 1$, the situation is much simpler since the restriction of the one dimensional Hausdorff measure to a continuum of finite measure is 1 rectifiable. (If $n = 2$, see [3] or, for a new treatment, [9]. For the general case see [8].) No comparable statement holds if $m \geq 2$. (See, for example, [11, 4.2.25].) This is the reason why even the case $m = 2$ and $n = 3$ of the above conjecture remained open. Nevertheless, a partial result follows from the work of J. M. Marstrand [17] (if $m = 2$ and $n = 3$) and of P. Mattila [18] (in the general case):

If a measure Φ over $\mathbf{R}^n$ fulfils

$$(\mathrm{BP}^*) \qquad 0 < \lim_{r \searrow 0} \Phi(B(x, r))/r^m = \limsup_{\operatorname{diam}(S) \searrow 0,\ S \ni x} \Phi(S)/((\operatorname{diam} S)/2)^m < \infty$$

almost everywhere, then it is m rectifiable.

An important observation used by these authors is that m rectifiability is implied by the existence of "m dimensional weak tangents". (It should be noted that the term "weak tangent" has been used differently, e.g. in [9, page 51]). We formulate this notion more explicitly than the original authors did, since it is one of the main starting points of our investigation:

(WT) A measure Φ over $\mathbf{R}^n$ is said to have m dimensional weak tangents if for Φ almost every x there is $c(x) \in (0, \infty)$ such that for every $\mu \in (0, 1)$ the following statement holds for all sufficiently small $r > 0$: There is an m dimensional affine subspace V or R^n passing through x such that

$$(\mathrm{WT}_1) \qquad (1 - \mu)c(x)s^m \leq \Phi(B(z, s)) \leq (1 + \mu)c(x)s^m$$

for every $z \in B(x, r) \cap V$ and for every $s \in [\mu r, r]$. (One should notice that V may depend not only upon x but also upon r!)

We shall not describe the ingenious argument of Marstrand used in [17] and [19] to prove that a measure satisfying the (BP*) has m dimensional weak tangents. Unfortunately, it does not seem to be applicable even if Hausdorff measures are replaced by spherical measures or, in our language, if (BP*) is replaced by

$$(\mathrm{BP}^{**}) \quad 0 < \lim_{r \searrow 0} \Phi(B(x, r))/r^m = \lim_{r \searrow 0} \sup\{ \Phi(B(z, s))/s^m;\ z \in B(x, r),\ s \in [\|z - x\|, r]\} < \infty.$$

Another corollary of our investigation is a generalization of the following result of J. M. Marstrand [15], [16], [18]: If α is a real number and a measure Φ over $\mathbf{R}^n$ fulfils

$$(\mathrm{BP}_{r^\alpha}) \qquad 0 < \lim_{r \searrow 0} \Phi(B(x, r))/r^\alpha < \infty$$

almost everywhere, then Φ almost every x fulfils:

(WT*) For every $\mu \in (0, 1)$ one can find arbitrarily small $r > 0$ for which there is an affine subspace V of $\mathbf{R}^n$ passing through x such that

$$(1 - \mu)t^{\dim V}\Phi(B(x, r)) \leq \Phi(B(z, tr)) \leq (1 + \mu)t^{\dim V}\Phi(B(x, r))$$

for every $z \in V \cap B(x, r)$ and for every $t \in [\mu, 1]$. (One should notice that (WT*) is a weaker form of (WT)!)

We prove (considerably more than) that (WT*) holds if Φ has the (BP_h) for some positive function h. (When Φ measures $\mathbf{R}^n$, $\liminf_{r \searrow 0} h(2r)/h(r) \geq 2^{n-1}$, and h fulfils a mild regularity condition, this was proved by M. Chlebík [6].)

A striking corollary of Marstrand's result is that, if (BP_{r^α}) holds for some nonzero measure Φ over $\mathbf{R}^n$ then α is an integer (and $0 \leq \alpha \leq n$). (When $n = 1$, this was proved already by A. S. Besicovitch [1]; see also [5].) Similarly, our results lead to substantial progress in the following more general problem: For which functions h is there a nonzero measure over $\mathbf{R}^n$ having the (BP_h)? When $n = 1$, P. Mattila [20] proved that $0 < \lim_{r \searrow 0} h(r) < \infty$ or $0 < \lim_{r \searrow 0} h(r)/r < \infty$. As we shall see, the situation is more intricate if $n \geq 2$.

It should be also noted that the above formulations of (WT) and (WT*) are stronger than necessary. For example, if Φ has the (BP*), an equivalent definition of (WT) is obtained by replacing (WT_1) by

$$(\mathrm{WT}_2) \qquad \Phi[B(x, r) - B(V, \mu r)] \leq \mu r^m.$$

A simple proof of this fact is given in [17] and [19]. If Φ is supposed to have the (BP) only, the question whether (WT_1) may be replaced by (WT_2) is more complicated. In fact, M. Chlebík's positive answer to this problem (private communication) and my attempt to give a different proof (see 3.13) were probably the main starting points for the results presented here.

Last but not least, the source of the ideas used in this paper were long discussions with Professor Casper Goffman as well as with other participants in the Special Year in Real Analysis at the University of California at Santa Barbara.

I would also like to thank Professors H. Federer and P. Mattila for a number of corrections to and comments on the preliminary version of this paper.

It might be worthwhile to show informally how our main arguments may be used to give relatively simple proofs of (BP_{r^α}) $\Rightarrow$ (WT*) and [(BP*) $\Rightarrow$](BP**) $\Rightarrow$ (WT). First we introduce the notion of "tangent measures": When Φ measures $\mathbf{R}^n$, $x \in \mathbf{R}^n$, and $r > 0$, we denote by $\Phi_{x,r}$ the measure defined by the formula $\Phi_{x,r}(E) = \Phi(x + rE)$. By $\mathrm{Tan}(\Phi, x)$ we denote the set of all tangent measures of Φ at x, i.e. the set of all nonzero, locally finite limits (in the vague topology) of the form

$$\lim_{k \to \infty} c_k \Phi_{x, r_k},$$

where $r_k \searrow 0$ and $c_k > 0$.

It is easy to see that (WT*) is equivalent to: There is a linear subspace V of $\mathbf{R}^n$ such that $\mathscr{H}^{\dim V} \llcorner V \in \mathrm{Tan}(\Phi, x)$. ($\mathscr{H}^m \llcorner E$ denotes the restriction of the m dimensional measure to the set E.) From (BP_h) we infer that for Φ almost every

x every $\Psi \in \operatorname{Tan}(\Phi, x)$ has the "Global Besicovitch Property":

(GBP*) $$\Psi(B(z, r)) = \Psi(B(0, r)) < \infty$$

for every z belonging to the support of Ψ and for every $r > 0$.

For every measure Ψ let $\mathfrak{M}(\Psi)$ denote the closure (in the vague topology) of the set $\{c\Psi_{z,r};\ z \in \operatorname{spt}\Psi,\ c > 0,\ r > 0\}$. Since one of the general properties of the tangent measures ensures that, for Φ almost every x, $\mathfrak{M}(\Psi) \subset \operatorname{Tan}(\Phi, x)$ whenever $\Psi \in \operatorname{Tan}(\Phi, x)$, we just need to prove that every measure having the (GBP*) fulfils:

(WT**) There is a linear space V such that $\mathscr{H}^{\dim V} \llcorner V \in \mathfrak{M}(\Psi)$. (We do not need the above property of tangent measures if we just want to prove that (BP_{r^α}) may hold for a nonzero measure only if α is an integer.)

If $h(r) = r^\alpha$, one also knows that $\Psi(B(0, r)) = cr^\alpha$, where c is a positive constant. Assuming that (WT**) is false even for such measures, we find the smallest nonnegative integer n for which there are α and a nonzero measure Ψ over $\mathbf{R}^n$ such that $\Psi(B(z, r)) = r^\alpha$ for each $z \in \operatorname{spt}\Psi$ and each $r > 0$ and such that

$$\mathfrak{M}(\Psi) \cap \{\mathscr{H}^{\dim V} \llcorner V;\ V \text{ is a linear subspace of } \mathbf{R}^n\} = \varnothing.$$

Then $\operatorname{spt}\Psi \neq \mathbf{R}^n$ since otherwise Ψ would be a constant multiple of the Lebesgue measure. Hence there are $u \in \mathbf{R}^n - \operatorname{spt}\Psi$ and $x \in \operatorname{spt}\Psi$ such that $\|u - x\| = \operatorname{dist}(u, \operatorname{spt}\Psi)$. We find $\tilde{\Psi} \in \operatorname{Tan}(\Psi, x)$ and we finish the proof by showing that the dimension of the linear span of $\operatorname{spt}\tilde{\Psi}$ is less than n. To prove this, we use a simplified version of Marstrand's argument [15], [16], [18]: For each $r > 0$, let $b(r) \in \mathbf{R}^n$ be the center of mass of the restriction of Ψ to $B(x, r)$; i.e.

$$\langle b(r), v\rangle = \int_{B(x,r)} \langle z, v\rangle\, d\Psi(z)/\Psi B(x, r) \quad \text{for every } v \in \mathbf{R}^n.$$

If $b(r) = x$ for every $r > 0$, we easily see that $\operatorname{spt}\tilde{\Psi}$ is a subset of the orthogonal complement of $u - x$. If $b(r) \neq x$ for some $r > 0$, we use the equality

$$\int_{B(y,r)} (r^2 - \|z - y\|^2)\, d\Psi(z) = \int_{B(x,r)} (r^2 - \|z - x\|^2)\, d\Psi(z)$$

(where $y \in \operatorname{spt}\Psi$) to conclude that there is a constant C such that $|\langle b(r) - x, y - x\rangle| \le C\|y - x\|^2$ for every $y \in \operatorname{spt}\Psi$. Consequently, $\operatorname{spt}\tilde{\Psi}$ is a subset of the orthogonal complement of $b(r) - x$.

It is not difficult to modify the above argument to show that (GBP*) $\Rightarrow$ (WT**). (Cf. 3.8.) Another way to prove this statement is to use the result of [13] according to which for every measure Ψ with the (GBP*) there are an integer $m = 0, \ldots, n$ and an m dimensional analytic submanifold E of $\mathbf{R}^n$ such that Ψ is a constant multiple of $\mathscr{H}^m \llcorner E$.

If we attempt to use the notion of tangent measures to prove (BP) $\Rightarrow$ (WT), we first note that, if (BP) holds, (WT) is equivalent to:

$$\operatorname{Tan}(\Phi, x) \subset \{ c\mathscr{H}^m \llcorner V;\ c > 0 \text{ and } V \text{ is an } m \text{ dimensional linear subspace of } \mathbf{R}^n \}$$

for Φ almost every x. To prove this equivalence, we could assume much less than (BP) (cf. 5.6(4)). The full strength of (BP) is used to prove that Φ almost every x has the property: Every tangent measure of Φ at x is a constant multiple of a measure Ψ for which

$$\text{(GBP)} \qquad \Psi(B(z, r)) = \Psi(B(0, r)) = \alpha(m) r^m$$

whenever $z \in \operatorname{spt} \Psi$ and $r > 0$. (Here $\alpha(m)$ denotes the m dimensional volume of the m dimensional unit ball.) Hence (BP) $\Rightarrow$ (WT) would be proved if the following seemingly plausible conjecture were true.

(C) If a nonzero measure Ψ over $\mathbf{R}^n$ has the (GBP) then there is an m dimensional linear subspace V of $\mathbf{R}^n$ such that $\Psi = \mathscr{H}^m \llcorner V$.

If $m = 0$ or $m = n$, (C) is easy. If $m = 1$, an elementary proof of (C) is an easy exercise. If $m = 2$, the proof seems to be more complicated, but (C) still holds. (See 3.17; for the case $m = 2$ and $n = 3$ see also [14].) Unfortunately, if $m = 3$ and $n = 4$, (C) fails. (See 3.20.) Because of this, the proof of (BP) $\Rightarrow$ (WT) needs a more detailed study of measures fulfilling the (GBP). On the other hand, if Ψ fulfils the (BP*) (or only the (BP**)), the above argument shows that we need to prove (C) only under the additional assumption that $\Psi(B(z, r)) \le \alpha(m) r^m$ for every $z \in \mathbf{R}^n$. This weaker form of (C) holds for every m. (See 3.18.) A relatively simple proof of it may be obtained in the following way:

Let $b_r \in \mathbf{R}^n$ be defined by

$$\langle b_r, v \rangle = \int_{B(0, r)} (r^2 - \|z\|^2) \langle z, v \rangle \, d\Psi(z) \Big/ \int_{B(0, r)} (r^2 - \|z\|^2) \, d\Psi(z)$$

for every $v \in \mathbf{R}^n$, and let Q_r be the quadratic forms on $\mathbf{R}^n$ defined by

$$Q_r(v) = 2 \int_{B(0, r)} \langle z, v \rangle^2 \, d\Psi(z) \Big/ \int_{B(0, r)} (r^2 - \|z\|^2) \, d\Psi(z)$$

for every $v \in \mathbf{R}^n$. One easily sees that $\operatorname{Trace}(Q_r) = m$. Thus there is a sequence $r_k \to \infty$ such that the sequence Q_{r_k} converges to a quadratic form Q such that

$\text{Trace}(Q) = m$. Using the inequality

$$\int_{B(u,r)} (r^2 - \|z-u\|^2)^2 \, d\Psi(z) \le \int_{B(0,r)} (r^2 - \|z\|^2)^2 \, d\Psi(z) \qquad (u \in R^n)$$

and the equality

$$\int_{B(x,r)} (r^2 - \|z-x\|^2)^2 \, d\Psi(z) = \int_{B(0,r)} (r^2 - \|z\|^2)^2 \, d\Psi(z) \qquad (x \in \operatorname{spt} \Psi),$$

we find a constant C such that

$$2\langle b_r, u\rangle + Q_r(u) - \|u\|^2 \le C\|u\|^3/r \qquad (u \in \mathbf{R}^n,\ r > 0)$$

and

$$\left|2\langle b_r, x\rangle + Q_r(x) - \|x\|^2\right| \le C\|x\|^3/r \qquad (x \in \operatorname{spt} \Psi,\ r > 0).$$

From the first of these inequalities we see that the b_r converge to zero as $r \to \infty$ and that $Q(u) \le \|u\|^2$ for every $u \in \mathbf{R}^n$. Using the second inequality, we see that

$$\operatorname{spt} \Psi \subset \left\{x \in \mathbf{R}^n;\ Q(x) = \|x\|^2\right\}.$$

Recalling that $\text{Trace}(Q) = m$ and $Q(u) \le \|u\|^2$, we infer that spt Ψ is a subset of an m dimensional linear subspace V, which easily implies that $\Psi = \mathscr{H}^m \llcorner V$.

We complete the introduction with a more detailed survey of the contents of the paper.

In the first chapter we set out the notation to be used (1.1–1.9) and we recall some of the results concerning the differentiation of measures (1.7), the distance between measures and the vague convergence of measures (1.10–1.14, where, for the convenience of the reader, we also give short proofs).

In the second chapter we study the general properties of tangent measures. We start with the concepts of a d-cone of measures (2.1(1)), of its basis (2.1(2)), and of the distance of a measure from a d-cone (2.1(3)). Some simple properties of this distance are given in 2.1(4–7). In 2.2 we prove various characterizations of d-cones with compact bases. The definition of the set $\text{Tan}(\Phi, x)$ of all tangent measures of Φ at x (where $x \in \mathbf{R}^n$ or $x = \infty$) is introduced in 2.3(1, 2). We notice that $\text{Tan}(\Phi, x)$ is a d-cone (this was the reason for studying d-cones separately) and that there are simple relations between tangent measures of Φ and of its multiple by a function (2.3(3, 4)). In 2.5 we prove the existence of the tangent measures. Theorem 2.6 establishes important connectedness properties of $\text{Tan}(\Phi, x)$. Its corollaries 2.7 and 2.8 show how the compactness of the basis of $\text{Tan}(\Phi, x)$ is connected with the behaviour of Φ near x. In 2.9–2.11 we study the situation to be encountered in Chapter 3, namely, measures Φ and Ψ with $\operatorname{Tan}(\Phi, x)$ equal to the cone generated by Ψ. Theorem 2.12 is an equivalent formulation of the general property of tangent measures mentioned in the sketch of the proof of $(\mathrm{BP}_{r^\alpha}) \Rightarrow (\mathrm{WT}^*)$. Finally, in 2.13 we give again some general properties of the d-cones and in 2.13(5) we use them to obtain a useful corollary of 2.12.

The third chapter is devoted to the study of uniformly distributed measures over $\mathbf{R}^n$. This notion (defined in 3.1(1)) is very close to (GBP*); in fact, the family $\mathfrak{U}(n)$ introduced in 3.1(2) is the family of all nonzero measures over $\mathbf{R}^n$ having the (GBP*). Theorem 3.3. compares uniformly distributed measures with close supports; its statement 3.3(2) assuring that uniformly distributed measures are (up to a constant multiple) determined uniquely by their supports holds even in arbitrary metric spaces. (See [7] or, for a further generalization, [23].) In 3.4 we introduce and give some estimates of the moments $b_{k,s}(\Phi)$ of measures $e^{-s\|z\|^2}\Phi$ ($\Phi \in \mathfrak{U}(n)$). Practically all our main results are based upon the existence of the asymptotic expansions of these moments for $s \to \infty$ and for $s \searrow 0$ (the latter being the more important) proved in 3.6. In 3.7 we introduce various d-cones of "flat" measures. Theorem 3.8 is a slightly stronger version of the implication $(\mathrm{GBP}^*) \Rightarrow (\mathrm{WT}^{**})$. In 3.10 and 3.11 we study the tangential behaviour of a uniformly distributed measure Φ at any point $x \in (\operatorname{spt}\Phi) \cup \{\infty\}$. It turns out that $\operatorname{Tan}(\Phi, x)$ is a cone generated by a single measure $\Psi \in \mathfrak{U}(n)$ and that Ψ has the (GBP). The corresponding integer $m = 0, 1, \dots, n$ is the same for all $x \in \operatorname{spt}\Phi$. Hence we introduce in 3.12(1, 2) two dimensions of a unformly distributed measure: $\dim_\infty\Phi$ and $\dim_0\Phi$. In 3.12(3–8) we notice some properties of these dimensions. (We should also remark that from 3.11(2; i, iv) and from 5.6 one sees that every uniformly distributed measure Φ is $\dim_0\Phi$ rectifiable. Since considerably more is proved in [13], we do not go into it here.) In dependence upon the "flatness" of $\operatorname{Tan}(\Phi, \infty)$, the uniformly distributed measures are partitioned into two subclasses: measures flat at ∞ and measures curved at ∞. The fact established in 3.14 that these classes are very far from each other turns out to be the first decisive step towards the proof that (BP) $\Rightarrow$ (WT). The second decisive step towards this proof are the algebraic properties of (the moments of) measures flat at ∞ proved in 3.15. In fact, from 3.15(x, v) one easily sees that a measure flat at ∞ having the (GBP) is of the form $\mathscr{H}^m \llcorner V$ for some m dimensional linear space V. (We prove more in 3.16.) Using this fact, 3.14, and 2.6, one can easily finish the proof that (BP) $\Rightarrow$ (WT). (Our proof in Chapter 5 is more involved since we give a more general result.) Corollary 3.17 proves (C) for those m and n for which it holds while Example 3.20 disproves it in the remaining cases. In 3.18 we give flatness criteria similar to those used in the above sketch of the proof of $(\mathrm{BP}^{**}) \Rightarrow$ (WT). From 3.19

and 3.12(3) one can infer some information about uniformly distributed measures over $\mathbf{R}^n$ with dimensions $\geq n - 1$. Finally, in 3.21 we pose some open problems.

In Chapter 4 we study tangential properties of (ε-) approximately uniformly distributed measures over $\mathbf{R}^n$. Roughly speaking, these are the measures for which the ratio of the measures of close balls with the same small radius is close to one. The formal definition and some criteria are given in 4.1–4.3. For example, according to 4.3(1) every measure having the (BP_h) for some h is approximately uniformly distributed. Another important subclass consists of those measures for which

$$\lim_{r \searrow 0} \Phi(B(x, 2r))/\Phi(B(x, r))$$

exists Φ almost everywhere (4.3(2)). We allow for the "error" ε because it enables us to study also measures for which

$$0 < \limsup_{r \searrow 0} \Phi(B(x, r))/h(r) < (1 + \varepsilon) \liminf_{r \searrow 0} \Phi(B(x, r))/h(r) < \infty$$

for almost every x. After establishing in 4.4 some elementary properties of measures approximable by "flat" measures (4.4(5, 6) are simple rectifiability criteria), we deduce in 4.5–4.6 the main tangential properties of general ε-approximately uniformly distributed measures. Corollary 4.7(1) shows that approximately uniformly distributed measures are precisely those for which tangent measures are uniformly distributed. (Its "error version" 4.6(1) and 4.2(5) imply that our "unilateral" definition 4.1 is, up to a rescaling of ε, equivalent to the "bilateral" one.) From 4.7(2) we see that every approximately uniformly distributed measure fulfils (WT*). In 4.8–4.10 we find sufficient conditions under which all tangent measures of a given measure are "flat". Some equivalent conditions are established in Theorem 4.11. Finally, in 4.12 we define the concept of the dimension of an approximately uniformly distributed measure at a point and we derive some of its properties.

In Chapter 5 we study the connections among m rectifiability (defined in 5.1(1)), m dimensional densities (defined in 5.1(2)), and "flatness" of tangent measures. Except for the rectifiability criterion 5.3 the results are obtained as an easy application of 4.8–4.11. (Since 5.3 is a more precise version of the rectifiability criterion used in [17] and [19], it may be proved also by an argument similar to that used by these authors. Our proof of 5.3 might seem to be more complicated, since Lemma 5.2 is formulated in a way which can be used also in 6.4.) The main rectifiability criteria involving the densities are given in

5.4. Corollary 5.5 establishes the existence of positive constants $\tilde{\omega}(n,m)$ such that every measure Φ over $\mathbf{R}^n$ fulfilling (at almost every point x)

$$0 < \limsup_{r \searrow 0} \Phi(B(x,r))/r^m < (1 + \tilde{\omega}(n,m)) \liminf_{r \searrow 0} \Phi(B(x,r))/r^m < \infty$$

is m rectifiable. This result is illustrated in 5.7 by demonstration that $\lim_{n \to \infty} \tilde{\omega}(n,2) = 0$. I believe that this is one of the reasons why the proof of (BP) $\Rightarrow$ (WT) for $m \geq 2$ differs so much from the case of $m = 1$ as well as from the proof of (BP*) $\Rightarrow$ (WT). (E. F. Moore [21] proved the estimate $\tilde{\omega}(n,m) \geq .01$ and M. Chlebík [6] improved the result of [19] by showing that there are positive constants $\tilde{\omega}(m)$ such that every measure in (any) $\mathbf{R}^n$ fulfillling (at almost every x)

$$\begin{aligned} 0 < &\limsup_{\operatorname{diam}(S) \searrow 0,\ S \ni x} \Phi(S)/((\operatorname{diam} S)/2)^m \\ &< (1 + \tilde{\omega}(m)) \liminf_{r \searrow 0} \Phi(B(x,r))/r^m < \infty \end{aligned}$$

is m rectifiable.) Theorem 5.6 reformulates some of the conditions of 5.4 as necessary and sufficient for m rectifiability and Examples 5.9 illustrate the need for the density assumptions in 5.6(3, 4). These examples are based on a more general construction 5.8 which is used also in 6.5 and in 6.11.

The final Chapter 6 studies density functions in $\mathbf{R}^n$, i.e. those positive functions h on $(0, \infty)$ for which there is a nonzero measure over $\mathbf{R}^n$ with finite and nonzero h density almost everywhere. (The definition 6.1(1) of the h density differs slightly from the usual one used in the formulation of the (BP_h). The functions h for which there is a nonzero measure over $\mathbf{R}^n$ with the (BP_h) are termed exact density functions in $\mathbf{R}^n$.) In 6.1(2–5) we notice some properties of the h densities. As a consequence, we formulate in 6.2(2) a simple connection between exact density functions and density functions. In 6.2(6) we define $\operatorname{Dim}(h)$ as the set of all integers m for which there are arbitrary small $r > 0$ such that $h(tr)/h(r)$ is close to t^m. Theorem 6.3 stating that $\operatorname{Dim}(h) \neq \varnothing$ is a considerable generalization of the results of [18] and of [6] mentioned earlier. We call a density function regular if $\lim_{r \searrow 0} h(tr)/h(r)$ exists for some (or, equivalently, all) $t > 0$, $t \neq 1$. (See 6.2(4,5).) Theorem 6.5 characterizes (exact) regular density functions. (One of the corollaries of 6.5 is that $r|\ln r|$ is not a density function in any $\mathbf{R}^n$ while $r/|\ln r|$ is an exact density function in $\mathbf{R}^2$.) Theorem 6.7 asserts that, if h is not a density function in $\mathbf{R}^n$ then there is $\varepsilon > 0$ such that for every measure Φ over $\mathbf{R}^n$ the upper h density is at least $(1 + \varepsilon)$ times the lower density. (I do not know whether an analogous statement holds if all h densities involved are replaced by the usual h densities.) In 6.11 we discuss the dependence of ε upon h and n and we formulate some open problems connected with it. In 6.8 and 6.9 we show for which sets M of nonnegative integers one can construct (irregular, or even more special) density functions h such that $\mathrm{Dim}(h) = M$. We just mention three particular cases: a (nonexact) irregular density function in $\mathbf{R}$ (6.9(2)), an exact irregular density function in $\mathbf{R}^2$ (6.9(3)), and an exact irregular density function in $\mathbf{R}^3$ weakly equivalent to r (6.9(1)). (In the language of [11] the last example may be reformulated as a construction of a purely $(\mathscr{H}^1, 1)$ unrectifiable set $E \subset \mathbf{R}^3$ and of a function h such that $0 < \mathscr{H}^1(E) < \infty$ and

$$0 < \lim_{r\to 0} \mathscr{H}^1(B(x,r) \cap E)/h(r) < \infty$$

for $\mathscr{H}^1$ almost every $x \in E$.) We have already seen that Mattila's result [20] is confined to the one dimensional space and to the exact density functions. Nevertheless, in 6.10 we connect it to an open problem concerning exact density functions in higher dimensional spaces.

1. Preliminaries

1.1. We shall denote by $\|\cdot\|$, $\mathrm{dist}(\cdot,\cdot)$, and $\langle \cdot,\cdot \rangle$ the Euclidean norm, distance, and inner product in $\mathbf{R}^n$, respectively. If E is a nonempty subset of $\mathbf{R}^n$ and $r > 0$, we denote

$$B(E,r) = \{x \in \mathbf{R}^n; \mathrm{dist}(x,E) \le r\},$$
$$B^0(E,r) = \{x \in \mathbf{R}^n; \mathrm{dist}(x,E) < r\},$$

$\mathrm{Int}(E)$ and $\mathrm{Clos}(E)$ the interior and closure of E, respectively, and $\mathrm{diam}(E)$ the diameter of E.
If $x \in \mathbf{R}^n$, we let $B(x,r) = B(\{x\}, r)$ and $B^0(x,r) = B^0(\{x\}, r)$. Sometimes we shall use the notation $B_n(x,r)$ to indicate that the ball is in $\mathbf{R}^n$.

1.2. Whenever V is a linear space of finite dimension, we denote by $\bigodot^k V$ the set of all symmetric k-linear forms on V. These forms will be naturally identified with the linear forms on the k-th symmetric tensor power $\bigodot_k V$ of V. The k-th symmetric tensor power of $x \in V$ will be denoted by x^k. If W is a subspace of V and $b \in \bigodot^k V$, we define the restriction of b to W by the formula $(b \llcorner W)(x^k) = b(x^k)$ for $x \in W$. (Since the notation $\langle \cdot,\cdot \rangle$ is reserved for the inner product, we use here the usual function notation.)
If V and W are linear spaces, $\mathrm{Hom}(V,W)$ denotes the space of all linear maps of V into W.
If V is an inner product space and $b \in \bigodot^2 V$, we denote by $\mathrm{Trace}(b)$ the trace of the linear automorphism of V which represents b.

1.3. The set of all m dimensional linear subspaces of $\mathbf{R}^n$ will be denoted by $G(n, m)$. If $V \in G(n, m)$, we denote by $V^{\perp}$ its orthogonal complement and by $x + V$ (where $x \in \mathbf{R}^n$) the m dimensional affine subspace of $\mathbf{R}^n$ passing through x and parallel to V.

1.4. If $E \subset \mathbf{R}^n$, χ_E denotes the characteristic function of E. Whenever f: $\mathbf{R}^n \mapsto \mathbf{R}$, we define $\mathrm{Lip}(f) \in [0, \infty]$ as the smallest constant for which $|f(x) - f(y)| \leq (\mathrm{Lip}(f))\|x - y\|$ for all $x, y \in \mathbf{R}^n$. The functions with $\mathrm{Lip}(f) < \infty$ are termed Lipschitzian. We also define the support of f by the formula

$$\mathrm{spt}(f) = \mathbf{R}^n - \{x \in \mathbf{R}^n;\ \text{there is } r > 0 \text{ such that } f(z) = 0 \\ \text{for every } z \in B(x, r)\}.$$

1.5. In this paper, a measure over $\mathbf{R}^n$ is a map Φ of the family of all subsets of $\mathbf{R}^n$ into $[0, \infty]$ such that

$$\Phi(A) = \inf\left\{ \sum_{B \in F} \Phi(B);\ F \text{ is a countable cover of } A \text{ by Borel subsets of } \mathbf{R}^n \right\}$$

for every set $A \subset \mathbf{R}^n$, and $\Phi(A \cup B) = \Phi(A) + \Phi(B)$ whenever A and B are disjoint Borel subsets of $\mathbf{R}^n$.

In other words, Φ is a Borel regular outer measure such that all Borel sets are Φ measurable. (Recall that E is called Φ measurable if $\Phi(T) = \Phi(T \cap E) + \Phi(T - E)$ for every set T.)

1.6. Suppose that Φ measures $\mathbf{R}^n$.

(1) The expression "Φ almost" is used in the usual way. For example, the phrase "f is defined Φ almost everywhere" means that the complement of the domain of f has Φ measure zero.

(2) A Φ measurable function is a map f of a subset of $\mathbf{R}^n$ into $[-\infty, \infty]$ for which $f^{-1}(G)$ is measurable whenever G is an open subset of $[-\infty, \infty]$.

(3) If f is a Φ measurable function defined Φ almost everywhere, we define

$$\int f\, d\Phi = \int f(z)\, d\Phi(z) = \int_0^{\infty} \Phi\{z \in \mathbf{R}^n;\ f(z) > r\}\, dr \\ - \int_0^{\infty} \Phi\{z \in \mathbf{R}^n;\ f(z) < -r\}\, dr$$

provided that at least one of the integrals on the right is finite. The equivalence of this definition with other definitions of the integral ([11, 2.4.4]) follows easily from the Fubini's theorem ([11, 2.6.2]).

(4) If $f\colon \mathbf{R}^n \to [0, \infty]$ is Φ measurable, the measure $f\Phi$ is defined by the formula $(f\Phi)(E) = \int f\chi_E \, d\Phi$ for every Φ measurable set $E \subset \mathbf{R}^n$. We note that

$$\int g(z)\, d(f\Phi)(z) = \int g(z) f(z)\, d\Phi(z).$$

(5) If $E \subset \mathbf{R}^n$, the restriction of Φ to E is defined by the formula $(\Phi \llcorner E)(A) = \Phi(A \cap E)$ for every Borel set $A \subset \mathbf{R}^n$. We also denote $\int_E f\, d\Phi = \int f\, d(\Phi \llcorner E)$.

(6) The support of Φ is defined by

$$\operatorname{spt} \Phi = \{x \in \mathbf{R}^n;\ \Phi(B(x, r)) > 0 \quad \text{for every } r > 0\}.$$

One easily sees that $\Phi(\mathbf{R}^n - \operatorname{spt} \Phi) = 0$ (or, equivalently, $\Phi = \Phi \llcorner \operatorname{spt} \Phi$) and that $\operatorname{spt} \Phi$ is the smallest closed set with this property.

(7) If $T\colon \mathbf{R}^n \mapsto \mathbf{R}^m$ is Borel measurable, the image of Φ under T is defined by the formula

$$T[\Phi](E) = \Phi(T^{-1}(E)) \quad \text{for every Borel set E} \subset \mathbf{R}^n.$$

Whenever E is a Borel subset of $\mathbf{R}^n$ and $g\colon E \mapsto [-\infty, \infty]$ is Borel measurable, then

$$\int_E g(u)\, dT[\Phi](u) = \int_{T^{-1}(E)} g(T(t))\, d\Phi(t).$$

(8) Φ is said to be locally finite (almost finite, respectively) if for every $x \in \mathbf{R}^n$ (for Φ almost every $x \in \mathbf{R}^n$, respectively) there is $r > 0$ such that $\Phi(B(x, r)) < \infty$. One readily sees that Φ is locally finite if and only if every compact subset of $\mathbf{R}^n$ has a finite measure and that Φ is almost finite if and only if Φ almost all of $\mathbf{R}^n$ can be covered by an (increasing) sequence of open sets with finite Φ measure.

1.7. When Φ measures $\mathbf{R}^n$, a point $x \in \mathbf{R}^n$ is called a Φ density point of a set $E \subset \mathbf{R}^n$ if $x \in \operatorname{spt} \Phi$, $\Phi(B(x, s)) < \infty$ for some $s > 0$, and

$$\lim_{r \searrow 0} \Phi(B(x, r) \cap E)/\Phi(B(x, r)) = 1.$$

The density theorem asserts that, if Φ is almost finite, almost every point of any set E is a Φ density point of E. This follows from the differentiation theorem, according to which, if Ψ and Φ measure $\mathbf{R}^n$ and $\Psi + \Phi$ is almost finite, then, with the convention $c/0 = \infty$ for $c \geq 0$, the function

$$f(x) = \lim_{r \searrow 0} \Psi(B(x, r))/\Phi(B(x, r))$$

is defined $\Psi + \Phi$ almost everywhere, $\Phi\{x; f(x) = \infty\} = 0$ and

$$\Psi(E) = \Psi(E \cap \{x; f(x) = \infty\}) + \int_E f(z)\, d\Phi(z)$$

for every Borel set E. (See [11, 2.9.15 and 2.9.7].)

1.8. We shall denote by $\mathscr{L}^n$ the Lebesgue measure over $\mathbf{R}^n$, by $\alpha(n)$ the $\mathscr{L}^n$ measure of $B(0, 1)$, and by $\mathscr{H}^m$ the m dimensional Hausdorff measure over $\mathbf{R}^n$. (See [11, 2.6.5 and 2.10.2].)

1.9. (1) When Φ measures $\mathbf{R}^n$ and D is a compact subset of $\mathbf{R}^n$, we denote $F_D(\Phi) = \int \operatorname{dist}(z, \mathbf{R}^n - D)\, d\Phi(z)$.

(2) When Φ and Ψ measure $\mathbf{R}^n$, D is a compact subset of $\mathbf{R}^n$, and $F_D(\Phi) + F_D(\Psi) < \infty$, we define

$$F_D(\Phi, \Psi) = \sup\left\{\left|\int f\, d\Phi - \int f\, d\Psi\right|; \operatorname{spt}(f) \subset D,\ f \geq 0,\ \operatorname{Lip}(f) \leq 1\right\}.$$

(We note that $F_D(\Phi) = F_D(\Phi, 0)$.)

(3) We shall also use the simplified notation $F_r = F_{B(0, r)}$.

(4) Whenever Φ_k is a sequence of measures over $\mathbf{R}^n$ and Φ measures $\mathbf{R}^n$, we shall say that the sequence Φ_k converges to Φ (and denote $\Phi = \lim_{k \to \infty} \Phi_k$) if

(i) Φ is locally finite,

(ii) $\limsup_{k \to \infty} F_D(\Phi_k) < \infty$ for every compact set $D \subset \mathbf{R}^n$, and

(iii) $\lim_{k \to \infty} F_D(\Phi_k, \Phi) = 0$ for every compact set $D \subset \mathbf{R}^n$.

(5) When $x \in \mathbf{R}^n$ and $r \in \mathbf{R} - \{0\}$, we define the maps $T_{x, r}\colon \mathbf{R}^n \mapsto \mathbf{R}^n$ by the formula $T_{x, r}(z) = (z - x)/r$. We note that

(i) $T_{x, r}[\Phi](B(0, s)) = \Phi(B(x, sr))$ for every $s > 0$,

(ii) $\int f(z)\, dT_{x, r}[\Phi](z) = \int f((z - x)/r)\, d\Phi(z)$ whenever at least one of these integrals is defined,

(iii) $F_{B(x, r)}(\Phi) = rF_1(T_{x, r}[\Phi])$, and

(iv) $F_{B(x, r)}(\Phi, \Psi) = rF_1(T_{x, r}[\Phi], T_{x, r}[\Psi])$.

(6) We also define $T_{\infty, r} = T_{0, 1/r}$.

1.10. Proposition. *Suppose that Φ and Ψ measure $\mathbf{R}^n$, D is a compact subset of $\mathbf{R}^n$, and $\Phi(D) + \Psi(D) < \infty$. Then*

(1) $F_D(\Phi, \Psi) \leq \max(F_D(\Phi), F_D(\Psi))$,

(2) $F_D(\Phi) \leq \frac{1}{2}\Phi(D)\operatorname{diam}(D)$ *and* $F_D(\Phi) \geq s\Phi(E)$ *whenever* $s > 0$ *and* $B(E, s) \subset D$;

(3) *If* $s > 0$, $E \subset \mathbf{R}^n$, *and* $B(E, s) \subset D$, *then*

$$\Phi(E) \leq \Psi(B(E, s)) + F_D(\Phi, \Psi)/s.$$

(4) *If* $0 < t \leq s \leq (1 + 1/2n)t$, $a > 0$, *and* $\Phi(B(z, t)) \leq \Psi(B(z, s)) + a$ *and* $\Psi(B(z, t)) \leq \Phi(B(z, s)) + a$ *whenever* $B(z, s) \subset D$, *then*

$$F_D(\Phi, \Psi) \leq \left[4s + \left((s/t)^n - 1\right)\operatorname{diam}(D)\right] \times [\Phi(D) + \Psi(D)] + at^{-n}(\operatorname{diam}(D))^{n+1},$$

and

(5) *If* $F_D(\Psi) > 0$, $\tau \in (0, 1)$, *and* $F_D(\Phi, \Psi) \leq \tau F_D(\Psi)$, *then* $F_D(\Phi) > 0$ *and* $F_D(\Phi/F_D(\Phi), \Psi/F_D(\Psi)) \leq 2\tau$.

Proof. (1) and (2) are obvious.

(3) Let $h(z) = \min(1, \operatorname{dist}(z, \mathbf{R}^n - B(E, s))/s)$. Then

$$\Phi(E) \leq \int h\, d\Phi \leq \int h\, d\Psi + (\operatorname{Lip}(h))F_D(\Phi, \Psi) \leq \Psi(B(E, s)) + F_D(\Phi, \Psi)/s.$$

(4) Suppose that f is a function on $\mathbf{R}^n$, $\operatorname{Lip}(f) \leq 1$, $\operatorname{spt}(f) \subset D$, and $f \geq 0$. Let $A = \{z \in D;\ B(z, s) \subset D\}$ and for every $t \leq r \leq s$ let $g_r(x) = (\alpha(n)r^n)^{-1}\int_{A \cap B(x, r)} f(z)\, d\mathscr{L}^n(z)$. We easily see that $|g_r(x) - f(x)| \leq 2s$ for every $x \in \mathbf{R}^n$. Hence

$$\begin{aligned}
\int f\, d\Phi &\leq 2s\Phi(D) + \int g_t(z)\, d\Phi(z) \\
&= 2s\Phi(D) + (\alpha(n)t^n)^{-1} \int_A f(z)\Phi(B(z, t))\, d\mathscr{L}^n(z) \\
&\leq 2s\Phi(D) + (\alpha(n)t^n)^{-1} \int_A f(z)\Psi(B(z, s))\, d\mathscr{L}^n(z) \\
&\quad + a(\alpha(n)t^n)^{-1}F_D(\mathscr{L}^n) \\
&\leq 2s\Phi(D) + (s/t)^n \int g_s(z)\, d\Psi(z) + at^{-n}(\operatorname{diam} D)^{n+1} \\
&\leq (s/t)^n \int f\, d\Psi + 2s\Phi(D) + 2s(s/t)^n\Psi(D) + at^{-n}(\operatorname{diam} D)^{n+1} \\
&\leq \int f\, d\Psi + \left((s/t)^n - 1\right)(\operatorname{diam} D)\Psi(D) + 2s\Phi(D) \\
&\quad + 4s\Psi(D) + at^{-n}(\operatorname{diam} D)^{n+1} \\
&\leq \int f\, d\Psi + \left[4s + \left((s/t)^n - 1\right)\operatorname{diam}(D)\right] \\
&\quad \times [\Phi(D) + \Psi(D)] + at^{-n}(\operatorname{diam} D)^{n+1}.
\end{aligned}$$

This and a similar estimate of $\int f\, d\Psi$ imply (4).

(5) $F_D(\Phi) > 0$ since $|F_D(\Phi) - F_D(\Psi)| < F_D(\Psi)$. The last statement follows from

$$F_D(\Phi/F_D(\Phi), \Psi/F_D(\Psi)) \le F_D(\Phi, \Psi)/F_D(\Psi) + |F_D(\Phi) - F_D(\Psi)|/F_D(\Psi) \le 2\tau.$$

1.11. PROPOSITION. *Let* Φ_k *be a sequence of measures over* $\mathbf{R}^n$ *such that* $\limsup_{k\to\infty} F_D(\Phi_k) < \infty$ *for every compact set* D *and let* Φ *measure* $\mathbf{R}^n$. *Then the following statements are equivalent.*

(1) $\lim_{k\to\infty}\Phi_k = \Phi$.

(2) $\lim_{k\to\infty}\int f\,d\Phi_k = \int f\,d\Phi$ *for every nonnegative Lipschitzian function with compact support.*

(3) $\lim_{k\to\infty}\int f\,d\Phi_k = \int f\,d\Phi$ *whenever the following condition holds: There is a function* $g\colon \mathbf{R}^n \mapsto [0, \infty]$ *such that*

(i) $\limsup_{k\to\infty}\int g\,d\Phi_k < \infty$,

(ii) *for every* $\varepsilon > 0$ *there is* $r > 0$ *such that* $|f(z)| \le \varepsilon|g(z)|$ *if* $|f(z)| > r$ *or* $\|z\| > r$, *and*

(iii) f *is continuous* Φ *almost everywhere.*

(4) $\Phi(D) \ge \limsup_{k\to\infty}\Phi_k(D)$ *for every compact set* $D \subset \mathbf{R}^n$ *and* $\Phi(G) \le \liminf_{k\to\infty}\Phi_k(F)$ *for every open set* $G \subset \mathbf{R}^n$.

(5) *There is a continuous function* $f\colon \mathbf{R}^n \mapsto (0, \infty)$ *such that* $\lim_{k\to\infty} f\Phi_k = f\Phi$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (4). If $\varepsilon > 0$ and $s > 0$ are such that $\Phi(B(D, s)) \le \Phi(D) + \varepsilon$, we let $f(z) = \min(1, \operatorname{dist}(z, \mathbf{R}^n - B(D, s))/s)$ and conclude

$$\Phi(D) + \varepsilon > \int f\,d\Phi = \lim_{k\to\infty}\int f\,d\Phi_k \ge \limsup_{k\to\infty}\Phi_k(D).$$

If $c < \Phi(G)$, $D \subset G$ is a compact set with $\Phi(D) > c$ and $s > 0$ is such that $B(D, s) \subset G$, we define f as above and conclude

$$c < \int f\,d\Phi = \lim_{k\to\infty}\int f\,d\Phi_k \le \liminf_{k\to\infty}\Phi_k(G).$$

(4) $\Rightarrow$ (3). We may assume that $f \ge 0$. From the continuity of f Φ almost everywhere we infer that

$$\int f\,d\Phi = \int_0^\infty \Phi(\operatorname{Int}\{z;\ f(z) \ge t\})\,dt = \int_0^\infty \Phi(\operatorname{Clos}\{z;\ f(z) \ge t\})\,dt.$$

Hence

$$\int f\,d\Phi \le \int_0^\infty \liminf_{k\to\infty} \Phi_k(\operatorname{Int}\{z;\ f(z) \ge t\})\,dt$$
$$\le \liminf_{k\to\infty} \int_0^\infty \Phi_k(\operatorname{Int}\{z;\ f(z) \ge t\})\,dt$$
$$\le \liminf_{k\to\infty} \int f\,d\Phi_k.$$

For every $\varepsilon > 0$ we find $r > 0$ with the property described in (ii) and we estimate

$$\limsup_{k\to\infty} \int f\,d\Phi_k \le \limsup_{k\to\infty} \int_{B(0,r)} \min(r, f(z))\,d\Phi_k(z)$$
$$+ \limsup_{k\to\infty} \varepsilon \int g\,d\Phi_k$$
$$\le \limsup_{k\to\infty} \int_0^r \Phi_k(\operatorname{Clos}\{z \in B(0,r);\ f(z) \ge t\})\,dt$$
$$+ \varepsilon \limsup_{k\to\infty} \int g\,d\Phi_k$$
$$\le \int_0^r \Phi(\operatorname{Clos}\{z \in B(0,r);\ f(z) \ge t\})\,dt$$
$$+ \varepsilon \limsup_{k\to\infty} \int g\,d\Phi_k$$
$$\le \int f\,d\Phi + \varepsilon \limsup_{k\to\infty} \int g\,d\Phi_k.$$

(2) $\Rightarrow$ (1). Let D be a compact subset of $\mathbf{R}^n$ and let

$$L = \{ f\colon \mathbf{R}^n \mapsto [0,\infty);\ \operatorname{spt}(f) \subset D \text{ and } \operatorname{Lip}(f) \le 1 \}.$$

For every $\varepsilon > 0$ there is a finite set $S \subset L$ such that for every $f \in L$ one may find $g \in L$ with $|f - g| < \varepsilon$. If f and g have this property, then

$$\left| \int f\,d\Phi_k - \int g\,d\Phi_k \right| \le \varepsilon \Phi_k(D) \le \varepsilon F_{B(D,1)}(\Phi_k)$$

and

$$\left| \int f\,d\Phi - \int g\,d\Phi \right| \le \varepsilon \Phi(D) \le \varepsilon F_{B(D,1)}(\Phi) = \varepsilon \lim_{k\to\infty} F_{B(D,1)}(\Phi_k).$$

Hence

$$\limsup_{k\to\infty} F_D(\Phi_k, \Phi) \le 2\varepsilon \limsup_{k\to\infty} F_{B(D,1)}(\Phi_k).$$

Since we have already proved that the statements (1)–(4) are equivalent, (1) $\Leftrightarrow$ (5) is obvious.

1.12. PROPOSITION. (1) *Let Φ_k be a sequence of measures over $\mathbf{R}^n$ such that* $\limsup_{k\to\infty}\Phi_k(B(0,r)) < \infty$ *for each* $r > 0$. *Then Φ_k has a convergent subsequence.*

(2) *The set of all locally finite measures with the metric*

$$\sum_{p=0}^{\infty} 2^{-p}\min\big(1, F_p(\Phi,\Psi)\big)$$

is a complete separable metric space. The notion of convergence in this space coincides with that described in 1.9(4).

Proof. (1) Let S be a countable family of continuous functions with compact supports satisfying:

$(*)$ Whenever $p = 1, 2, \dots, f$ is a continuous function with $\operatorname{spt}(f) \subset B(0,p)$, and $\varepsilon > 0$, then there is $g \in S$ such that $\operatorname{spt}(g) \subset B(0,p)$ and $|f-g| < \varepsilon$.

Using the usual diagonal procedure, we find a subsequence Φ_{k_j} such that $\lim_{j\to\infty}\int g\, d\Phi_{k_j}$ exists for every $g \in S$. From $(*)$ we infer that $\lambda(f) = \lim_{j\to\infty}\int f d\Phi_{k_j}$ is well defined for every continuous function with compact support. Hence the Riesz representation theorem ([11, 2.5.13]) provides us with a measure Φ over $\mathbf{R}^n$ such that $\lambda(f) = \int f d\Phi$. Clearly, 1.11 shows that $\Phi = \lim_{j\to\infty}\Phi_{k_j}$.

(2) follows easily from (1).

1.13. LEMMA. *Suppose that Φ_1 and Φ_2 measure $\mathbf{R}^n$, $q = 0, 1, \dots, n$, P and Q are the orthogonal projections of $\mathbf{R}^n = \mathbf{R}^q \times \mathbf{R}^{n-q}$ onto R^q and R^{n-q}, respectively, $a > 0$, $\int e^{a\|z\|} d(\Phi_1+\Phi_2)(z) < \infty$, and*

$$(*) \qquad \int e^{\langle z, Px\rangle}\langle z, Qx\rangle^p\, d\Phi_1(z) = \int e^{\langle z, Px\rangle}\langle z, Qx\rangle^p\, d\Phi_2(z)$$

for every $p = 0, 1, \dots$ and every $x \in \mathbf{R}^n$ with $\|x\| < a$. Then $\Phi_1 = \Phi_2$.

Proof. Dividing the p-th equality in $(*)$ by $p!$ and summing up, we get

$$\int e^{\langle z, x\rangle}\, d\Phi_1(z) = \int e^{\langle z, x\rangle}\, d\Phi_2(z)$$

for every $x \in \mathbf{R}^n$ with $\|x\| < a/2$. Hence

$$\int e^{s\langle z, y\rangle}\, d\Phi_1(z) = \int e^{\langle z, sy\rangle}\, d\Phi_1(z) = \int e^{\langle z, sy\rangle}\, d\Phi_2(z)$$
$$= \int e^{s\langle z, y\rangle}\, d\Phi_2(z)$$

whenever $y \in \mathbf{R}^n$, $s \in \mathbf{R}$, and $|s|\,\|y\| < a/2$. By analyticity, this equality extends to complex s such that $|\mathrm{Re}\, s|\,\|y\| < a/2$. Thus the Fourier transforms of Φ_1 and of Φ_2 coincide.

1.14. PROPOSITION. *Let Φ_k be measures over $\mathbf{R}^n$ such that $\limsup_{k\to\infty} \int e^{a\|z\|}\, d\Phi_k(z) < \infty$ for each $a > 0$. Then the sequence Φ_k converges to a measure Φ if and only if $\int e^{a\|z\|}\, d\Phi(z) < \infty$ for each $a > 0$, and*

$$\lim_{k\to\infty} \int \langle z, x\rangle^p \, d\Phi_k(z) = \int \langle z, x\rangle^p \, d\Phi(z)$$

for each $p = 0, 1, \ldots$ and for each $x \in \mathbf{R}^n$.

Proof. The implication $\Rightarrow$ follows from 1.11(3). To prove the converse, we note that 1.13 (with $q = 0$) and 1.11(3) imply that, if a subsequence of the sequence Φ_k has a limit, then this limit equals Φ. Thus the statement follows from 1.12(1).

2. Tangent measures

2.1. (1) A set $\mathfrak{M}$ of nonzero locally finite measures over $\mathbf{R}^n$ will be called a cone if $c\Psi \in \mathfrak{M}$ whenever $\Psi \in \mathfrak{M}$ and $c > 0$. It will be called a *d*-cone if it is a cone and if $T_{0,r}[\Psi] \in \mathfrak{M}$ whenever $\Psi \in \mathfrak{M}$ and $r > 0$.

(2) The basis of a *d*-cone $\mathfrak{M}$ is, by definition, the set $\{\Psi \in \mathfrak{M};\ F_1(\Psi) = 1\}$. We shall say that $\mathfrak{M}$ has a closed (compact, respectively) basis, if its basis is closed (compact, respectively) in the topology from 1.12(2). We also observe that a *d*-cone has a closed basis if and only if it is a relatively closed subset of the set of all nonzero locally finite measures over $\mathbf{R}^n$.

(3) Whenever $\mathfrak{M}$ is a nonempty *d*-cone of measures over $\mathbf{R}^n$, Φ measures $\mathbf{R}^n$, $s > 0$, and $0 < F_s(\Phi) < \infty$, we define

$$d_s(\Phi, \mathfrak{M}) = \inf\{F_s(\Phi/F_s(\Phi), \Psi);\ \Psi \in \mathfrak{M} \text{ and } F_s(\Psi) = 1\}.$$

We also define

$$d_s(\Phi, \mathfrak{M}) = 1 \quad \text{if } F_s(\Phi) = 0 \text{ or } F_s(\Phi) = \infty.$$

We easily see that

(4) $d_s(\Phi, \mathfrak{M}) \le 1$ (1.10(1)),

(5) $d_s(\Phi, \mathfrak{M}) = d_1(T_{0,s}[\Phi], \mathfrak{M})$ (1.9(5; iii, iv)), and

(6) if $\Phi = \lim_{k\to\infty} \Phi_k$ and $F_s(\Phi) > 0$, then $d_s(\Phi, \mathfrak{M}) = \lim_{k\to\infty} d_s(\Phi_k, \mathfrak{M})$.

2.2. PROPOSITION. *If a d-cone $\mathfrak{M}$ of measures over $\mathbf{R}^n$ has a closed basis, then the following statements are equivalent.*

(1) *$\mathfrak{M}$ has a compact basis.*

(2) *Whenever $\Psi_k \in \mathfrak{M}$ and $\lim_{k\to\infty} F_1(\Psi_k) = 0$ then $\lim_{k\to\infty}\Psi_k = 0$.*

(3) *There are $q \in (0, \infty)$ and bounded sets $C, D \subset \mathbf{R}^n$ such that*

$$0 \in \operatorname{Int}(C) \subset \operatorname{Clos}(C) \subset \operatorname{Int}(D) \quad \textit{and} \quad \Psi(D) \leq q\Psi(C) \quad \textit{for every } \Psi \in \mathfrak{M}.$$

(4) *There is $q \in (0, \infty)$ such that $\Psi(B(0,2r)) \leq q\Psi(B(0,r))$ for every $r > 0$ and every $\Psi \in \mathfrak{M}$.*

(5) *For every $\lambda > 1$ there is $\tau > 1$ such that $F_{\tau r}(\Psi) \leq \lambda F_r(\Psi)$ for every $\Psi \in \mathfrak{M}$ and every $r > 0$.*

Moreover, these statements imply

(6) *0 belongs to the support of every $\Psi \in \mathfrak{M}$, and*

(7) *the integral $\int e^{-a\|z\|}\, d\Psi(z)$ converges for every $a > 0$ and every $\Psi \in \mathfrak{M}$.*

Proof. (1) ⇒ (2). Let $\Psi_k \in \mathfrak{M}$, $\lim_{k\to\infty} F_1(\Psi_k) = 0$, $t > 1$, $\varepsilon > 0$, and $F_t(\Psi_k) > \varepsilon$ $(k = 1, 2, \ldots)$. Let $\tilde{\Psi}_k = T_{0, r_k}[\Psi_k]$, where

$$r_k = \sup\{r \in [1, t];\ F_r(\Psi_k) \leq F_1(\Psi_k) + 1/k\}.$$

Then $F_1(\tilde{\Psi}_k) > 0$ and $\lim_{k\to\infty} F_{t/r_k}(\tilde{\Psi}_k)/F_1(\tilde{\Psi}_k) = \infty$. Hence the sequence $\tilde{\Psi}_k/F_1(\tilde{\Psi}_k)$ has no convergent subsequence.

(2) ⇒ (3). If $\Psi_k \in \mathfrak{M}$ and $\Psi_k(B(0,2)) > k\Psi_k(B(0,1))$, then

$$\lim_{k\to\infty} F_1(\Psi_k/\Psi_k(B(0,2))) = 0 \quad \text{and} \quad F_3(\Psi_k/\Psi_k(B(0,2))) \geq 1.$$

Since this contradicts (2), (3) holds with $C = B(0,1)$ and $D = B(0,2)$.

(3) ⇒ (4). Let $\lambda > 1$ be such that $\lambda C \subset D$ and let $0 < t < s < \infty$ be such that $B(0,t) \subset C \subset B(0,s)$. We also note that

$$\Psi(\lambda r C) = T_{0,r}[\Psi](\lambda C) \leq q T_{0,r}[\Psi](C) = q\Psi(rC)$$

for every $r > 0$ and every $\Psi \in \mathfrak{M}$. Hence, if $p \geq 1$ is an integer such that $\lambda^p > 2s/t$, then

$$\Psi(B(0,2r)) \leq \Psi(2rC/t) \leq q^p \Psi(2rC/\lambda^p t) \leq q^p \Psi(B(0,r))$$

for every $r > 0$ and every $\Psi \in \mathfrak{M}$.

(4) ⇒ (1). This implication follows easily from 1.12(1).

(1) ⇒ (5). Suppose that $\lambda > 1$, $\tau_k \searrow 1$, $r_k > 0$, $\Psi_k \in \mathfrak{M}$, and $F_{\tau_k r_k}(\Psi_k) > \lambda F_{r_k}(\Psi_k)$. Let Ψ be the limit of some subsequence of the sequence $T_{0,\tau_k r_k}[\Psi_k]/F_1(T_{0,\tau_k r_k}[\Psi_k])$. We easily see that $1 = F_1(\Psi) \geq \lambda F_r(\Psi)$ for every

$r \in (0,1)$. But this is impossible, since the monotone convergence theorem implies that $\lim_{r \nearrow 1} F_r(\Psi) = F_1(\Psi)$.

The implications (5) $\Rightarrow$ (3), (4) $\Rightarrow$ (6), and (4) $\Rightarrow$ (7) are obvious.

2.3. (1) Suppose that Φ measures $\mathbf{R}^n$ and $x \in \mathbf{R}^n \cup \{\infty\}$. A nonzero locally finite measure Ψ over $\mathbf{R}^n$ is said to be a tangent measure of Φ at x if there are sequences $r_k \searrow 0$ and $c_k > 0$ such that $\Psi = \lim_{k\to\infty} c_k T_{x, r_k}[\Phi]$.

(2) The set of all tangent measures of Φ at x will be denoted by $\operatorname{Tan}(\Phi, x)$.

(3) We observe that $\operatorname{Tan}(\Phi, x)$ is a d-cone with a closed basis.

(4) If Φ is an almost finite measure over $\mathbf{R}^n$, we infer from 1.7 that

(i) whenever $E \subset \mathbf{R}^n$ then $\operatorname{Tan}(\Phi \llcorner E, x) = \operatorname{Tan}(\Phi, x)$ for Φ almost every $x \in E$, and

(ii) whenever $f \geq 0$ is a Φ measurable function such that the measure $\tilde{\Phi} = f\Phi$ is almost finite, then $\operatorname{Tan}(\tilde{\Phi}, x) = \operatorname{Tan}(\Phi, x)$ for $\tilde{\Phi}$ almost every x.

2.4. Lemma. *If Φ measures $\mathbf{R}^n$, $t > 1$, and E is an open subset of $\mathbf{R}^n$ with $\Phi(E) < \infty$, then*

$$\lim_{c\to\infty} \limsup_{r\searrow 0} \Phi\{x \in E;\ \Phi(B(x, tr)) \geq c\Phi(B(x, r))\} = 0.$$

Proof. For each $\varepsilon > 0$ we find a compact set $D \subset E$ such that $\Phi(E - D) < \varepsilon/2$. If $c > (2(t+1))^{n+1}\Phi(E)/\varepsilon$,

$$r < \operatorname{dist}(D, \mathbf{R}^n - E)/4t, \quad \text{and} \quad A = \{x \in E;\ \Phi(B(x, tr)) \geq c\Phi(B(x, r))\},$$

then $\Phi(B(z, (t+1)r)) \geq c\Phi(B(z, r/2))$ whenever $B(z, r/2) \cap A \neq \varnothing$. Hence

$$\begin{aligned}
&\Phi(A \cap D)\\
&= \alpha(n)^{-1}(r/2)^{-n} \int_{B(A\cap D, r/2)} \Phi(A \cap D \cap B(z, r/2))\, d\mathscr{L}^n(z)\\
&\leq (2(t+1))^n c^{-1} \alpha(n)^{-1}((t+1)r)^{-n} \int \Phi(E \cap B(z, (t+1)r))\, d\mathscr{L}^n(z)\\
&\leq (2(t+1))^n c^{-1} \Phi(E) < \varepsilon/2.
\end{aligned}$$

Thus $\Phi(A) < \varepsilon$, which proves the statement of the lemma.

2.5. Theorem. *If Φ is an almost finite measure over $\mathbf{R}^n$ then $\operatorname{Tan}(\Phi, x) \neq \varnothing$ for Φ almost every $x \in \mathbf{R}^n$.*

Proof. Whenever $\varepsilon > 0$ and $E \subset \mathbf{R}^n$ is an open set with $\Phi(E) < \infty$, we use 2.4 to find $c_k > 0$ and $s_k > 0$ such that

$$\Phi\{x \in E;\ \Phi(B(x, kr)) \geq c_k \Phi(B(x, r))\} < \varepsilon/2^k$$

for every $0 < r < s_k$ $(k = 1, 2, \dots)$. Let

$$A_r = \{x \in E;\ \Phi(B(x, kr)) \geq c_k \Phi(B(x, r)) \quad \text{for some } k = 1, 2, \dots \text{ such that } kr < s_k\}, \quad \text{and}$$

$$A = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_{1/i}.$$

Since $\Phi(A_r) < \varepsilon$, $\Phi(A) \leq \varepsilon$. Finally, we observe that for every $x \in E - A$ there is a sequence $r_j \searrow 0$ such that

$$\limsup_{j \to \infty} \Phi(B(x, kr_j))/\Phi(B(x, r_j)) \leq c_k$$

for each $k = 1, 2, \dots$. Hence we may use 1.12(1) to infer that some subsequence of the sequence $T_{x, r_j}[\Phi]/\Phi(B(x, r_j))$ converges to some (obviously nonzero) locally finite measure.

2.6. THEOREM. *Let Φ measure $\mathbf{R}^n$ and $x \in \mathbf{R}^n \cup \{\infty\}$.*

(1) *If $\mathfrak{M}$ is a d-cone of measures over $\mathbf{R}^n$ with a compact basis, $\mathfrak{M} \cap \operatorname{Tan}(\Phi, x) \neq \varnothing$, and $0 < \varepsilon < \limsup_{r \searrow 0} d_1(T_{x,r}[\Phi], \mathfrak{M})$, then there is $\Psi \in \operatorname{Tan}(\Phi, x)$ such that $d_1(\Psi, \mathfrak{M}) = \varepsilon$ and $d_r(\Psi, \mathfrak{M}) \leq \varepsilon$ for every $r \geq 1$.*

(2) *If the basis of* $\operatorname{Tan}(\Phi, x)$ *is compact then* $\operatorname{Tan}(\Phi, x)$ *is a connected set.*

Proof. (1) Let $\tilde{\Psi} \in \mathfrak{M} \cap \operatorname{Tan}(\Phi, x)$, $s_k \searrow 0$ and $c_k > 0$ be such that $\tilde{\Psi} = \lim_{k \to \infty} c_k T_{x, s_k}[\Phi]$. For each $k = 1, 2, \dots$ we find the smallest $r_k \in [0, s_k]$ such that $d_1(T_{x,r}[\Phi], \mathfrak{M}) < \varepsilon$ for every $r_k < r \leq s_k$. Since $\limsup_{r \searrow 0} d_1(T_{x,r}[\Phi], \mathfrak{M}) > \varepsilon$, we have $r_k > 0$ and, denoting $\Phi_k = T_{x, r_k}[\Phi]$, $d_1(\Phi_k, \mathfrak{M}) \geq \varepsilon$ if k is sufficiently large.

If there is a sequence $k_1 < k_2 < \cdots$ such that $\lim_{j \to \infty} r_{k_j}/s_{k_j} = t > 0$, we infer that $d_1(T_{0,t}[\tilde{\Psi}], \mathfrak{M}) \geq \varepsilon$. Since $T_{0,t}[\tilde{\Psi}] \in \mathfrak{M}$, this is impossible. Hence $\lim_{k \to \infty} r_k/s_k = 0$, which implies that $\lim_{k \to \infty} d_1(\Phi_k, \mathfrak{M}) = \varepsilon$ and $\limsup_{k \to \infty} d_r(\Phi_k, \mathfrak{M}) \leq \varepsilon$ for every $r \geq 1$. We also note that for every $r > 0$, $0 < F_r(\Phi_k) < \infty$ if k is sufficiently large.

Since $\varepsilon < 1$ (2.1(4)), $\lambda = 2/(1 + \varepsilon) > 1$ and there is $\tau > 1$ with $F_{\tau s}(\Psi) \leq \lambda F_s(\Psi)$ for every $\Psi \in \mathfrak{M}$ and every $s > 0$ (2.2(5)). Whenever $r \geq 1$ and k is sufficiently large, there is $\Psi \in \mathfrak{M}$ such that $F_{\tau r}(\Psi) = 1$ and $F_{\tau r}(\Phi_k/F_{\tau r}(\Phi_k), \Psi) < \varepsilon$. Hence $F_r(\Phi_k)/F_{\tau r}(\Phi_k) \geq F_r(\Psi) - \varepsilon \geq (1 - \varepsilon)/2$. Thus $\limsup_{k \to \infty} F_{\tau^p}(\Phi_k)/F_1(\Phi_k) \leq ((1 - \varepsilon)/2)^{-p}$ for each $p = 1, 2, \dots$ and 1.12(1) implies that the sequence $\Phi_k/F_1(\Phi_k)$ has a convergent subsequence. Clearly, this subsequence converges to a measure having the desired properties.

(2) We easily see that each component of $\operatorname{Tan}(\Phi, x)$ is a d-cone with a compact basis. Hence the statement follows from (1).

2.7. Corollary. *Suppose that* Φ *measures* $\mathbf{R}^n$, $x \in \mathbf{R}^n$, *and* $\operatorname{Tan}(\Phi, x) \neq \varnothing$. *Then* $x \in \operatorname{spt} \Phi$, $\Phi(B(x, s)) < \infty$ *for some* $s > 0$, *and the following statements are equivalent.*

(1) $\limsup_{r \searrow 0} \Phi(B(x, 2r))/\Phi(B(x, r)) < \infty$.

(2) *There are bounded sets* $C, D \subset \mathbf{R}^n$ *such that* $0 \in \operatorname{Int}(C) \subset \operatorname{Clos}(C) \subset \operatorname{Int}(D)$ *and* $\limsup_{r \searrow 0} \Phi(x + rD)/\Phi(x + rC) < \infty$.

(3) $\operatorname{Tan}(\Phi, x)$ *has a compact basis.*

Proof. One easily sees that $x \in \operatorname{spt} \Phi$ and $\Phi(B(x, s)) < \infty$ for some $s > 0$. The implication (1) $\Rightarrow$ (2) is obvious and (2) $\Rightarrow$ (3) follows from 2.2(3).

(3) $\Rightarrow$ (1). Let $q > 0$ be such that $\Psi(B(0, 2r)) \leq q\Psi(B(0, r))$ for every $\Psi \in \operatorname{Tan}(\Phi, x)$ and every $r > 0$ (2.2(4)). We prove that

$$\limsup_{r \searrow 0} F_1(T_{x,2r}[\Phi])/F_1(T_{x,r}[\Phi]) \leq 2q.$$

Indeed, if $r_k \searrow 0$ and $F_1(T_{x,2r_k}[\Phi]) \geq 2qF_1(T_{x,r_k}[\Phi])$, then

$$\begin{aligned} F_1\left(T_{x,2r_k}[\Phi]/F_1\left(T_{x,2r_k}[\Phi]\right), \Psi\right) &\geq F_{1/2}(\Psi) \\ &\quad - F_{1/2}\left(T_{x,2r_k}[\Phi]\right)/F_1\left(T_{x,2r_k}[\Phi]\right) \\ &\geq 1/2q - 1/4q = 1/4q \end{aligned}$$

whenever $\Psi \in \operatorname{Tan}(\Phi, x)$ and $F_1(\Psi) = 1$. Hence 2.6 with $\mathfrak{M} = \operatorname{Tan}(\Phi, x)$ provides us with a measure $\Psi \in \operatorname{Tan}(\Phi, x) - \operatorname{Tan}(\Phi, x)$, which is impossible.

Now, we easily estimate

$$\limsup_{r \searrow 0} \Phi(B(x, 2r))/\Phi(B(x, r)) \leq \limsup_{r \searrow 0} F_1(T_{x,4r}[\Phi])/F_1(T_{x,r}[\Phi]) \leq (2q)^2.$$

2.8. Corollary. *Suppose that* Φ *measures* $\mathbf{R}^n$ *and* $\operatorname{Tan}(\Phi, \infty) \neq \varnothing$. *Then* Φ *is nonzero and locally finite and the following statements are equivalent.*

(1) $\limsup_{r \to \infty} \Phi(B(0, 2r))/\Phi(B(0, r)) < \infty$.

(2) *There are bounded sets* $C, D \subset \mathbf{R}^n$ *such that* $0 \in \operatorname{Int}(C) \subset \operatorname{Clos}(C) \subset \operatorname{Int}(D)$ *such that* $\limsup_{r \to \infty} \Phi(rD)/\Phi(rC) < \infty$.

(3) $\operatorname{Tan}(\Phi, \infty)$ *has a compact basis.*

The proof is similar to that of 2.7.

2.9. Corollary. *Suppose that* Φ *measures* $\mathbf{R}^n$ *and* $x \in \mathbf{R}^n \cup \{\infty\}$. *Then the following statements are equivalent.*

(1) *There is a measure* Ψ *such that* $\operatorname{Tan}(\Phi, x) = \{c\Psi;\ c > 0\}$.

(2) *There is a map* $r \mapsto c(r)$ $(r \in (0, \infty))$ *such that* $\lim_{r \searrow 0} c(r) T_{x,r}[\Phi]$ *exists and is a nonzero measure.*

(3) $\lim_{r \searrow 0} T_{x,r}[\Phi]/F_1(T_{x,r}[\Phi])$ *exists.*

Proof. (1) $\Rightarrow$ (3). For every sequence $r_k \searrow 0$ we use 1.12(1), 2.7, and 2.8 to find $k_1 < k_2 < \cdots$ such that the sequence $T_{x, r_{k_j}}[\Phi]/F_1(T_{x, r_{k_j}}[\Phi])$ converges. Since its limit belongs to $\operatorname{Tan}(\Phi, x)$, it equals $\Psi/F_1(\Psi)$. Hence

$$\lim_{r \searrow 0} T_{x,r}[\Phi]/F_1(T_{x,r}[\Phi]) = \Psi/F_1(\Psi).$$

The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.

2.10. THEOREM. *Let Ψ be a nonzero locally finite measure over $\mathbf{R}^n$. Then the following statements are equivalent.*

(1) *There are a measure Φ over $\mathbf{R}^n$ and $x \in \mathbf{R}^n \cup \{\infty\}$ such that* $\operatorname{Tan}(\Phi, x) = \{c\Psi;\ c > 0\}$.

(2) $\operatorname{Tan}(\Psi, 0) = \{c\Psi;\ c > 0\}$.

(3) $\operatorname{Tan}(\Psi, \infty) = \{c\Psi;\ c > 0\}$.

(4) *For every $t > 0$ there is $c(t) > 0$ such that $T_{0,t}[\Psi] = c(t)\Psi$.*

(5) *There is $\alpha \geq 0$ such that $T_{0,t}[\Psi] = t^\alpha \Psi$ for every $t > 0$.*

(6) *The set $\{c\Psi;\ c > 0\}$ is a d-cone.*

Proof. Since the implications (1) $\Rightarrow$ (6) $\Rightarrow$ (4), (5) $\Rightarrow$ (2) $\Rightarrow$ (1), and (5) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious, it suffices to prove (4) $\Rightarrow$ (5). From 1.9(5; iii) we see that $F_t(\Psi)/t = c(t)F_1(\Psi)$. Hence $F_1(\Psi) > 0$ and $c(t)$ is a continuous function on $(0, \infty)$. Using 1.9(5; iii) once more, we infer that $c(ts) = c(t)c(s)$ for $t, s > 0$. Thus there is $\alpha \in R$ such that $c(t) = t^\alpha$ for every $t > 0$. Moreover, $\alpha \geq 0$ since $t^\alpha \Psi(B(0,1)) = \Psi(B(0,t))$ is a nondecreasing function of t.

2.11. PROPOSITION. *Suppose that Ψ is a nonzero locally finite measure over $\mathbf{R}^n$, $\alpha \geq 0$, and $T_{0,t}[\Psi] = t^\alpha \Psi$ for every $t > 0$.*

(1) *If $\alpha = 0$, there is $\sigma > 0$ such that $\Psi = \sigma \mathscr{H}^0 \llcorner \{0\}$.*

(2) *If $\alpha > 0$, $k = 0, 1, \ldots$, $u \in \mathbf{R}^n$, and f is a Borel function on $(0, \infty)$ such that $|f(t)| \leq Ke^{-a|t|}$ for some $K > 0$ and $a > 0$, then*

$$\int f(\|z\|)\langle z, u\rangle^k \, d\Psi(z) = (k + \alpha)\int_0^\infty f(t) t^{k+\alpha-1}\, dt \int_{B(0,1)} \langle z, u\rangle^k \, d\Psi(z).$$

(3) *If Φ measures $\mathbf{R}^n$,* $\operatorname{Tan}(\Phi, \infty) = \{c\Psi;\ c > 0\}$ *and there are $\beta \in [0, \infty)$ and $\gamma > 0$ such that*

$$s\int e^{-s\|z\|^2}\|z\|^2 \, d\Phi(z) \Big/ \int e^{-s\|z\|^2}\, d\Phi(z) = \beta + o(s^\gamma) \quad \text{as } s \searrow 0,$$

then there is $C \in (0, \infty)$ such that

$$\lim_{r \to \infty} r^{-\alpha} T_{0,r}[\Phi] = C\Psi.$$

(4) *If* Φ *measures* $\mathbf{R}^n$, $x \in \operatorname{spt}\Phi$, $\sup\{\Phi(B(x,2r))/\Phi(B(x,r)); r>0\} < \infty$, $\operatorname{Tan}(\Phi, x) = \{c\Psi;\ c>0\}$, *and there are* $\beta \in [0,\infty)$ *and* $\gamma > 0$ *such that*

$$s\int e^{-s\|z-x\|^2}\|z-x\|^2\,d\Phi(z)\Big/\int e^{-s\|z-x\|^2}\,d\Phi(z) = \beta + o(s^{-\gamma})$$

as $s \to \infty$, *then there is* $C \in (0,\infty)$ *such that*

$$\lim_{r\searrow 0} r^{-\alpha}T_{x,r}[\Phi] = C\Psi.$$

Proof. (1) is obvious.

(2) If f is the characteristic function of $[0,r]$, we compute

$$\begin{aligned}\int f(\|z\|)\langle z,u\rangle^k\,d\Psi(z) &= \int_{B(0,r)}\langle z,u\rangle^k\,d\Psi(z)\\ &= r^k\int_{B(0,1)}\langle z,u\rangle^k\,dT_{0,r}[\Psi](z)\\ &= r^{k+\alpha}\int_{B(0,1)}\langle z,u\rangle^k\,d\Psi(z)\\ &= (k+\alpha)\int_0^\infty f(t)t^{k+\alpha-1}\,dt\int_{B(0,1)}\langle z,u\rangle^k\,d\Psi(z).\end{aligned}$$

Hence the statement holds if f is a linear combination of such functions. Since $\int e^{-a\|z\|}|\langle z,u\rangle|^k\,d\Psi(z) < \infty$ (2.2(7)), the general statement follows by approximation.

(3) Let $g(s) = \ln\int e^{-s\|z\|^2}\,d\Phi(z)$. From 2.8(1) we easily see that $g(s)$ is well defined and differentiable for $s>0$. Our condition says that $-sg'(s) = \beta + o(s^{\gamma})$ as $s \searrow 0$. Hence there is $a>0$ such that

$$(*)\qquad g(s) = a - \beta\ln s + o(1)\quad \text{as } s\searrow 0.$$

Let $\Phi_r = T_{\infty,r}[\Phi]/F_1(T_{\infty,r}[\Phi])$. From 2.8(1) we infer that $\limsup_{r\searrow 0}\int e^{-c\|z\|^2}\,d\Phi_r(z) < \infty$ for every $c>0$. Hence 1.11(3) implies that

$$(**)\qquad \lim_{r\searrow 0} e^{g(s^2r^2)}/F_1(T_{\infty,r}[\Phi]) = \lim_{r\searrow 0}\int e^{-s^2\|z\|^2}\,d\Phi_r(z)$$

$$= \int e^{-s^2\|z\|^2}\,d\Psi(z)/F_1(\Psi) = C_1 s^{-\alpha}$$

$$\left(C_1 = \int e^{-\|z\|^2}\,d\Psi(z)/F_1(\Psi)\right)$$

whenever $s > 0$. Consequently,

$$\lim_{r \searrow 0} e^{g(4r^2)}/e^{g(r^2)} = 2^{-\alpha}.$$

On the other hand, $(*)$ implies that

$$\lim_{r \searrow 0} e^{g(4r^2)}/e^{g(r^2)} = 4^{-\beta}.$$

Hence $\beta = \alpha/2$ and, using this in $(**)$ with $s = 1$, we obtain $\lim_{r \searrow 0} e^a r^{-\alpha}/F_1(T_{\infty,r}[\Phi]) = C_1$. Thus

$$\lim_{r \searrow 0} r^\alpha T_{\infty,r}[\Phi] = e^a \Psi / C_1 F_1(\Psi),$$

which is the statement of (3).

The proof of (4) is similar to that of (3).

2.12. Theorem. *Let Φ measure $\mathbf{R}^n$. Then Φ almost every $x \in \mathbf{R}^n$ has the following property. Whenever $\Psi \in \operatorname{Tan}(\Phi, x)$ and $u \in \operatorname{spt} \Psi$ then*

$$T_{u,1}[\Psi] \in \operatorname{Tan}(\Phi, x).$$

Proof. For each $p = 1, 2, \ldots$ and each $q = 1, 2, \ldots$ let $E_{p,q}$ be the set of all $x \in \mathbf{R}^n$ for which there are $\Psi_x \in \operatorname{Tan}(\Phi, x)$ and $u_x \in \operatorname{spt} \Psi_x$ such that

$$F_p\big(T_{u_x,1}[\Psi_x], cT_{x,r}[\Phi]\big) > 1/p$$

for every $c > 0$ and every $0 < r < 1/q$.

If $\Phi(E_{p,q}) > 0$, we use the separability of the space of all locally finite measures to find a set $E \subset E_{p,q}$ such that

$$\Phi(E) > 0 \quad \text{and} \quad F_p\big(T_{u_x,1}[\Psi_x], T_{u_y,1}[\Psi_y]\big) < 1/2p$$

for every $x, y \in E$. Since $\operatorname{Tan}(\Phi, z) \neq \varnothing$ for each $z \in E$, each point of E is contained in an open set with finite Φ measure and we may use 1.7 to find $x \in E$ which is a Φ density point of E. Let $c_k > 0$ and $r_k \searrow 0$ be such that $\Psi_x = \lim_{k\to\infty} c_k T_{x,r_k}[\Phi]$, and let $x_k \in E$ be such that

$$\|x_k - (x + r_k u_x)\| < \operatorname{dist}(x + r_k u_x, E) + r_k/k.$$

We prove that

$$(*) \qquad \lim_{k\to\infty} \operatorname{dist}(x + r_k u_x, E)/r_k = 0.$$

Assuming that that is not the case, we find $\delta \in (0, \|u_x\|)$ such that $\operatorname{dist}(x + r_k u_x, E) > \delta r_k$ for infinitely many values of k. Since x is a Φ density point of E,

$$\begin{aligned} 1 &= \lim_{k\to\infty} \Phi(E \cap B(x, 2r_k\|u_x\|))/\Phi(B(x, 2r_k\|u_x\|)) \\ &\le 1 - \liminf_{k\to\infty} \Phi(B(x + r_k u_x, \delta r_k))/\Phi(B(x, 2r_k\|u_x\|)) \\ &\le 1 - \Psi_x(B^0(u_x, \delta))/\Psi(B(0, 2\|u_x\|)) < 1. \end{aligned}$$

Thus $(*)$ holds and, consequently, $\lim_{k\to\infty}\|x_k - (x + r_k u_x)\|/r_k = 0$. Hence

$$\begin{aligned} \lim_{k\to\infty} c_k T_{x_k, r_k}[\Phi] &= \lim_{k\to\infty} T_{(x_k - x)/r_k, 1}\left[c_k T_{x, r_k}[\Phi]\right] \\ &= T_{u_x, 1}[\Psi_x]. \end{aligned}$$

Therefore there is k such that $r_k < 1/q$ and

$$F_p\left(T_{u_x, 1}[\Psi_x], c_k T_{x_k, r_k}[\Phi]\right) < 1/2p.$$

Consequently,

$$\begin{aligned} 1/p &< F_p\left(T_{u_{x_k}, 1}[\Psi_{x_k}], c_k T_{x_k, r_k}[\Phi]\right) \\ &\le F_p\left(T_{u_{x_k}, 1}[\Psi_{x_k}], T_{u_x, 1}[\Psi_x]\right) + F_p\left(T_{u_x, 1}[\Psi_x], c_k T_{x_k, r_k}[\Phi]\right) \\ &< 1/p. \end{aligned}$$

Hence $\Phi(E_{p,q}) = 0$ for each p and q, which implies 2.12.

2.13. (1) Whenever $\mathfrak{M}$ is a d-cone of measures over $\mathbf{R}^n$ and $\sigma \ge 0$, we let $\mathfrak{M}[\sigma]$ denote the set of all nonzero locally finite measures Ψ over $\mathbf{R}^n$ such that $d_1(T_{u,r}[\Psi], \mathfrak{M}) \le \sigma$ for every $u \in \{0\} \cup \operatorname{spt}\Psi$ and every $r > 0$. Clearly $\mathfrak{M}[\sigma]$ is a d-cone with a closed basis.

We prove the following facts.

(2) If $\mathfrak{M}$ has a compact basis and $\sigma \in [0, 1)$ then

(i) the set of all locally finite measures Ψ over $\mathbf{R}^n$ such that $d_r(\Psi, \mathfrak{M}) \le \sigma$ for each $r > 0$ is a d-cone with a compact basis, and

(ii) $\mathfrak{M}[\sigma]$ has a compact basis.

(3) If $\mathfrak{M}$ has a closed basis, Ψ_k and Ψ are nonzero measures over $\mathbf{R}^n$, $\lim_{k\to\infty}\Psi_k = \Psi$ and $\lim_{k\to\infty} F_k(\Psi_k) d_k(\Psi_k, \mathfrak{M}) = 0$, then $\Psi \in \mathfrak{M}$.

(4) If $\mathfrak{M}$ has a closed basis then

$$\bigcap_{\sigma>0} \mathfrak{M}[\sigma] = \mathfrak{M}[0] = \{\Psi;\ T_{u,1}[\Psi] \in \mathfrak{M} \quad \text{for every } u \in \{0\} \cup \operatorname{spt}\Psi\} \subset \mathfrak{M}.$$

(5) If Φ measures $\mathbf{R}^n$ then $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}[0]$ for Φ almost every $x \in \mathbf{R}^n$ for which $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}$.

Proof. (2) According to 2.2(5) there is $\tau > 1$ such that

$$F_\tau(\Psi) \le 4F_1(\Psi)/(3 + \sigma) \quad \text{for every} \quad \Psi \in \mathfrak{M}.$$

Whenever $d_\tau(\tilde{\Psi}, \mathfrak{M}) < \sigma$ and $F_\tau(\tilde{\Psi}) = 1$, we find $\Psi \in \mathfrak{M}$ such that $F_\tau(\Psi) = 1$ and $F_\tau(\Psi, \tilde{\Psi}) < (1 + \sigma)/2$. Then

$$F_1(\tilde{\Psi}) \geq F_1(\Psi) - (1 + \sigma)/2 \geq (1 - \sigma)/4.$$

Hence $F_\tau(\tilde{\Psi}) \leq 4F_1(\tilde{\Psi})/(1 - \sigma)$, which easily implies (i). The statement (ii) follows from (i).

(3) Let $\Phi_k \in \mathfrak{M}$ be such that $F_k(\Phi_k) = 1$ and

$$\lim_{k \to \infty} F_k(\Psi_k) F_k(\Psi_k / F_k(\Psi_k), \Phi_k) = 0.$$

Then $\lim_{k \to \infty} F_k(\Psi_k, F_k(\Psi_k)\Phi_k) = 0$ and, consequently,

$$\Psi = \lim_{k \to \infty} F_k(\Psi_k)\Phi_k.$$

(4) This statement follows easily from (3).

(5) It suffices to use (4) and (2.12).

3. Uniformly distributed measures over $\mathbf{R}^n$

3.1. (1) A measure Φ over $\mathbf{R}^n$ is said to be uniformly distributed if $\Phi(B(x, r)) = \Phi(B(y, r)) < \infty$ whenever $x, y \in \operatorname{spt} \Phi$ and $r > 0$.

(2) The set of all uniformly distributed measures Φ over $\mathbf{R}^n$ with $0 \in \operatorname{spt} \Phi$ will be denoted by $\mathfrak{U}(n)$.

(3) If $\Phi \in \mathfrak{U}(n)$, we shall denote $I(s) = \int e^{-s\|z\|^2}\, d\Phi(z)$.

(4) We note that $cT_{x,r}[\Phi]$ is uniformly distributed whenever Φ is uniformly distributed, $x \in \mathbf{R}^n$, $r \neq 0$, and $c > 0$.

3.2. LEMMA. *Let $\Phi \in \mathfrak{U}(n)$. Then*

(1) $\Phi(B(x, r)) \leq (1 + 2r/s)^n \Phi(B(0, s))$ *whenever* $0 < s \leq r < \infty$,

(2) $0 < I(s) < \infty$ *for every* $s > 0$,

(3) $I(s/4) \leq 5^n I(s)$ *for every* $s > 0$,

(4) $\int e^{-s\|z\|^2} \|z\|^k\, d\Phi(z) \leq 5^n k^{k/2} s^{-k/2} I(s)$ *for every* $s > 0$ *and every* $k > 0$, *and*

(5) $\sum_{j=1}^{\infty} (s^j/j!) \int e^{-s\|z\|^2} (2^j \langle z, x \rangle^j - \|x\|^{2j})\, d\Phi(z) = 0$ *for every* $s > 0$ *and every* $x \in \operatorname{spt} \Phi$.

Proof. (1)

$$\begin{aligned}\Phi(B(x, r)) &\leq \left(\alpha(n)(s/2)^n\right)^{-1} \int_{B(x, r+s/2)} \Phi(B(z, s/2))\, d\mathscr{L}^n(z) \\ &\leq \left(\alpha(n)(s/2)^n\right)^{-1} \int_{B(x, r+s/2)} \Phi(B(0, s))\, d\mathscr{L}^n(z) \\ &= (1 + 2r/s)^n \Phi(B(0, s)).\end{aligned}$$

The statements (2) and (3) follow easily from (1).

(4) Since the function $z \mapsto -\frac{3}{4}s\|z\|^2 + k\ln\|z\|$ attains its maximum for $\|z\| = (2k/3s)^{1/2}$, it does not exceed the value $(k\ln k - k\ln s)/2$. Hence

$$\int e^{-s\|z\|^2 + k\ln\|z\|}\,d\Phi(z) \le \int e^{-s\|z\|^2/4 + (k\ln k - k\ln s)/2}\,d\Phi(z)$$
$$\le k^{k/2}s^{-k/2}I(s/4) \le 5^n k^{k/2}s^{-k/2}I(s).$$

(5) Since Φ is uniformly distributed and since $0 \in \operatorname{spt}\Phi$,

$$\int e^{-s\|z-x\|^2}\,d\Phi(z) = \int e^{-s\|z\|^2}\,d\Phi(z)$$

for every $x \in \operatorname{spt}\Phi$. Hence

$$\int\left[\sum_{j=1}^{\infty}(s^j/j!)e^{-s\|z\|^2}(2^j\langle z, x\rangle^j - \|x\|^{2j})\right]d\Phi(z)$$
$$= \int\left[e^{-s\|z\|^2 + 2s\langle z,x\rangle} - e^{-s\|z\|^2 + s\|x\|^2}\right]d\Phi(z)$$
$$= e^{s\|x\|^2}\int\left(e^{-s\|z-x\|^2} - e^{-s\|z\|^2}\right)d\Phi(z) = 0 \qquad (x \in \operatorname{spt}\Phi)$$

and it suffices to show that one may interchange the integration and the summation. But this follows from

$$\sum_{j=1}^{\infty}(s^j/j!)\int e^{-s\|z\|^2}\left|2^j\langle z, x\rangle^j - \|x\|^{2j}\right|d\Phi(z)$$
$$\le \sum_{j=1}^{\infty}(2s\|x\|)^j\int e^{-s\|z\|^2}\|z\|^j\,d\Phi(z)/j! + e^{s\|x\|^2}I(s)$$
$$\le 5^n I(s)\sum_{j=1}^{\infty}(2^j j^{j/2}s^{j/2}\|x\|^j)/j! + e^{s\|x\|^2}I(s) < \infty.$$

3.3. THEOREM. *Let Φ be a uniformly distributed measure over $\mathbf{R}^n$ and let Ψ measure $\mathbf{R}^n$.*

(1) *If G is an open subset of $\mathbf{R}^n$, $G \cap \operatorname{spt}\Psi \subset \operatorname{spt}\Phi$ and $\Psi(B(x,r)) = \Psi(B(y,r))$ whenever $x, y \in G \cap \operatorname{spt}\Psi$, $r > 0$, and $B(x,r) \cup B(y,r) \subset G$, then either*

(i) *$\lim_{r\searrow 0}\Psi(B(x,r))/\Phi(B(y,r)) = \infty$ whenever $x \in G \cap \operatorname{spt}\Psi$ and $y \in \operatorname{spt}\Phi$, or*

(ii) *$G \cap \operatorname{spt}\Psi$ is a relatively open subset of $\operatorname{spt}\Phi$ and $\Psi\llcorner G = c\Phi\llcorner(G \cap \operatorname{spt}\Psi)$ for some $c > 0$.*

(2) *If Ψ is uniformly distributed and $\operatorname{spt}\Psi = \operatorname{spt}\Phi$, then there is $c > 0$ such that $\Psi = c\Phi$.*

(3) *If* $\Phi \neq 0$ *and if* Φ_k *is a sequence of uniformly distributed measures over* R^n *then the following statements are equivalent.*

(i) *There is a sequence* $c_k > 0$ *such that* $\lim_{k\to\infty} c_k \Phi_k = \Phi$.

(ii) *For every infinite set* N *of positive integers*

$$\operatorname{spt} \Phi = \bigcap_{p=1}^{\infty} \operatorname{Clos}\left(\bigcup_{N \ni k \geq p} \operatorname{spt} \Phi_k \right).$$

Proof. (1) We may assume that $\Phi(G) > 0$ and $\Phi(G) > 0$. Let $u \in G \cap \operatorname{spt} \Psi$. If $\lim_{r\searrow 0} \Psi(B(u, r))/\Phi(B(u, r)) = \infty$, (i) holds. If $\liminf_{r\searrow 0} \Psi(B(u, r))/\Phi(B(u, r)) < \infty$, 1.7 implies that $0 < c = \lim_{r\searrow 0} \Psi(B(u, r))/\Phi(B(u, r)) < \infty$ and $\Psi \llcorner G = c\Phi \llcorner (G \cap \operatorname{spt} \Psi)$.

Let $x \in G \cap \operatorname{spt} \Psi$ and let $s > 0$ be such that $B(x, 4s) \subset G$ and $\Psi(B(u, r))/\Phi(B(u, r)) > (1 - 7^{-n})c$ for every $0 < r \leq 2s$. If $y \in (B(x, s) \cap \operatorname{spt} \Phi) - \operatorname{spt} \Psi$, we find the largest $r > 0$ such that $B^0(y, r) \cap \operatorname{spt} \Psi = \varnothing$. We also note that $r \leq s$. Let $z \in B(y, r) \cap \operatorname{spt} \Psi$. Then

$$\begin{aligned} \Phi(B(z, 2r) \cap \operatorname{spt} \Psi) &= \Psi(B(z, 2r))/c = \Psi(B(u, 2r))/c \\ &> (1 - 7^{-n})\Phi(B(u, 2r)). \end{aligned}$$

Since 3.2(1) and 3.1(4) imply $\Phi(B(z, 2r) - \operatorname{spt} \Psi) \geq \Phi(B^0(y, r)) \geq 7^{-n}\Phi(B(u, 2r))$,

$$\Phi(B(u, 2r)) = \Phi(B(z, 2r) - \operatorname{spt} \Psi) + \Phi(B(z, 2r) \cap \operatorname{spt} \Psi) > \Phi(B(u, 2r)).$$

This contradiction proves that $G \cap \operatorname{spt} \Psi$ is a relatively open subset of $\operatorname{spt} \Phi$.

(2) We use (1) with $G = \mathbf{R}^n$. Since one may interchange the roles of Φ and Ψ, (1; i) is impossible. Consequently, (1; ii) implies (2).

(3) We may assume that $\Phi \in \mathfrak{U}(n)$.

(i) $\Rightarrow$ (ii). If k_j is an increasing sequence of positive integers, $x_j \in \operatorname{spt} \Phi_{k_j}$, $x = \lim_{j\to\infty} x_j$, and $r > 0$, then

$$\Phi(B(x, r)) \geq \limsup_{j\to\infty} c_{k_j} \Phi_{k_j}\big(B(x_j, r/2)\big) \geq \Phi(B(0, r/4)).$$

Hence $\operatorname{spt} \Phi$ contains the set on the right hand side of (ii). The opposite inclusion is obvious.

(ii) $\Rightarrow$ (i). We note first that $\lim_{k\to\infty} \operatorname{dist}(0, \operatorname{spt} \Phi_k) = 0$. Hence $F_1(\Phi_k) > 0$ for sufficiently large k and, according to 3.2(1) and 1.12(1), each subsequence of the sequence $\Phi_k/F_1(\Phi_k)$ has a convergent subsequence. If Ψ is the limit of such a subsequence, we may use the implication (i) $\Rightarrow$ (ii) to conclude that $\operatorname{spt} \Psi = \operatorname{spt} \Phi$. Since $F_1(\Psi) = 1$, (2) implies that $\Psi = \Phi/F_1(\Phi)$, which proves (i).

3.4. (1) Whenever $\Phi \in \mathfrak{U}(n)$, $s > 0$, and $k = 1, 2, \ldots,$ we define symmetric k linear forms $b_{k,s} \in \bigodot^k \mathbf{R}^n$ by the formula

$$b_{k,s}(u_1 \odot \cdots \odot u_k) = (2s)^k (I(s)k!)^{-1} \int e^{-s\|z\|^2} \langle z, u_1 \rangle \ldots \langle z, u_k \rangle \, d\Phi(z).$$

If it is necessary to explain to which measure the forms $b_{k,s}$ belong, we shall write $b_{k,s}(\Phi)(u_1 \odot \cdots \odot u_k)$ instead of $b_{k,s}(u_1 \odot \cdots \odot u_k)$.

We prove that

$$(2) \qquad |b_{k,s}(u_1 \odot \cdots \odot u_k)| \le 2^k 5^n k^{k/2} s^{k/2} \|u_1\| \ldots \|u_k\| / k!$$

whenever $k = 1, 2, \ldots,$ $u_1, \ldots, u_k \in \mathbf{R}^n$, and $s > 0$;

$$(3) \qquad \left| \sum_{k=1}^{2q} b_{k,s}(x^k) - \sum_{k=1}^{q} s^k \|x\|^{2k}/k! \right| \le 5^{n+9} (s\|x\|^2)^{q+1/2}$$

whenever $s > 0$, $q = 1, 2, \ldots,$ and $x \in \operatorname{spt} \Phi$; and

$$(4) \qquad b_{k,s}(T_{0,t}[\Phi]) = t^k b_{k,st^{-2}}(\Phi)$$

whenever $s > 0$, $t > 0$, and $k = 1, 2, \ldots$.

Proof. (2) follows immediately from 3.2(4).

(3) We note that

$$\sum_{k=1}^{\infty} 2^k k^{k/2}/k! = \sum_{k=1}^{\infty} 8^k k^k/(2k)!$$
$$+ 8^{1/2} \sum_{k=0}^{\infty} 8^k (k + 1/2)^{k+1/2}/(2k+1)!$$
$$\le \sum_{k=1}^{\infty} 8^k/k! + 8^{1/2} \sum_{k=0}^{\infty} 8^k/k! \le 4e^8.$$

If $s\|x\|^2 > 1$, we use (2) to compute

$$\sum_{k=1}^{2q} |b_{k,s}(x^k)| + \sum_{k=1}^{q} s^k \|x\|^{2k}/k!$$
$$\le 5^n \sum_{k=1}^{2q} 2^k k^{k/2} (s\|x\|^2)^{k/2}/k! + (s\|x\|^2)^q \sum_{k=1}^{\infty} 1/k!$$
$$\le 5^n (s\|x\|^2)^q 4e^8 + (s\|x\|^2)^q e \le 5^{n+9} (s\|x\|^2)^q \le 5^{n+9} (s\|x\|^2)^{q+1/2}.$$

If $s\|x\|^2 \le 1$, we use 3.2(5) to estimate

$$\begin{aligned}
&\left|\sum_{k=1}^{2q} b_{k,s}(x^k) - \sum_{k=1}^{q} s^k\|x\|^{2k}/k!\right| \\
&\quad\le \sum_{k=2q+1}^{\infty} \left|b_{k,s}(x^k)\right| + \sum_{k=q+1}^{\infty} s^k\|x\|^{2k}/k! \\
&\quad\le 5^n \sum_{k=2q+1}^{\infty} 2^k k^{k/2}\left(s\|x\|^2\right)^{k/2}/k! + \left(s\|x\|^2\right)^{q+1} e \\
&\quad\le 5^n\left(s\|x\|^2\right)^{q+1/2} 4e^8 + \left(s\|x\|^2\right)^{q+1} e \le 5^{n+9}\left(s\|x\|^2\right)^{q+1/2}.
\end{aligned}$$

(4) is obvious.

3.5. LEMMA. *Let V be a finite dimensional linear space with an inner product. Let $P_1, \ldots, P_q \in$ Hom(V, V) be orthogonal projections (i.e. $P_jP_j = P_j$ and* Ker$(P_j) = ($Range$(P_j))^{\perp}$) *such that $\bigcap_{j=1}^{q}$Ker$(P_j) = 0$. Then for every linear subspace E of V there are $a > 0$ and an analytic map $s \in (-a, a) \mapsto Q_s \in$* Hom$(V, V)$ *such that*

(i) *for each $s \in (-a, a)$, Q_s is a projection onto E (i.e. $Q_sQ_s = Q_s$ and* Range$(Q_s) = E$), *and*

(ii) *for each $s \in (-a, a) - \{0\}$*

$$\operatorname{Ker}(Q_s) = \left\{\sum_{k=1}^{q} s^{-k}P_k(x);\ x \in E^{\perp}\right\}.$$

Proof. For $k = 1, 2, \ldots, q$ let

$$F_k = E^{\perp} \cap \bigcap_{k<j\le q}\left(\operatorname{Ker}(P_j)\right) \cap \left[E^{\perp} \cap \bigcap_{k\le j\le q}\left(\operatorname{Ker}(P_j)\right)\right]^{\perp}$$

and let E_k be the image of F_k under P_k. (As usual, we let $\bigcap_{q<j\le q}($Ker$(P_j)) = V$.) We prove that

$$(\alpha)\qquad V = E \oplus \bigoplus_{k=1}^{q} E_k.$$

In fact, if $x \ne 0$ is orthogonal to $E \cup \bigcup_{j=1}^{q} E_j$, let $k = 1, \ldots, q$ be the smallest number such that $x \in \bigcap_{k<j\le q}(Ker(P_j))$. Since $x \in E^{\perp}$, there is $u \in E^{\perp} \cap \bigcap_{k\le j\le q}(Ker(P_j))$ such that $x - u \in F_k$. Thus $P_k(x) = P_k(x-u) \in E_k$. Hence x is orthogonal to $P_k(x)$. Consequently, $P_k(x) = 0$, which contradicts the choice of k. Counting the dimensions, we prove (α) and

$$(\beta)\qquad E^{\perp} = \bigoplus_{k=1}^{q} F_k.$$

Since (α) holds, there are projections $T, T_1, \ldots, T_q \in$ Hom(V, V) onto $E, E_1, \ldots, E_q$, respectively, such that $x = T(x) + \sum_{k=1}^{q} T_k(x)$ for every $x \in V$.

Also, since $F_k \cap \operatorname{Ker}(P_k) = \{0\}$, there are $S_k \in \operatorname{Hom}(E_k, F_k)$ such that $P_k S_k(x) = x$ for every $x \in E_k$.

Let

$$A_s(x) = T(x) + \sum_{k=1}^{q} \sum_{j=1}^{k} s^{k-j} P_j S_k T_k(x).$$

Then $s \in R \mapsto A_s \in \operatorname{Hom}(V, V)$ is an analytic map and A_0 is the identity. Since the set of all invertible operators on V is open and since the map assigning to each invertible operator its inverse is analytic, there is $a > 0$ such that the map $s \mapsto A_s^{-1}$ is well defined and analytic on $(-a, a)$.

We put $Q_s = TA_s^{-1}$. Since $A_s(x) = x$ and $T(x) = x$ for $x \in E$ and since $\operatorname{Range}(T) = E$, (i) holds. Let $s \in (-a, a) - \{0\}$. Then $x \in \operatorname{Ker}(Q_s)$ if and only if there are $z_1 \in F_1, \dots, z_q \in F_q$ such that

$$x = A_s\big(P_1(z_1) + \cdots + P_q(z_q)\big) = \sum_{k=1}^{q} \sum_{j=1}^{k} s^{k-j} P_j(z_k)$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{q} s^{k-j} P_j(z_k)$$

(since $P_j(z_k) = 0$ if $j > k$). Denoting $s^k z_k = u_k$, we see that $x \in \operatorname{Ker}(Q_s)$ if and only if there are $u_1 \in F_1, \dots, u_q \in F_q$ such that

$$x = \sum_{k=1}^{q} \sum_{j=1}^{q} s^{-j} P_j(u_k) = \sum_{j=1}^{q} s^{-j} P_j\left(\sum_{k=1}^{q} u_k\right).$$

In view of (β), this implies (ii).

3.6. THEOREM. *Let* $\Phi \in \mathfrak{U}(n)$.

(1) *There are symmetric forms* $b_k^{(j)} = b_k^{(j)}(\Phi) \in \bigodot^k \mathbf{R}^n$ $(k = 1, 2, \dots,\ j = 1, 2, \dots)$ *such that*

(i) $b_{k,s} = \sum_{j=1}^{q} s^j b_k^{(j)}/j! + o(s^q)$ *as* $s \searrow 0$ *for every* $k = 1, 2, \dots$ *and every* $q = 1, 2, \dots$,

(ii) $b_k^{(i)} = 0$ *whenever* $k > 2i$, *and*

(iii) $\sum_{k=1}^{2q} b_k^{(q)}(x^k) = \|x\|^{2q}$ *for every* $q = 1, 2, \dots$ *and every* $x \in \operatorname{spt} \Phi$.

Moreover, the forms $b_k^{(j)}$ *are determined uniquely by* (i).

(2) *There are symmetric forms* $\hat{b}_k^{(j)} = \hat{b}_k^{(j)}(\Phi) \in \bigodot^k \mathbf{R}^n$ $(k = 1, 2, \dots,\ j = 1, 2, \dots)$ *such that*

(i) $s^{-k} b_{k,s} = \sum_{j=1}^{q} s^{-j} \hat{b}^{(j)}/j! + o(s^{-q})$ *as* $s \to \infty$ *for every* $k = 1, 2, \dots$ *and every* $q = 1, 2, \dots$, *and*

(ii) $\hat{b}_k^{(i)} = 0$ *whenever* $k > 2i$.

Moreover, the forms $\hat{b}_k^{(j)}$ *are determined uniquely by* (i).

Proof. Since the asymptotic expansions are determined uniquely, it suffices to fix $q = 1, 2, \ldots$ and to find forms $b_k^{(j)}, \hat{b}_k^{(j)}$ $(k = 1, \ldots, 2q,\ j = 1, \ldots q)$ depending possibly upon q such that the statements of 3.6 hold if $k \le q$ and $i \le q$.

Let $V = \bigoplus_{j=1}^{2q} \bigodot_j \mathbf{R}^n$ and let $P_k \in \operatorname{Hom}(V, V)$ be the canonical projection onto $\bigodot_k \mathbf{R}^n$ $(k = 1, \ldots, 2q)$. Multiplying the usual inner product on $\bigodot_j R^n$ ([11, 1.10.5]), by $2^j/(j!)^2$, we construct an inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on V such that each P_k is an orthogonal projection and

$$\langle\langle u_1 \odot \cdots \odot u_k, x^k \rangle\rangle = 2^k \langle u_1, x\rangle \ldots \langle u_k, x\rangle / k!$$

whenever $k = 1, \ldots, 2q$, $u_1, \ldots, u_k \in \mathbf{R}^n$, and $x \in \mathbf{R}^n$.

Let $\hat{w}_{2k} \in \bigodot^{2k} \mathbf{R}^n$ be such that $\hat{w}_{2k}(x^{2k}) = \|x\|^{2k}/k!$ $(k = 1, \ldots, q)$ and let $w_s \in \operatorname{Hom}(V, R)$ $(s \in R)$ be defined by $w_s(u) = \sum_{k=1}^q s^k \hat{w}_{2k}(P_{2k}(u))$.

For each $s > 0$ let $b_s \in \operatorname{Hom}(V, R)$ be defined by the formula

$$b_s(u) = \sum_{k=1}^{2q} b_{k,s}(P_k(u))$$

$$= I(s)^{-1} \int e^{-s\|z\|^2} \left\langle\left\langle \sum_{k=1}^{2q} s^k P_k(u), \sum_{k=1}^{2q} z^k \right\rangle\right\rangle d\Phi(z).$$

(1) Let $E \subset V$ be the linear span of the set $\{x + x^2 + \cdots + x^{2q};\ x \in \operatorname{spt} \Phi\}$ and let Q_s be the projections constructed in 3.5 (with q replaced by $2q$). Since $b_s(u) = 0$ if $u \in \operatorname{Ker}(Q_s)$ and since $b_s(u) = w_s(u) + o(s^q)$ if $u \in E$ (3.4(3)),

$$b_s = b_s Q_s = w_s Q_s + o(s^q) \quad \text{as } s \searrow 0.$$

Since the maps $s \mapsto w_s/s$ and $s \mapsto Q_s$ are analytic at $s = 0$, (i) holds. The statement (ii) follows from 3.4(2). To prove (iii), we use 3.4(3) to infer that, for every $x \in \operatorname{spt} \Phi$,

$$\sum_{k=1}^{2q} \sum_{j=1}^{q} s^j b_k^{(j)}(x^k)/j! = \sum_{k=1}^{q} s^k \|x\|^{2k}/k! + o(s^q) \quad \text{as } s \searrow 0.$$

Hence $\sum_{k=1}^{2q} b_k^{(q)}(x^k)/q! = \|x\|^{2q}/q!$, which is (iii).

(2) Let $E \subset V$ be the orthogonal complement of $\{u \in V;\ b_t(u) = w_t(u)$ for every $t > 0\}$ and let Q_s be the projections constructed in 3.5 (with q replaced by $2q$). Also, let

$$\hat{b}_s(u) = \sum_{k=1}^{2q} s^k b_{k,1/s}(P_k(u))$$

$$= I(1/s)^{-1} \int e^{-\|z\|^2/s} \left\langle\left\langle u, \sum_{k=1}^{2q} z^k \right\rangle\right\rangle d\Phi(z).$$

If $u \in \mathrm{Ker}(Q_s)$, we find $v \in E^{\perp}$ such that $u = \sum_{k=1}^{2q} s^{-k} P_k(v)$ and we infer that

$$\hat{b}_s(u) = \hat{b}_s\left(\sum_{k=1}^{2q} s^{-k} P_k(v)\right) = b_{1/s}(v) = w_{1/s}(v)$$
$$= w_s\left(\sum_{k=1}^{2q} s^{-k} P_k(v)\right) = w_s(u).$$

Let $v_t \in V$ be such that $\langle\langle v_t, u\rangle\rangle = b_t(u) - w_t(u)$ for every $u \in V$ ($t > 0$). Using 3.4(3) and 3.2(4), we obtain

$$|\hat{b}_s(v_t)| = \left| I(1/s)^{-1} \int e^{-\|z\|^2/s} (b_t - w_t)\left(\sum_{k=1}^{2q} z^k\right) d\Phi(z)\right|$$
$$\leq 5^{n+9} t^{q+1/2} I(1/s)^{-1} \int e^{-\|z\|^2/s} \|z\|^{2q+1}\, d\Phi(z)$$
$$\leq 5^{2n+9} t^{q+1/2} (2q+1)^{q+1/2} s^{q+1/2}.$$

Since E is spanned by $\{v_t;\ t > 0\}$, $\hat{b}_s(u) = o(s^q)$ as $s \searrow 0$ for every $u \in E$. Hence

$$\hat{b}_s = \hat{b}_s(\mathrm{Id} - Q_s) + \hat{b}_s Q_s = w_s(\mathrm{Id} - Q_s) + o(s^q) \quad \text{as } s \searrow 0,$$

which implies (i). The statement (ii) follows from 3.4(2).

3.7. (1) Whenever $V \in G(n, m)$, we define

$$\mathfrak{M}_{n,V} = \{c\mathscr{H}^m \llcorner V;\ c > 0\}.$$

We also denote $\mathfrak{M}_{n,m} = \bigcup_{V \in G(n,m)} \mathfrak{M}_{n,V}$, and $\mathfrak{M}_n = \bigcup_{m=1}^{n} \mathfrak{M}_{n,m}$. Using 2.2(4) and 3.2(1), we see that

(2) each of the sets $\mathfrak{M}_n$, $\mathfrak{M}_{n,m}$, $\mathfrak{M}_{n,V}$, and $\mathfrak{U}(n)$ is a d-cone with a compact basis.

3.8. Theorem. *For every nonnegative integer n and every positive number μ there is a constant $\omega_1 = \omega_1(n, \mu) \in (0, 1]$ with the following property: Whenever Φ is a uniformly distributed measure over $\mathbf{R}^n$, $x \in \operatorname{spt} \Phi$ and $r > 0$, then there are $y \in B(x, r) \cap \operatorname{spt} \Phi$ and $s \in [\omega_1 r, r]$ such that $d_1(T_{y,s}[\Phi], \mathfrak{M}_n) < \mu$.*

Proof. Assume, to the contrary, that one can find the smallest nonnegative integer n such that the statement of the theorem does not hold for some $\mu > 0$. Clearly, $n \geq 1$ since $\omega_1(0, \mu) = 1$ has the desired properties. (We observe that $\mathbf{R}^0 = \{0\}$; hence $\operatorname{spt} \Phi = \{0\}$ and $d_1(T_{0,r}[\Phi], \mathfrak{M}_0) = 0$.)

Let Φ_k be uniformly distributed measures over $\mathbf{R}^n$ such that, for some $x_k \in \operatorname{spt} \Phi_k$ and $r_k > 0$, $d_1(T_{y,s}[\Phi_k], \mathfrak{M}_n) \geq \mu$ whenever $y \in B(x_k, r_k) \cap \operatorname{spt} \Phi_k$ and $s \in [2^{-2k} r_k, r_k]$.

Let Φ be the limit of some subsequence of the sequence $T_{x_k, 2^{-k} r_k}[\Phi_k]/F_1(T_{x_k, 2^{-k} r_k}[\Phi_k])$. (See 3.7(2).) Then $d_1(T_{y,s}[\Phi], \mathfrak{M}_n) \geq \mu$ whenever $y \in \operatorname{spt} \Phi$ and $s > 0$. From 3.3(2) we see that $\operatorname{spt} \Phi \neq \operatorname{spt} \mathscr{L}^n = \mathbf{R}^n$. Let $z \in \mathbf{R}^n - \operatorname{spt} \Phi$ and let $q > 0$ be the largest number such that $B^0(z, q) \cap \operatorname{spt} \Phi = \varnothing$. If $x \in B(z, q) \cap \operatorname{spt} \Phi$ and $\Psi \in \operatorname{Tan}(\Phi, x)$, then

$$(\alpha) \qquad \operatorname{spt} \Psi \subset \{u \in R^n;\ \langle u, z - x\rangle \leq 0\}, \quad \text{and}$$

$$(\beta) \qquad d_1\big(T_{y,s}[\Psi], \mathfrak{M}_n\big) \geq \mu \quad \text{whenever } y \in \operatorname{spt} \Psi \text{ and } s > 0.$$

Using 3.2(1) and 1.11(3), we see that

$$\int e^{-\|u\|^2} \langle u, z - x\rangle \, d\Psi(u) \Big/ \int e^{-\|u\|^2} \, d\Psi(u)$$

$$= \lim_{s \to \infty} \frac{1}{2s} b_{1, s^2}(T_{x,1}[\Phi])(z - x) = 0$$

according to 3.6(2; i). Hence (α) implies $\operatorname{spt} \Psi \subset \{u \in \mathbf{R}^n;\ \langle u, z - x\rangle = 0\}$ which, in view of (β), contradicts the minimality of n.

3.9. Lemma. *If Φ is a uniformly distributed measure over $\mathbf{R}^n$ then $\mathfrak{M}_n \cap \operatorname{Tan}(\Phi, x) \neq \varnothing$ for Φ almost every $x \in \mathbf{R}^n$.*

Proof. Since $\operatorname{Tan}(\Phi, x) \subset \mathfrak{U}(n)$ for every $x \in \operatorname{spt} \Phi$, the statement follows directly from 2.12 and 3.8.

3.10. Theorem. *Let $\Psi \in \mathfrak{U}(n)$ be such that $\{c\Psi;\ c > 0\}$ is a d-cone. Then there are an integer $m = 0, 1, \ldots, n$ and a constant $C \in (0, \infty)$ such that*

(1) *$T_{0,t}[\Psi] = |t|^m \Psi$ for every $t \neq 0$,*

(2) *$\Psi(B(0, r)) = C\alpha(m) r^m$ for every $r > 0$,*

(3) *whenever $u \in \operatorname{spt} \Phi$, $e \in \mathbf{R}^m$, $\|u\| = \|e\|$ and whenever f is a non-negative Borel function on $\mathbf{R}^2$, then*

$$\int_{\mathbf{R}^n} f(\|z\|^2, \langle z, u\rangle) \, d\Psi(z) = C \int_{\mathbf{R}^m} f(\|z\|^2, \langle z, e\rangle) \, d\mathscr{L}^m(z),$$

(4) *$b_{2k-1,s}(\Psi) = 0$ and $b_{2k,s}(\Psi) = s^k b_{2k}^{(k)}(\Psi)/k!$ for every $s > 0$ and every $k = 1, 2, \ldots,$*

(5) *$\operatorname{spt} \Psi \subset \{x \in \mathbf{R}^n;\ b_{2k}^{(k)}(\Psi)(x^{2k}) = \|x\|^{2k}$ for each $k = 1, 2, \ldots\}$, and*

(6) *if $x \in \operatorname{spt} \Psi$ and $\lambda \in \mathbf{R}$, then $\lambda x \in \operatorname{spt} \Psi$.*

Proof. According to 2.10 there is $m \geq 0$ such that $T_{0,t}[\Psi] = t^m\Psi$ for every $t > 0$. Let $x \in \operatorname{spt}\Psi$ be such that $\mathfrak{M}_n \cap \operatorname{Tan}(\Psi, x) \neq \varnothing$. Since $\Psi \in \mathfrak{U}(n)$, any measure $\hat{\Psi} \in \mathfrak{M}_n \cap \operatorname{Tan}(\Psi, x)$ fulfils $\hat{\Psi}(B(0, r)) = cr^m$ for some $c > 0$. Hence m is an integer and $m \leq n$. This proves (1) for $t > 0$ and (2).

(4) Since, for each $t > 0$, $T_{0,t}[\Psi]$ is a constant multiple of Ψ, 3.4(4) implies $b_{k,s} = s^{k/2}b_{k,1}$ for each $k = 1, 2, \ldots$ and each $s > 0$. If k is odd, we see from 3.6(1; i) that $b_{k,s} = 0$ for each $s > 0$. If k is even, the statement follows again from 3.6(1; i).

(5) It suffices to use (4) and 3.6(1; iii).

(1) Since we already know that this statement holds for $t > 0$, it suffices to prove that $T_{0,-1}[\Psi] = \Psi$. But this follows from 1.13 since, according to (4),

$$\int e^{-\|z\|^2}\langle z, u\rangle^k \, d\Psi(z) = \int e^{-\|z\|^2}\langle z, u\rangle^k \, dT_{0,-1}[\Psi](z),$$

for each $k = 0, 1, \ldots$.

(3) We prove that

$$(*) \qquad \int e^{-s\|z\|^2}\langle z, u\rangle^p \, d\Psi(z) = C\int e^{-s\|z\|^2}\langle z, e\rangle^p \, d\mathscr{L}^m(z)$$

for every $s > 0$ and every $p = 0, 1, \ldots$.

If $p = 0$, $(*)$ follows from (2).

If p is odd, we use (4) to deduce that the left-hand side of $(*)$ equals zero.

If $p = 2k$, $k = 1, 2, \ldots$, we use (4) and (5) to compute

$$\begin{aligned}
&(2s)^{2k}\int e^{-s\|z\|^2}\langle z, u\rangle^{2k} \, d\Psi(z) \Big/ \left[(2k)!\int e^{-s\|z\|^2} \, d\Psi(z)\right] \\
&\quad = b_{2k,s}(\Psi)(u^{2k}) = s^k b_{2k}^{(k)}(u^{2k})/k! = s^k\|u\|^{2k}/k! = s^k\|e\|^{2k}/k! \\
&\quad = (2s)^{2k}\int e^{-s\|z\|^2}\langle z, e\rangle^{2k} \, d\mathscr{L}^m(z) \Big/ \left[(2k)!\int e^{-s\|z\|^2} \, d\mathscr{L}^m(z)\right].
\end{aligned}$$

Hence, $(*)$ follows from the case $p = 0$.

Finally, we use 1.13 to deduce (3).

(6) follows easily from (1).

3.11. THEOREM. *Let Φ be a nonzero uniformly distributed measure over $\mathbf{R}^n$.*

(1) *There are an integer $m = 0, 1, \ldots, n$, a constant $C \in (0, \infty)$, and a measure $\Psi \in \mathfrak{U}(n)$ such that*

(i) $\lim_{r\to\infty}\Phi(B(x, r))/r^m = C$ *for each* $x \in \mathbf{R}^n$,

(ii) $\lim_{r\to\infty}T_{x,r}[\Phi]/r^m = \Psi$ *for each* $x \in \mathbf{R}^n$, *and*

(iii) $\operatorname{Tan}(\Phi, \infty) = \{c\Psi;\ c > 0\}$.

(2) *There are an integer* $p = 0, 1, \ldots, n$ *and a constant* $\tilde{C} \in (0, \infty)$ *such that for every* $x \in \operatorname{spt} \Phi$ *there is* $\tilde{\Psi} \in \mathfrak{U}(n)$ *with the following properties.*

(i) $\lim_{r \searrow 0} \Phi(B(x, r))/r^p = \tilde{C}$,

(ii) $\lim_{r \searrow 0} T_{x,r}[\Phi]/r^p = \tilde{\Psi}$, *and*

(iii) $\operatorname{Tan}(\Phi, x) = \{c\tilde{\Psi};\ c > 0\}$.

Moreover,

(iv) $\tilde{\Psi} \in \mathfrak{M}_{n,p}$ *for* Φ *almost every* $x \in \mathbf{R}^n$.

Proof. We may assume that $\Phi \in \mathfrak{U}(n)$. For each $r > 0$ let

$$\Phi_r = e^{-\|\cdot\|^2} T_{0,r}[\Phi] \Big/ \int e^{-\|z\|^2/r^2}\, d\Phi(z).$$

From 3.2(3) we see that

$$\limsup_{r \to \infty} \int e^{a\|z\|}\, d\Phi_r(z) < \infty \quad \text{and} \quad \limsup_{r \searrow 0} \int e^{a\|z\|}\, d\Phi_r(z) < \infty$$

for each $a > 0$. Moreover, 3.6(1; i, ii) implies

$$\begin{aligned} \lim_{r \to \infty} \int \langle z, u \rangle^k\, d\Phi_r(z) &= 1 && \text{if } k = 0, \\ &= 2^{-k} k! \lim_{r \to \infty} r^k b_{k, r^{-2}}(\Phi)(u^k) = 0 && \text{if } k \text{ is odd, and} \\ &= 2^{-k} k!\, b_k^{(k/2)}(\Phi)(u^k) && \text{if } k \geq 2 \text{ is even.} \end{aligned}$$

From 1.14 we infer that $\lim_{r \to \infty} \Phi_r$ exists. Consequently, (see 1.11(5)) there is $\hat{\Psi} \in \mathfrak{U}(n)$ such that

$$\lim_{r \to \infty} T_{0,r}[\Phi] \Big/ \int e^{-\|z\|^2/r^2}\, d\Phi(z) = \hat{\Psi}.$$

Clearly, $\operatorname{Tan}(\Phi, \infty) = \{c\hat{\Psi};\ c > 0\}$. According to 3.10 there is $m = 0, 1, \ldots, n$ such that $T_{0,r}[\hat{\Psi}] = r^m \hat{\Psi}$ for each $r > 0$. Moreover, since 3.6(1; i) implies

$$\begin{aligned} s \int e^{-s\|z\|^2} \|z\|^2\, d\Phi(z) \Big/ \int e^{-s\|z\|^2}\, d\Phi(z) &= \operatorname{Trace}(b_{2,s})/2s \\ &= \operatorname{Trace}\big(b_2^{(1)}\big)/2 + o(s^{1/2}) \quad \text{as } s \searrow 0, \end{aligned}$$

we infer from 2.11(3) that there is $\hat{c} \in (0, \infty)$ such that

$$\lim_{r \to \infty} T_{0,r}[\Phi]/r^m = \hat{c}\hat{\Psi}.$$

It is clear now that (1; i) and (1; ii) hold if $x = 0$. Moreover, if $x \in R^n$, $t > 0$, $f\colon \mathbf{R}^n \mapsto \mathbf{R}$, $\operatorname{spt}(f) \subset B(0, t)$ and $\operatorname{Lip}(f) \leq 1$, then

$$\begin{aligned}\lim_{r\to\infty} r^{-m}&\left|\int f(z)\,dT_{x,r}[\Phi](z) - \int f(z)\,dT_{0,r}[\Phi](z)\right|\\ &\leq \lim_{r\to\infty} r^{-m}\int |f((z-x)/r) - f(z/r)|\,d\Phi(z)\\ &\leq \lim_{r\to\infty} r^{-m-1}\|x\|\Phi(B(0,(t+\|x\|)r)) = 0.\end{aligned}$$

Hence (1; ii) and, consequently, (1, i) hold for each $x \in \mathbf{R}^n$. The statement (1; iii) is obvious.

To prove (2), we first note that, since Φ is uniformly distributed, the values of p and $\tilde{C}$ cannot depend upon $x \in \operatorname{spt}\Phi$. Hence it suffices to consider $x = 0$. Arguing similarly as in the proof of (1), we use 3.6(2; i, ii), 1.14, 1.11(5), 3.10 and 2.11(4) to conclude that there are $p = 0, 1, \dots, n$ and $\tilde{\Psi} \in \mathfrak{U}(n)$ such that $\lim_{r\searrow 0} T_{0,r}[\Phi]/r^p = \tilde{\Psi}$. This and 3.9 imply all the statements of (2).

3.12. Let Φ be a nonzero uniformly distributed measure over $\mathbf{R}^n$.

(1) The number m from 3.11(1) will be denoted by $\dim_\infty\Phi$.

(2) The number p from 3.11(2) will be denoted by $\dim_0\Phi$.

(3) If $\dim_0\Phi = n$, 3.3(1) implies that Φ is a constant multiple of $\mathscr{L}^n$.

(4) If $\Psi \in \operatorname{Tan}(\Phi, \infty)$, we use 3.11(1; ii) and 3.10 to infer that $\dim_\infty\Phi = \dim_0\Psi = \dim_\infty\Psi$.

(5) If $x \in \operatorname{spt}\Phi$ and $\Psi \in \operatorname{Tan}(\Phi, x)$, we use 3.11(2; ii) and 3.10 to infer that $\dim_0\Phi = \dim_0\Psi = \dim_\infty\Psi$.

(6) If $x \in \mathbf{R}^n$ and $t \neq 0$, then $\dim_\infty T_{x,t}[\Phi] = \dim_\infty\Phi$, $\operatorname{Tan}(T_{x,t}[\Phi], \infty) = \operatorname{Tan}(\Phi, \infty)$, and $\dim_0 T_{x,t}[\Phi] = \dim_0\Phi$.

(7) If $\Phi \in \mathfrak{U}(n)$, $\dim_\infty\Phi = m$, and $\Psi \in \operatorname{Tan}(\Phi, \infty)$, we see from 1.11(3) that

$$b_{2k}^{(k)}(\Phi)(x^k) = 2^{2k}k!\int e^{-\|z\|^2}\langle z, x\rangle^{2k}\,d\Psi(z)\Big/\left[(2k)!\int e^{-\|z\|^2}\,d\Psi(z)\right]$$

for each $k = 1, 2, \dots$ and $x \in \mathbf{R}^n$. Hence (4) and 3.10(3) imply

$$\operatorname{Trace}\left(b_2^{(1)}(\Phi)\right) = 2\int e^{-\|z\|^2}\|z\|^2\,d\mathscr{L}^m(z)\Big/\int e^{-\|z\|^2}\,d\mathscr{L}^m(z) = \dim_\infty\Phi.$$

(8) If $\dim_\infty\Phi = m$ and if $\tilde{\Phi}$ is a nonzero uniformly distributed measure over $\mathbf{R}^n$ such that $\dim_\infty\tilde{\Phi} \neq m$, then

$$\liminf_{r\to\infty} F_r\left(\Phi/F_r(\Phi), \tilde{\Phi}/F_r(\tilde{\Phi})\right) > 1/3(m+2).$$

Indeed, let $\dim_\infty \tilde{\Phi} = k > m$, $\Psi \in \operatorname{Tan}(\Phi, \infty)$, $\tilde{\Psi} \in \operatorname{Tan}(\tilde{\Phi}, \infty)$, $F_1(\Psi) = F_1(\tilde{\Psi}) = 1$, and $t = (m+1)/(m+2)$. Then

$$\lim_{r \to \infty} F_r\big(\Phi/F_r(\Phi), \tilde{\Phi}/F_r(\tilde{\Phi})\big) = F_1(\Psi, \tilde{\Psi}) \geq F_t(\Psi) - F_t(\tilde{\Psi})$$
$$= t^{m+1} - t^{k+1} \geq t^{m+1}(1-t) > 1/3(m+2).$$

(9) We shall say that the measure Φ is flat at ∞ if $\operatorname{Tan}(\Phi, \infty) \subset \mathfrak{M}_n$. If the measure is not flat at ∞, we shall say that it is curved at ∞.

We note that, if $x \in \mathbf{R}^n$ and $t \neq 0$, then Φ is flat at ∞ if and only if $T_{x,t}[\Phi]$ is.

3.13. Lemma. *For every nonnegative integer m and every $\tau > 0$ there is a number $\kappa = \kappa(m, \tau) \in (0, 1/2)$ with the following property. Whenever $n \geq m \geq 0$ are integers, Φ measures $\mathbf{R}^n$, $V \in G(n, m)$, $x \in \operatorname{spt} \Phi$, $r > 0$, $C > 0$,*

$$C(1-\kappa)\alpha(m)s^m \leq \Phi(B(z,s)) \leq C(1+\kappa)\alpha(m)s^m$$

for every $s \in (\kappa r, r)$ and every $z \in B(x, r) \cap \operatorname{spt} \Phi$, and

$$\int_{B(x,r)} \operatorname{dist}^2(z - x, V)\, d\Phi(z) \leq \kappa C \alpha(m) r^{m+2},$$

then

(i) *$d_1(T_{x,r}[\Phi], \mathfrak{M}_{n,V}) < \tau$, and*
(ii) *$B(z, 2\tau^{1/(m+1)}r) \cap \operatorname{spt} \Phi \neq \varnothing$ whenever $z \in (x + V) \cap B(x, r)$.*

Proof. Assume, to the contrary, that there are a nonnegative integer m, a number $\tau > 0$, a sequence Φ_k of measures over $\mathbf{R}^{n_k}(n_k \geq m)$, a sequence $V_k \in G(n_k, m)$, and sequences $x_k \in \operatorname{spt} \Phi_k$, $r_k > 0$, and $C_k > 0$ such that

$$C_k(1 - 2^{-(m+3)(k+1)})\alpha(m)s^m \leq \Phi_k\big(B_{n_k}(z,s)\big)$$
$$\leq C_k(1 + 2^{-(m+3)(k+1)})\alpha(m)s^m$$

for every $z \in B(x_k, r_k) \cap \operatorname{spt} \Phi_k$ and every $s \in (2^{-(m+3)(k+1)}r_k, r_k)$,

$$\int_{B(x_k, r_k)} \operatorname{dist}^2(z - x_k, V_k)\, d\Phi_k(z) \leq 2^{-(m+3)(k+1)} C_k \alpha(m) r_k^{m+2},$$

and $d_1(T_{x_k, r_k}[\Phi_k], \mathfrak{M}_{n_k, V_k}) \geq \tau$.

We may shift, dilate, and rotate the measures Φ_k and multiply them by suitable constants to achieve that $x_k = 0$, $r_k = 1$, $C_k = 1$, and V_k is spanned by the first m vectors of the standard basis. Hence each V_k may be naturally identified with $\mathbf{R}^m$.

If $y \in [B_{n_k}(0, 1 - 2^{-k}) - B_{n_k}(\mathbf{R}^m, 2^{-k})] \cap \operatorname{spt} \Phi$, then

$$\begin{aligned}\int_{B(0,1)} \operatorname{dist}^2(z, V_k)\, d\Phi_k(z) &\geq 2^{-2(k+1)} \Phi_k\big(B(y, 2^{-(k+1)})\big)\\ &\geq 2^{-2(k+1)}\big(1 - 2^{-(m+3)(k+1)}\big)\alpha(m) 2^{-m(k+1)}\\ &> 2^{-(m+3)(k+1)}\alpha(m),\end{aligned}$$

which contradicts our assumptions. Hence

$$B_{n_k}(0, 1 - 2^{-k}) \cap \operatorname{spt} \Phi_k \subset B_{n_k}(\mathbf{R}^m, 2^{-k}).$$

Let P_k be the orthogonal projection of $\mathbf{R}^{n_k}$ onto $\mathbf{R}^m$ and let $\Psi_k = P_k[\Phi_k \llcorner B_{n_k}(0, 1 - 2^{-k})]$.

Whenever f is a function on $\mathbf{R}^{n_k}$ such that $\operatorname{spt}(f) \subset B(0,1)$ and $\operatorname{Lip}(f) \leq 1$, then

$$\begin{aligned}&\left|\int f(z)\, d\Phi_k(z) - \int f(z)\, d\Psi_k(z)\right|\\ &\quad\leq \int_{B(0, 1-2^{-k})} |f(z) - f(P_k(z))|\, d\Phi_k(z) + 2^{-k}\Phi_k\big[B^0_{n_k}(0,1)\big]\\ &\quad\leq 2^{-k+2}\alpha(m).\end{aligned}$$

Consequently, $\lim_{k\to\infty} F_1(\Phi_k, \Psi_k) = 0$.

On the other hand, since $\Psi_k(\mathbf{R}^m) = \Psi_k(B_m(0, 1 - 2^{-k})) \leq 2\alpha(m)$, some subsequence of the sequence Ψ_k converges to a measure Ψ over $\mathbf{R}^m$. We easily see that $0 \in \operatorname{spt}\Psi$ and $\Psi(B_m(z,r)) = \alpha(m) r^m$ whenever $z \in \operatorname{spt}\Psi$, $\|z\| < 1$, and $r < 1 - \|z\|$. Using 3.3(1), we see that $\Psi \llcorner B^0_m(0,1) = \mathscr{L}^m \llcorner B^0_m(0,1)$. Hence

$$\lim_{k\to\infty} F_1(\Phi_k, \mathscr{H}^m \llcorner \mathbf{R}^m) = 0 < \tau F_1(\mathscr{H}^m \llcorner \mathbf{R}^m)/2,$$

which, in view of 1.10(5), gives a contradiction.

To prove (ii), we note that $z \in (x + V) \cap B(x, r)$ and $B(z, 2\tau^{1/(m+1)} r) \cap \operatorname{spt}\Phi = \varnothing$ imply $\tau < 1$ and

$$d_1(T_{x,r}[\Phi], \mathfrak{M}_{n,V}) \geq F_{\tau^{1/(m+1)}}(\mathscr{L}^m)/F_1(\mathscr{L}^m) = \tau.$$

3.14. Theorem. *For every nonnegative integer m there is a constant $\omega(m) \in (0,1)$ such that the following two statements hold.*

(1) *If $n \geq m \geq 0$ are integers, $\Phi \in \mathfrak{U}(n)$, $\dim_\infty \Phi = m$, and $m \leq 2$, or $m = n$, or there is $W \in G(n, n-m)$ such that* $\operatorname{Trace}(b_2^{(1)}(\Phi) \llcorner W) < \omega(m)$, *then Φ is flat at ∞.*

(2) *If* $n \geq m \geq 0$ *are integers,* Φ *is a uniformly distributed measure over* $\mathbf{R}^n$, $\dim_\infty \Phi = m$, *and* Φ *is curved at* ∞, *then*

(i) $n > m \geq 3$,

(ii) $\operatorname{Trace}(b_2^{(1)}(\Phi) \llcorner W) \geq \omega(m)$ *for every* $W \in G(n, n-m)$,

(iii) $\liminf_{r \to \infty} d_r(\Phi, \mathfrak{M}_n) \geq \omega(m)/3(m+2)$, *and*

(iv) *there is* $r_0 > 0$ *such that*

(a) $[B(0, r) - B(V, \omega(m)r/(m+2))] \cap \operatorname{spt} \Phi \neq \varnothing$ *for each* $r > r_0$ *and each* $V \in \bigcup_{j=1}^m G(n, j)$, *and*

(b) $[V \cap B(0, r)] - B(\operatorname{spt} \Phi, \omega^2(m)r/(m+2)^6) \neq \varnothing$ *for each* $r > r_0$ *and each* $V \in \bigcup_{j=m}^n G(n, j)$.

Proof. We show that the statement holds with

$$\omega(m) = \kappa\big(m, [2(m+1)]^{-m-1}\big),$$

where $\kappa(m, \tau)$ are the constants from 3.13.

Suppose that $\Phi \in \mathfrak{U}(n)$ and $\dim_\infty \Phi = m$. Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$ be the eigenvalues of $b_2^{(1)}(\Phi)$ and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of $\mathbf{R}^n$ formed by the corresponding eigenvectors. Let $\tilde{V}$ be the linear span of $e_1, \ldots, e_m$ and let $\Psi = c \lim_{r \to \infty} T_{0,r}[\Phi]/r^m$, where $c > 0$ is chosen so that $\Psi(B(0, t)) = \alpha(m)t^m$ for each $t > 0$. (See 3.12(4), 3.11(1; i), and 3.10(2).)

(1) We claim that $\alpha_j = 0$ if $j > m$.

If $m = 0$, this follows from $\sum_{j=1}^n \alpha_j = \operatorname{Trace} b_2^{(1)} = 0$.

If $m = 1$, we note that $\operatorname{spt} \Psi \neq \{0\}$ and we use 3.10(6) to find $y \in \operatorname{spt} \Psi$ with $\|y\| = 1$. Hence $\alpha_1 \geq b_2^{(1)}(y^2) = 1$ (3.10(5)) and, since $\sum_{k=1}^n \alpha_k = 1$, $\alpha_j = 0$ for $j = 2, 3, \ldots, n$.

If $m = 2$, we use the same argument as in the case $m = 1$ to find $y_1 \in \operatorname{spt} \Psi$ with $\|y_1\| = 1$. From 3.10(3) we deduce that $\Psi\{z \in \mathbf{R}^n;\ |\langle z, y_1\rangle| \leq 1\} = \infty$. Using also 3.10(6), we infer that there is $y_2 \in \operatorname{spt} \Psi$ such that $\|y_2\| = 1$ and $\langle y_1, y_2\rangle = 0$. Hence $\alpha_1 + \alpha_2 \geq b_2^{(1)}(y_1^2) + b_2^{(1)}(y_2^2) = 2$ (3.10(5)) and, since $\sum_{k=1}^n \alpha_k = 2$, $\alpha_j = 0$ for $j = 3, 4, \ldots, n$.

If $m = n$, the validity of our claim is obvious.

Finally, suppose that $3 \leq m < n$ and that $\sum_{j=m+1}^n \alpha_j < \omega(m)$. From 3.12(7), 3.11(1) and 2.11(2) we infer that

$$b_2^{(1)}(x^2) = (m+2)\alpha(m)^{-1} \int_{B(0,1)} \langle z, x\rangle^2 \, d\Psi(z)$$

for every $x \in \mathbf{R}^n$. Hence

$$\int_{B(0,1)} \operatorname{dist}^2(z, \tilde{V}) \, d\Psi(z) \leq \alpha(m) \sum_{j=m+1}^n \alpha_j$$

$$< \kappa\big(m, [2(m+1)]^{-m-1}\big)\alpha(m)$$

and 3.13(ii) shows that there is $x \in \operatorname{spt} \Psi$ such that $\|x - e_m\| \leq 1/(m+1)$. If

$\alpha_m < 1$, we note that

$$\alpha_i - 1 \le (m-1)(1-\alpha_m) \quad \text{for } i = 1, \dots, m-1, \quad \text{and}$$
$$\alpha_j \le \alpha_m < 1 \qquad \text{for } j = m+1, \dots, n.$$

Since $x \in \operatorname{spt} \Psi$, we use 3.10(5) to infer

$$0 = \sum_{i=1}^{n} (\alpha_i - 1)\langle x, e_i \rangle^2 \le \sum_{i=1}^{m} (\alpha_i - 1)\langle x, e_i \rangle^2$$
$$\le (m-1)(1-\alpha_m) \sum_{i=1}^{m-1} \langle x, e_i \rangle^2 + (\alpha_m - 1)\langle x, e_m \rangle^2$$
$$\le (1-\alpha_m)\left[(m-1)(m+1)^{-2} - (1 - 1/(m+1))^2\right]$$
$$= -(1-\alpha_m)(m^2 - m + 1)(m+1)^{-2} < 0.$$

Since this is impossible, $\alpha_m \ge 1$. Using also $\sum_{j=1}^{n} \alpha_j = m$, we finish the proof of our claim.

From 1.11(3) we infer that

$$2 \int e^{-\|z\|^2} \operatorname{dist}^2(z, \tilde{V})\, d\Psi(z) \Big/ \int e^{-\|z\|^2}\, d\Psi(z)$$
$$= \lim_{s \searrow 0} s^{-1} \operatorname{Trace}\left(b_{2,s}(\Phi) \llcorner \tilde{V}^{\perp}\right)$$
$$= \operatorname{Trace}\left(b_2^{(1)} \llcorner \tilde{V}^{\perp}\right) = 0.$$

Hence $\operatorname{spt} \Psi \subset \tilde{V}$ and 3.12(3) implies that $\Psi = \mathscr{H}^m \llcorner \tilde{V}$. Thus Φ is flat at ∞.

(2) The statements (i) and (ii) follow from (1).

Before proving (iii) and (iv) we note that (1) implies:

($*$) If $\int_{B(0,1)} \operatorname{dist}^2(z, V)(1 - \|z\|)\, d\Psi(z) < 2\omega(m)F_1(\Psi)/3(m+2)$ for some $V \in G(n, m)$, then Φ is flat at ∞.

In fact, using $m \ge 3$, 3.12(7), and 2.11(2), we see that

$$\operatorname{Trace}\left(b_2^{(1)}(\Phi) \llcorner V^{\perp}\right) = (m+2)(m+3)(m+1)^{-1}$$
$$\times \int_{B(0,1)} \operatorname{dist}^2(z, V)(1 - \|z\|)\, d\Psi(z)/F_1(\Psi) < \omega(m).$$

(iii) If $d_1(\Psi, \mathfrak{M}_n) = \lim_{r \to \infty} d_r(\Phi, \mathfrak{M}_n) < \omega(m)/3(m+2)$, we deduce from 3.12(8) that there is $V \in G(n, m)$ such that

$$d_1(\Psi, \mathfrak{M}_{n,V}) < \omega(m)/3(m+2).$$

Hence

$$\int_{B(0,1)} \operatorname{dist}^2(z, V)(1 - \|z\|)\, d\Psi(z)/F_1(\Psi)$$
$$\leq \int_{B(0,1)} \operatorname{dist}(z, V)(1 - \|z\|)\, d\Psi(z)/F_1(\Psi)$$
$$\leq 2d_1(\Psi, \mathfrak{M}_{n,V}) < 2\omega(m)/3(m+2)$$

and the statement follows from $(*)$.

(iv) Clearly, it suffices to prove (a) and (b) for $V \in G(n, m)$ only. If (a) does not hold, there is $V \in G(n, m)$ such that $B^0(0, 1) \cap \operatorname{spt} \Psi \subset B(V, \omega(m)/(m+2))$. Hence

$$\int_{B(0,1)} \operatorname{dist}^2(z, V)(1 - \|z\|)\, d\Psi(z) \leq \omega(m)^2 F_1(\Psi)/(m+2)^2$$
$$< 2\omega(m) F_1(\Psi)/3(m+2).$$

If (b) does not hold, we denote $s = \omega(m)/(m+2)^3$ and we infer from 3.3(3) that there is $V \in G(n, m)$ such that $B(\operatorname{spt} \Psi, s^2) \supset V \cap B(0,1)$. Hence

$$\Psi\{z \in B(0,1); \operatorname{dist}(z, V) \leq s\} \geq \alpha(m)^{-1} s^{-m} \int_{V \cap B(0, 1-s)} \Psi(B(z, s))\, d\mathscr{H}^m(z)$$
$$\geq s^{-m}(1-s)^m \alpha(m)(s - s^2)^m$$
$$= \alpha(m)(1-s)^{2m} \geq \alpha(m)(1 - 2ms).$$

Thus

$$\int_{B(0,1)} \operatorname{dist}^2(z, V)(1 - \|z\|)\, d\Psi(z) \leq s^2 F_1(\Psi) + 2ms\alpha(m)/4$$
$$= s^2 F_1(\Psi) + m(m+1)sF_1(\Psi)/2$$
$$< 2\omega(m)F_1(\Psi)/3(m+2).$$

Consequently, the statement follows from $(*)$.

3.15. THEOREM. *If $\Phi \in \mathfrak{U}(n)$ is flat at ∞ and $\dim_\infty \Phi = m$, then there are orthogonal projections P and Q of $\mathbf{R}^n$ onto some $V \in G(n, m)$ and onto $V^\perp$, respectively, such that*

(i) $\operatorname{Tan}(\Phi, \infty) = \mathfrak{M}_{n,V}$.
(ii) $b_{2k}^{(k)}(x^{2k}) = \|P(x)\|^{2k}$ *for each* $x \in \mathbf{R}^n$ *and each* $k = 1, 2, \ldots$.
(iii) $b_{2k-1}^{(k)} \llcorner V = 0$ *for each* $k = 1, 2, \ldots$.
(iv) $\operatorname{spt} \Phi \subset \{x \in \mathbf{R}^n;\ b_1^{(1)}(x) = \|Q(x)\|^2\}$.
(v) $\operatorname{spt} \Phi \subset \{x \in \mathbf{R}^n;\ \|Q(x)\| \leq \|b_1^{(1)}\|\}$.

(vi) *There is* $r_0 > 0$ *such that* $B(u, r_0) \cap \operatorname{spt} \Phi \neq \varnothing$ *for every* $u \in V$.

(vii) $b_k^{(j)}(u^i \odot v^{k-i}) = 0$ *whenever* $u \in V$, $v \in V^{\perp}$, $k = 1, 2, \ldots$, $j = 1, 2, \ldots, k-1$, *and* $i = 0, 1, \ldots, 2(k-j) - 1$.

(viii)
$$\sum_{j=1}^{2k} b_j^{(k+1)}(z^j) + (2k+1) b_{2k+1}^{(k+1)}\big((P(z))^{2k} \odot Q(z)\big) - \sum_{j=0}^{k} \binom{k+1}{j} \|P(z)\|^{2j} \big(b_1^{(1)}[Q(z)]\big)^{k+1-j} = 0$$

for every $z \in \operatorname{spt} \Phi$ *and every* $k = 1, 2, \ldots$.

(ix) *For every* $u \in V$ *there is* $v \in V^{\perp}$ *such that* $b_1^{(1)}(v) = \|v\|^2$ *and*

$$b_{2k}^{(k+1)}(u^{2k}) + (2k+1) b_{2k+1}^{(k+1)}(u^{2k} \odot v) - (k+1)\|u\|^{2k} b_1^{(1)}(v) = 0$$

for each $k = 1, 2, \ldots$.

(x) $4\|b_1^{(1)}\|^2/(m+2) \leq \operatorname{Trace}(b_2^{(2)}) \leq 2\|b_1^{(1)}\|^2$, *and*

(xi) *If* $b_1^{(1)} = 0$ *then* $b_k^{(j)} = 0$ *whenever* $k = 1, 2, \ldots$, $j = 1, 2, \ldots$, *and* $k \neq 2j$.

Proof. Let $\Psi = \lim_{r \to \infty} T_{0,r}[\Phi]/\alpha(m) r^m$. (See 3.11(1; ii).) Since Φ is flat at ∞ and $\dim_0 \Psi = m$ (3.12(4)), there are $V \in G(n, m)$ and $C > 0$ such that $\Psi = C \mathscr{H}^m \llcorner V$. Let P and Q be the orthogonal projections onto V and onto $V^{\perp}$, respectively. Then (i) holds and (ii) follows from 3.12(7) and 3.10(4, 5), since $\operatorname{spt} \Psi = V$.

(iii) From (ii) and 3.6(1; iii) we see that there is $c > 0$ such that

$$b_{2k-1}^{(k)}(x^{2k-1}) \geq \|x\|^{2k} - \|P(x)\|^{2k} - c(1 + \|x\|^{2k-2})$$

if $x \in \operatorname{spt} \Phi$. Hence 3.3(3) implies that $b_{2k-1}^{(k)}(u^{2k-1}) \geq 0$ for every $u \in V$.

(iv) Use 3.6(1; iii) with $q = 1$.

(v) This statement follows from (iv), since $b_1^{(1)}(x) = b_1^{(1)}(Q(x))$ according to (iii).

(vi) Let $\kappa = \kappa(m, 4^{-m-1})$. (See 3.13.) Let $\tilde{r} > 0$ be such that

$$C(1-\kappa)\alpha(m) s^m \leq \Phi(B(0, s)) \leq C(1+\kappa)\alpha(m) s^m$$

for every $s > \tilde{r}$. Finally, let $r_0 = \max(\tilde{r}, 2\|b_1^{(1)}\|(1+\kappa))/\kappa$.

If $u \in V$ and $B(u, r_0) \cap \operatorname{spt} \Phi = \varnothing$, we find the smallest $r > 0$ such that $B(u, r) \cap \operatorname{spt} \Phi \neq \varnothing$. Let $x \in B(u, r) \cap \operatorname{spt} \Phi$. Since (v) implies

$$\int_{B(x,r)} \operatorname{dist}^2(z - x, V)\, d\Phi(z) \leq 4\|b_1^{(1)}\|^2 \Phi(B(x, r)) \leq \kappa C \alpha(m) r^{m+2},$$

we infer from 3.13(ii) that

$$B^0(u, r) \cap \operatorname{spt} \Phi \supset B(x + u - P(x), r/2) \cap \operatorname{spt} \Phi \neq \varnothing.$$

But this contradicts the minimality of r.

(vii) We use (v) and 3.2(4) to estimate

$$\lim_{s \searrow 0} \left| s^{k-j} I(s)^{-1} \int e^{-s\|z\|^2} \langle z, u \rangle^i \langle z, v \rangle^{k-i} \, d\Phi(z) \right|$$

$$\leq \|b_1^{(1)}\|^{k-i} \|v\|^{k-i} \|u\|^i \lim_{s \searrow 0} s^{k-j} I(s)^{-1} \int e^{-s\|z\|^2} \|z\|^i \, d\Phi(z)$$

$$\leq 5^n i^{i/2} \|b_1^{(1)}\|^{k-i} \|v\|^{k-i} \|u\|^i \lim_{s \searrow 0} s^{k-j-i/2} = 0.$$

(viii) For every $z \in \operatorname{spt} \Phi$ we have

$$\sum_{j=0}^{2k+2} b_j^{(k+1)}(z^j) - \sum_{j=0}^{k+1} \binom{k+1}{j} \|P(z)\|^{2j} \|Q(z)\|^{2(k+1-j)} = 0$$

(see 3.6(1; iii)). Moreover, (ii), (iii), (iv), and (vii) imply

$$b_{2k+2}^{(k+1)}(z^{2k+2}) = \|P(z)\|^{2k+2}, \qquad \|Q(z)\|^2 = b_1^{(1)}(Q(z)), \quad \text{and}$$

$$b_{2k+1}^{(k+2)}(z^{2k+1}) = (2k+1) b_{2k+1}^{(k+1)}\left((P(z))^{2k} \odot Q(z)\right).$$

(ix) For every $u \in V$ we use (vi) to find sequences t_i of positive numbers, $u_i \in V$, and $v_i \in V^\perp$ such that $\lim_{i \to \infty} t_i = \infty$, $t_i u + u_i + v_i \in \operatorname{spt} \Phi$, and $\|u_i + v_i\| \leq r_0$. Since $\|v_i\| \leq r_0$, we may also assume that the sequence v_i converges to some $v \in V^\perp$.

For each $k = 1, 2, \ldots$ we use $t_i u + u_i + v_i \in \operatorname{spt} \Phi$ and (viii) to infer that

$$\sum_{j=1}^{2k} b_j^{(k+1)}\left((t_i u + u_i + v_i)^j\right) + (2k+1) b_{2k+1}^{(k+1)}\left((t_i u + u_i)^{2k} \odot v_i\right)$$

$$- \sum_{j=0}^{k} \binom{k+1}{j} \|t_i u + u_i\|^{2j} \left(b_1^{(1)}(v_i)\right)^{k+1-j} = 0.$$

Dividing this equation by t_i^{2k} and taking the limit $i \to \infty$, we get (ix).

(x) Let $b \in V^\perp$ be such that $b_1^{(1)}(x) = 2\langle b, x \rangle$ for every $x \in \mathbf{R}^n$. (See (iii).) Since (ii) implies that $b_2^{(1)} \llcorner V^\perp = 0$,

$$\operatorname{Trace}\left(b_2^{(2)} \llcorner V^\perp\right) = \lim_{s \searrow 0} 4 \int e^{-s\|z\|^2} \|Q(z)\|^2 \, d\Phi(z) / I(s).$$

Using also (iv), we infer that

$$\operatorname{Trace}\left(b_2^{(2)} \llcorner V^{\perp}\right) = \lim_{s \searrow 0} 8 \int e^{-s\|z\|^2} \langle z, b\rangle \, d\Phi(z)/I(s)$$
$$= 4 b_1^{(1)}(b) = 2\|b_1^{(1)}\|^2.$$

Since $\operatorname{Trace}(b_2^{(2)}) = \operatorname{Trace}(b_2^{(2)} \llcorner V) + \operatorname{Trace}(b_2^{(2)} \llcorner V^{\perp})$, we need to show only that

$$(*) \qquad 0 \geq \operatorname{Trace}\left(b_2^{(2)} \llcorner V\right) \geq -2m\|b_1^{(1)}\|^2/(m+2).$$

Since this is obvious if $m = 0$, we shall assume that $m \geq 1$.

Let w be the linear map of $\bigodot_2 V$ into $V^{\perp}$ defined by the formula

$$\langle w(u^2), v\rangle = 3b_3^{(2)}(u^2 \odot v) - 4\|u\|^2 \langle b, v\rangle$$

for every $v \in V^{\perp}$. Also, let $\hat{b} \in \bigodot^2 V$ be defined by the formula

$$\hat{b}(u^2) = b_2^{(2)}(u^2) + \langle w(u^2), b\rangle \qquad \text{for every } u \in V.$$

Whenever $u \in V$ and $\hat{v} \in V^{\perp}$, we note that

$$\hat{b}(u^2) + \langle w(u^2), \hat{v}\rangle = b_2^{(2)}(u^2) + 3b_3^{(2)}(u^2 \odot (b + \hat{v}))$$
$$- 2\|u\|^2 b_1^{(1)}(b + \hat{v}).$$

If $v \in V^{\perp}$ fulfils (ix), we let $\hat{v} = v - b$ and we infer that $\|\hat{v}\| = \|b\|$ and $\hat{b}(u^2) + \langle w(u^2), \hat{v}\rangle = 0$. Consequently,

$$(\alpha) \qquad \left(\hat{b}(u^2)\right)^2 \leq \left\|w(u^2)\right\|^2 \|b\|^2 \quad \text{for each } u \in V.$$

Whenever $z \in \operatorname{spt} \Phi$, we may use the case $k = 1$ of (viii) to conclude that

$$0 = b_1^{(2)}(z) + b_2^{(2)}(z^2) + 3b_3^{(2)}\left((P(z))^2 \odot Q(z)\right)$$
$$- \left(b_1^{(1)}(Qz)\right)^2 - 2\left\|P(z)\right\|^2\left(b_1^{(1)}(Q(z))\right)$$
$$= \hat{b}\left((P(z))^2\right) + \left\langle w\left((P(z))^2\right), Q(z) - b\right\rangle + b_1^{(2)}(z)$$
$$+ 2b_2^{(2)}(P(z) \odot Q(z)) + b_2^{(2)}\left((Q(z))^2\right) - \left(b_1^{(1)}(Q(z))\right)^2.$$

Hence there is $K \in (0, \infty)$ such that

$$\left|\hat{b}\left((P(z))^2\right) + \left\langle w\left((P(z))^2\right), Q(z) - b\right\rangle\right| \leq K(\|z\| + 1)$$

for every $z \in \operatorname{spt} \Phi$. Using 3.2(4), we infer that

$$(\beta)\quad \lim_{s \searrow 0} \left| s \int e^{-s\|z\|^2} \left[\hat{b}\big((P(z))^2\big) + \left\langle w\big((P(z))^2\big), Q(z) - b \right\rangle \right] d\Phi(z)/I(s) \right|$$

$$\leq \lim_{s \searrow 0} Ks \int e^{-s\|z\|^2}(\|z\| + 1)\, d\Phi(z)/I(s)$$

$$\leq \lim_{s \searrow 0} (5^n K s^{1/2} + Ks) = 0.$$

Next we prove that

$$(\gamma)\qquad \lim_{s \searrow 0} s \int e^{-s\|z\|^2} \| P(z) \|^2 \langle z - b, v \rangle\, d\Phi(z)/I(s) = 0$$

for every $v \in V^{\perp}$.

In fact, the limit in question exists, since, in view of 3.6(1; i, ii), it can be computed with the help of $b_2^{(1)}$ and $b_3^{(2)}$. On the other hand, it is equal to

$$(\gamma')\qquad \lim_{s \searrow 0} C^{-1} \pi^{-m/2} s^{1+m/2} \int e^{-s\|z\|^2} \|z\|^2 \langle z - b, v \rangle\, d\Phi(z),$$

since

$$\lim_{s \searrow 0} s \int e^{-s\|z\|^2} \| Q(z) \|^2 | \langle z - b, v \rangle |\, d\Phi(z)/I(s)$$

$$\leq \lim_{s \searrow 0} \left(s \|b_1^{(1)}\|^2 \|b\|\, \|v\| \right) = 0$$

because of (v) and (iv), and since $\lim_{s \searrow 0} s^{m/2} I(s) = C\pi^{m/2}$. Since $m > 0$, we may compute (γ') using the L'Hôpital rule and

$$\lim_{s \searrow 0} C^{-1} \pi^{-m/2} s^{m/2} \int e^{-s\|z\|^2} \langle z - b, v \rangle\, d\Phi(z)$$

$$= b_1^{(1)}(v)/2 - \langle b, v \rangle = 0.$$

Whenever $u \in V$ and $v \in V^{\perp}$, we use (ii) and the definition of $b_3^{(2)}$ and $b_2^{(1)}$ to infer that

$$(\delta)\qquad \lim_{s \searrow 0} 8s \int e^{-s\|z\|^2} \langle z, u \rangle^2 \langle z - b, v \rangle\, d\Phi(z)/I(s)$$

$$= 3 b_3^{(2)}(u^2 \odot v) - 4\|u\|^2 \langle b, v \rangle = \langle w(u^2), v \rangle.$$

Let γ be the measure over V defined by the formula $\gamma(E) = (2\pi)^{-m/2} \int_E e^{-\|z\|^2/2}\, d\mathscr{H}^m(z)$. Since $\int_V \langle y, u \rangle^4\, d\gamma(u) = 3\|y\|^4$ for each $y \in V$, the polarization formula implies that

$$\int_V \langle y, u\rangle^2 \langle z, u\rangle^2 \, d\gamma(u) = 2\langle y, z\rangle^2 + \|y\|^2\|z\|^2$$

for each pair y, $z \in V$. Consequently, whenever $y \in V$ and $v \in V^\perp$, we may use (δ) and (γ) to compute

$$\begin{aligned}
&\int_V \langle y, u\rangle^2 \langle w(u^2), v\rangle \, d\gamma(u) \\
&\quad = \lim_{s \searrow 0} 8s \int e^{-s\|z\|^2} \langle z - b, v\rangle \left[\int_V \langle y, u\rangle^2 \langle z, u\rangle^2 \, d\gamma(u)\right] d\Phi(z)/I(s) \\
&\quad = \lim_{s \searrow 0} 16s \int e^{-s\|z\|^2} \langle z - b, v\rangle \langle z, y\rangle^2 \, d\Phi(z)/I(s) \\
&\qquad + \lim_{s \searrow 0} 8s\|y\|^2 \int e^{-s\|z\|^2} \|P(z)\|^2 \langle z - b, v\rangle \, d\Phi(z)/I(s) \\
&\quad = 2\langle w(y^2), v\rangle.
\end{aligned}$$

Using the last result, (α), (δ), (β), 1.11(3) and $\Psi = C\mathscr{H}^m \llcorner V$, we obtain

$$\begin{aligned}
(\varepsilon)\quad &\int_V \left(\hat{b}(u^2)\right)^2 d\gamma(u) \\
&\quad \le \|b\|^2 \int_V \|w(u^2)\|^2 \, d\gamma(u) \\
&\quad = \|b\|^2 \lim_{s \searrow 0} \int_V \left[8s \int e^{-s\|z\|^2} \langle z, u\rangle^2 \langle z - b, w(u^2)\rangle \, d\Phi(z)/I(s)\right] d\gamma(u) \\
&\quad = \|b\|^2 \lim_{s \searrow 0} 8s \int e^{-s\|z\|^2} \left[\int_V \langle z, u\rangle^2 \langle z - b, w(u^2)\rangle \, d\gamma(u)\right] d\Phi(z)/I(s) \\
&\quad = \|b\|^2 \lim_{s \searrow 0} 16s \int e^{-s\|z\|^2} \left\langle w\left((P(z))^2\right), Q(z) - b\right\rangle d\Phi(z)/I(s) \\
&\quad = -16\|b\|^2 \lim_{s \searrow 0} s \int e^{-s\|z\|^2} \hat{b}\left((P(z))^2\right) d\Phi(z)/I(s) \\
&\quad = -8\|b\|^2 \lim_{s \searrow 0} s \int e^{-s\|z\|^2/2} \hat{b}\left((P(z))^2\right) d\Phi(z)/I(s/2) \\
&\quad = -8\|b\|^2 \int e^{-\|z\|^2/2} \hat{b}\left((P(z))^2\right) d\Psi(z) \Big/ \int e^{-\|z\|^2/2} \, d\Psi(z) \\
&\quad = -8\|b\|^2 \int_V \hat{b}(u^2) \, d\gamma(u).
\end{aligned}$$

Let $\hat{e}_1, \ldots, \hat{e}_m$ be an orthonormal basis of V formed by the eigenvectors of $\hat{b}$ and let $\beta_1, \ldots, \beta_m$ be the corresponding eigenvalues. Then

$$\begin{aligned}\int_V \left(\hat{b}(u^2)\right)^2 d\gamma(u) &= 3\sum_{i=1}^m \beta_i^2 + 2\sum_{1\le i<j\le m} \beta_i\beta_j \\ &= \left(\sum_{i=1}^m \beta_i\right)^2 + 2\sum_{i=1}^m \beta_i^2 \ge (1+2/m)\left(\sum_{i=1}^m \beta_i\right)^2 \\ &= (1+2/m)\left(\int_V \hat{b}(u^2)\, d\gamma(u)\right)^2.\end{aligned}$$

Using this inequality in (ε), we obtain

$$(1+2/m)\left(\int_V \hat{b}(u^2)\, d\gamma(u)\right)^2 \le -8\|b\|^2 \int_V \hat{b}(u^2)\, d\gamma(u).$$

Hence

$$0 \ge \int_V \hat{b}(u^2)\, d\gamma(u) \ge -8m\|b\|^2/(m+2) = -2m\|b_1^{(1)}\|^2/(m+2).$$

Since, according to (γ) and (δ),

$$\begin{aligned}&\operatorname{Trace}\left(b_2^{(2)} \llcorner V\right) \\ &= \int_V b_2^{(2)}(u^2)\, d\gamma(u) \\ &= \int_V \left[b_2^{(2)}(u^2) + \lim_{s\searrow 0} 8s\int e^{-s\|z\|^2}\langle z, u\rangle^2 \langle z-b, b\rangle\, d\Phi(z)/I(s)\right] d\gamma(u) \\ &= \int_V \left[b_2^{(2)}(u^2) + \langle w(u^2), b\rangle\right] d\gamma(u) = \int_V \hat{b}(u^2)\, d\gamma(u),\end{aligned}$$

$$0 \ge \operatorname{Trace}\left(b_2^{(2)} \llcorner V\right) \ge -2m\|b_1^{(1)}\|^2/(m+2).$$

Thus $(*)$ holds and the proof of (x) is finished.

(xi) If $b_1^{(1)} = 0$, (iv) implies that $\operatorname{spt}\Phi \subset V$. Hence $b_{k,s}(x^k) = b_{k,s}((P(x))^k)$ for every $k = 1, \ldots,$ every $s > 0$, and every $x \in \mathbf{R}^n$. It follows that $b_k^{(j)}(x^k) = b_k^{(j)}((P(x))^k)$ for every $x \in \mathbf{R}^n$, every j, and every k.

Whenever $j = 1, 2, \ldots$ and $x \in \operatorname{spt}\Phi$, we use

$$\sum_{k=1}^{2j} b_k^{(j)}(x^k) - \|x\|^{2j} = 0 \qquad (3.6(1;\text{iii})),$$

$$b_{2j}^{(j)}(x^{2j}) = \|P(x)\|^{2j} \qquad \text{(ii), and}$$

$$\|x\|^{2j} = \|P(x)\|^{2j} \qquad (\operatorname{spt}\Phi \subset V)$$

to conclude that

$$\sum_{k=1}^{2j-1} b_k^{(j)}(x^k) = 0.$$

Hence (vi) and spt $\Phi \subset V$ imply that $b_k^{(j)} \llcorner V = 0$ whenever $1 \leq k \leq 2j - 1$. Since $b_k^{(j)}(x^k) = b_k^{(j)}((P(x))^k)$ for every $x \in \mathbf{R}^n$, this proves the statement of (xi) if $1 \leq k \leq 2j - 1$. If $k \geq 2j + 1$, the statement holds because of 3.6(1; ii).

3.16. Corollary. *If $0 \leq m \leq n$ are integers, $\Phi \in \mathfrak{U}(n)$ is flat at ∞, $\dim_\infty \Phi = m$, and*

$$\Phi(B(0, r))/r^m \geq \lim_{s \to \infty} \Phi(B(0, s))/s^m$$

for every $r > 0$, then there is $V \in G(n, m)$ such that Φ is a constant multiple of $\mathscr{H}^m \llcorner V$.

Proof. Let $h(r) = \Phi(B(0, r))$, $C = \lim_{r \to \infty} \Phi(B(0, r))/r^m$, and $f(s) = s^{1+m/2} \int_0^\infty e^{-sr^2} r h(r)\, dr$. Using

$$\begin{aligned} 4f'(s) &= s^{-1+m/2} \int_0^\infty \left[\frac{d}{dr}\left(e^{-sr^2}(2sr^2 - m)\right)\right] h(r)\, dr \\ &= -s^{-1+m/2} \int e^{-s\|z\|^2}(2s\|z\|^2 - m)\, d\Phi(z) \end{aligned}$$

and $\mathrm{Trace}(b_2^{(1)}) = m$ (3.12(7)), we infer that

$$\mathrm{Trace}\left(s^{-1} b_{2,s}\right) - \mathrm{Trace}\left(b_2^{(1)}\right) = -4I(s)^{-1} s^{1-m/2} f'(s).$$

Since $\lim_{s \searrow 0} s^{m/2} I(s) = C\alpha(m)^{-1} \pi^{m/2}$ (3.11(1), 3.2(3), and 1.11(3), the L'Hôpital rule implies

$$\begin{aligned} \mathrm{Trace}\left(b_2^{(2)}\right) &= -8C^{-1}\alpha(m)\pi^{-m/2} \lim_{s \searrow 0} f'(s) \\ &= -8C^{-1}\alpha(m)\pi^{-m/2} \lim_{s \searrow 0} s^{m/2} \int_0^\infty e^{-sr^2} r^{m+1}(r^{-m}h(r) - C)\, dr. \end{aligned}$$

Since $\mathrm{Trace}(b_2^{(2)}) \geq 0$ according to 3.15(x), we infer that $\mathrm{Trace}(b_2^{(2)}) = 0$. Hence 3.15(x, xi) imply that $b_1^{(1)} = 0$ and $\mathrm{Trace}(b_2^{(k)}) = 0$ for each $k \geq 2$. Using the last fact with $k \geq 2 + m/2$, we conclude that

$$\begin{aligned} 0 &= -\mathrm{Trace}\left(b_2^{(k)}\right)/\left(4k! C^{-1}\alpha(m)\pi^{-m/2}\right) \\ &= \lim_{s \searrow 0} \int_0^\infty s^{-k+2+m/2} e^{-sr^2} r^{m+1}(r^{-m}h(r) - C)\, dr \\ &\geq \int_0^\infty r^{m+1}(r^{-m}h(r) - C)\, dr. \end{aligned}$$

Since h is right continuous, $h(r) = Cr^m$ for each $r > 0$. Consequently, $\dim_0\Phi = m$.

Finally, we use $b_1^{(1)} = 0$ and 3.15(iv) to find $V \in G(n, m)$ such that $\operatorname{spt}\Phi \subset V$ and we infer from 3.3(1) that Φ is a positive multiple of $\mathscr{H}^m \llcorner V$.

3.17. Corollary. *Suppose that* $0 \le m \le n$, Φ *is a nonzero measure over* $\mathbf{R}^n$, *and* $\Phi(B(x, r)) = \alpha(m) r^m$ *for every* $x \in \operatorname{spt}\Phi$ *and every* $r > 0$. *If* $m = 0$, 1, 2, *or* n, *then there is an* m *dimensional affine subspace* V *of* $\mathbf{R}^n$ *such that* $\Phi = \mathscr{H}^m \llcorner V$.

Proof. Since we may assume that $0 \in \operatorname{spt}\Phi$, the statement follows directly from 3.14(1) and 3.16.

3.18. Theorem. (1) *If* $\Phi \in \mathfrak{U}(n)$, *each of the following four conditions implies that* Φ *is flat at* ∞ *and* $b_1^{(1)}(\Phi) = 0$.

(i) $\Phi(B(z, r)) \le \Phi(B(0, r))$ *for every* $z \in \mathbf{R}^n$ *and every* $r > 0$.

(ii) *Whenever* $D \subset \mathbf{R}^n$ *is a compact convex set with* $0 \in \operatorname{Int}(D)$ *and whenever* $x \in \operatorname{spt}\Phi$, *then* $\Phi(x + D) = \Phi(D)$.

(iii) $x + \operatorname{spt}\Phi = \operatorname{spt}\Phi$ *for every* $x \in \operatorname{spt}\Phi$.

(iv) $b_1^{(1)}(T_{x,1}[\Phi]) = 0$ *for every* $x \in \operatorname{spt}\Phi$.

(2) *If* $\Phi \in \mathfrak{U}(n)$ *is flat at* ∞, $b_1^{(1)}(\Phi) = 0$, *and* $\dim_0\Phi \ge \dim_\infty\Phi$, *then* $\Phi \in \mathfrak{M}_{n, \dim_\infty\Phi}$.

Proof. (1). (i) $\Rightarrow$ (iv). For every $x \in \operatorname{spt}\Phi$ and every $s > 0$, the function $z \mapsto \int e^{-s\|z-x\|^2}\, d\Phi(z)$ attains its maximum at $z = x$. Considering its first derivative, we get

$$\int e^{-s\|z-x\|^2} \langle z - x, u\rangle \, d\Phi(z) = 0$$

for each $u \in \mathbf{R}^n$. Hence (iv) holds.

(ii) $\Rightarrow$ (iii). Let $x \in \operatorname{spt}\Phi$ and $y \in \mathbf{R}^n - \{0\}$. For each $s \in (0, \|y\|/2)$ let D_s be the convex hull of $B(0, s) \cup B(y, s)$ and let $\tilde{D}_s = \{z \in D_s;\ \langle z, y\rangle \le (1 - s)\|y\|^2\}$. Then $\Phi(D_s - \tilde{D}_s) = \Phi(x + (D_s - \tilde{D}_s))$ for each $s \in (0, \|y\|/2)$. Hence $y \in \operatorname{spt}\Phi$ if and only if $x + y \in \operatorname{spt}\Phi$.

(iii) $\Rightarrow$ (iv). Let $x \in \operatorname{spt}\Phi$ and let $\Psi = T_{x,1}[\Phi]$. From the assumption of (iii) we see that $\operatorname{spt}\Psi$ is a symmetric set (i.e., $u \in \operatorname{spt}\Psi$ implies $-u \in \operatorname{spt}\Psi$). For each $u \in \operatorname{spt}\Psi$ we use 3.6(1; iii) to infer that

$$b_1^{(1)}(\Psi)(u) + b_2^{(1)}(\Psi)(u^2) = \|u\|^2, \quad \text{and}$$

$$b_1^{(1)}(\Psi)(-u) + b_2^{(1)}(\Psi)\big((-u)^2\big) = \|-u\|^2.$$

Hence $b_1^{(1)}(\Psi)(u) = 0$ for each $u \in \operatorname{spt}\Psi$. Since $b_1^{(1)}(\Psi)(u) = 0$ whenever u is orthogonal to $\operatorname{spt}\Psi$, this implies (iv).

(iv) For every $x \in \operatorname{spt} \Phi$ and every $u \in \mathbf{R}^n$ we use 3.2(4) and 3.4(2) to compute

$$\begin{aligned} 2I(s)^{-1} &\int e^{-s\|z-x\|^2} \langle z, u\rangle \, d\Phi(z) \\ &= e^{-s\|x\|^2} I(s)^{-1} \int e^{-s\|z\|^2} \sum_{k=0}^{\infty} 2^{k+1} s^k \langle z, x\rangle^k \langle z, u\rangle / k! \, d\Phi(z) \\ &= e^{-s\|x\|^2} \sum_{k=0}^{\infty} (k+1) b_{k+1,s}(x^k \odot u)/s \\ &= b_{1,s}(u)/s + 2b_{2,s}(x \odot u)/s + o(1) \quad \text{as } s \to 0. \end{aligned}$$

Hence

$$b_1^{(1)}(T_{x,1}[\Phi])(u) = b_1^{(1)}(\Phi)(u) + 2b_2^{(1)}(\Phi)(x \odot u) - 2\langle x, u\rangle$$

for every $x \in \operatorname{spt} \Phi$ and every $u \in \mathbf{R}^n$. Thus the assumption of (iv) implies that

$$\operatorname{spt} \Phi \subset V = \{x \in \mathbf{R}^n;\ b_2^{(1)}(x \odot u) = \langle x, u\rangle \quad \text{for every } u \in \mathbf{R}^n\}.$$

Since $b_2^{(1)}$ is a nonnegative quadratic form and since $\operatorname{Trace}(b_2^{(1)}) = \dim_\infty \Phi$, $V \in G(n, m)$ for some $m \le \dim_\infty \Phi$. Hence Φ is flat at ∞ according to 3.14(2; iv, a).

(2) From 3.15(v) we see that there is $V \in G(n, \dim_\infty \Phi)$ such that $\operatorname{spt} \Phi \subset V$. Hence $\Phi \in \mathfrak{M}_{n,V}$ according to 3.12(3).

3.19. Theorem. *If $\Phi \in \mathfrak{U}(n)$ is flat at ∞ and if $\dim_0 \Phi \ge \dim_\infty \Phi = n - 1$, then there are $V \in G(n, n-1)$ and $e \in V^\perp$ such that Φ is a constant multiple of $\mathscr{H}^{n-1} \llcorner (V \cup (e + V))$.*

Proof. If $b_1^{(1)}(\Phi) = 0$, $\Phi \in \mathfrak{M}_{n,n-1}$ according to 3.18(2). If $b_1^{(1)}(\Phi) \ne 0$, we conclude from 3.15(iv) that there are $V \in G(n, n-1)$ and $e \in V^\perp - \{0\}$ such that $\operatorname{spt} \Phi \subset V \cup (e + V)$. Moreover, 3.15(iii) implies that

$$(e + V) \cap \operatorname{spt} \Phi \ne \varnothing.$$

We prove that $\operatorname{spt} \Phi = V \cup (e + V)$. In fact, otherwise we find $x \in V \cup (e + V)$ and $z \in \operatorname{spt} \Phi$ such that $\|z - x\| < \|e\|/4$ and $B^0(x, \|z - x\|) \cap \operatorname{spt} \Phi = \varnothing$. But then 3.3(1) implies that $\Phi \llcorner B^0(z, \|e\|/2)$ is a constant multiple of $\mathscr{H}^m \llcorner (V \cup (e + V))$, which contradicts $x \notin \operatorname{spt} \Phi$. Hence $\operatorname{spt} \Phi = V \cup (e + V)$ and the statement follows from 3.3(2).

3.20. If $C = \{x \in \mathbf{R}^4;\ x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ and if $\Phi = \mathscr{H}^3 \llcorner C$, an elementary calculation shows that $\Phi(B(x, r)) = 4\pi r^3/3$ for every $x \in C$ and every $r > 0$. (See also [14].) This example illustrates the need for the alternative assumption of 3.14(1) as well as the need for considering the special values of m

in 3.17. Moreover, considering the direct product of Φ and $\mathscr{L}^k$ and embedding the resulting measure into a space of higher dimension, we see that 3.17 gives actually all pairs m, n for which its statement holds.

3.21. *Problems.* (1) It is not difficult to give a simple description of $\mathfrak{U}(1)$ and of $\{\Phi \in \mathfrak{U}(2);\ \operatorname{spt}\Phi \text{ is bounded}\}$. (See [13].) To give a similar description of uniformly distributed measures in higher dimensional spaces is a natural, though probably hopeless, question.

(2) A special case of (1) is to describe all measures $\Psi \in \mathfrak{U}(n)$ such that $\Psi(B(0, r)) = \alpha(m)r^m$ for every $r > 0$. If $m = 0, 1, 2$, or n, this is done in 3.17. For $m = n - 1$ the problem is solved in [14], where it is shown that, if Ψ is not flat, then (after a rotation) $\Psi = \Phi \otimes \mathscr{L}^{n-4}$, where Φ is the measure from 3.20. In fact, all the known examples of measures $\Psi \in \mathfrak{U}(n)$ with $\Psi(B(0, r)) = \alpha(m)r^m$ are obtained from 3.20 and from $\mathscr{L}^k$ by direct products.

(3) The reason for the following question should be clear from the results of Chapter 4: If $\Phi \in \mathfrak{U}(n)$ is flat at ∞, $x \in \operatorname{spt}\Phi$, and $\Psi \in \operatorname{Tan}(\Phi, x)$, is Ψ flat at ∞?

4. Approximately uniformly distributed measures over $\mathbf{R}^n$

4.1. Let Φ measure $\mathbf{R}^n$.

(1) For each $r > 0$ and $\varepsilon \in (0, 1)$ we denote by $Y_r(\Phi, \varepsilon)$ the set of all points $x \in \mathbf{R}^n$ such that

$$\Phi\{y \in B(x, r/\varepsilon);\ \Phi(B(y, r/(1+\varepsilon))) \geq (1+\varepsilon)\Phi(B(x, r))\}$$
$$\leq \varepsilon\Phi(B(x, r)) < \infty.$$

(2) Let $\varepsilon \in (0, 1)$. The measure Φ is said to be ε-approximately uniformly distributed if Φ almost every point of $\mathbf{R}^n$ belongs to the set

$$\bigcup_{s>0} \bigcap_{r\in(0,s)} Y_r(\Phi, \varepsilon).$$

(3) The measure Φ is said to be approximately uniformly distributed if it is ε-approximately uniformly distributed for every $\varepsilon \in (0, 1)$.

4.2. We make the following observations.

(1) If Φ is ε-approximately uniformly distributed and if $\varepsilon < \hat{\varepsilon} < 1$, then Φ is $\hat{\varepsilon}$-approximately uniformly distributed.

(2) If $x \in Y_r(\Phi, \varepsilon)$ and $s \in [r, r/\varepsilon]$, then

$$\Phi(B(x, s)) \leq 5^{n+1}(s/r)^n\Phi(B(x, r)).$$

(3) If

$$\Phi\{y \in B(x, r/\varepsilon); \Phi(B(y, r/(1+\varepsilon))) \geq (1+\varepsilon)\Phi(B(x,r))\}$$
$$\leq (\varepsilon/5)^{n+1}\Phi(B(x, r/\varepsilon)) < \infty$$

then $x \in Y_r(\Phi, \varepsilon)$.

(4) If $\Phi(B(x, 2r/\varepsilon)) < \infty$ and

$$\Phi\{y \in B(x, r/\varepsilon); \Phi(B(y, r/(1+\varepsilon)) < (1+\varepsilon)\Phi(B(x,r))\}$$
$$\geq \left(1 - (\varepsilon/5)^{n+1}\right)\Phi(B(x, r/\varepsilon))$$

then $x \in Y_r(\Phi, \varepsilon)$.

(5) If $d_1(T_{x, 2r/\varepsilon}[\Phi], \mathfrak{U}(n)) < (\varepsilon/6)^{n+3}$ then $x \in Y_r(\Phi, \varepsilon)$. Moreover,

$$\Phi\{y \in B(x, r/\varepsilon); \Phi(B(y, r/(1+\varepsilon)) \geq (1+\varepsilon)\Phi(B(x,r)) \quad \text{or}$$
$$\Phi(B(y, (1+\varepsilon)r)) \leq \Phi(B(x,r))/(1+\varepsilon)\}$$
$$\leq \varepsilon\Phi(B(x,r)).$$

Proof. The statement (1) is obvious.

(2)

$$\Phi(B(x,s)) \leq \Phi(B(x,r)) + \Phi\{y \in B(x,s); \Phi(B(y, r/2)) \leq 2\Phi(B(x,r))\}$$
$$\leq \Phi(B(x,r))$$
$$+\alpha(n)^{-1}(r/4)^{-n}\int_{B(x,s+r/4)} \Phi\{y \in B(z, r/4);$$
$$\Phi(B(y, r/2)) \leq 2\Phi(B(x,r))\}\, d\mathscr{L}^n(z)$$
$$\leq \Phi(B(x,r)) + 2(r/4)^{-n}(s + r/4)^n\Phi(B(x,r))$$
$$\leq 5^{n+1}(s/r)^n\Phi(B(x,r)).$$

(3) Similarly as in the proof of (2) we see that

$$\Phi(B(x, r/\varepsilon)) \leq (\varepsilon/5)^{n+1}\Phi(B(x, r/\varepsilon))$$
$$+2(r/4)^{-n}(r/\varepsilon + r/4)^n\Phi(B(x,r)).$$

Hence $\Phi(B(x, r/\varepsilon)) \leq 5^{n+1}\varepsilon^{-n}\Phi(B(x,r))$, which implies $x \in Y_r(\Phi, \varepsilon)$.

(4) The set

$$\{y \in B(x, r/\varepsilon); \Phi(B(y, r/(1+\varepsilon))) < (1+\varepsilon)\Phi(B(x,r))\}$$

is relatively open in $B(x, r/\varepsilon)$. Hence it is Φ measurable and we may use (3).

(5) Let $\tilde{\Phi} = T_{x,2r/\varepsilon}[\Phi]/F_1(T_{x,2r/\varepsilon}[\Phi])$ and let $\Psi \in \mathfrak{U}(n)$ be such that $F_1(\Psi) = 1$ and $F_1(\tilde{\Phi}, \Psi) < (\varepsilon/6)^{n+3}$. Also, let $s = \varepsilon/2$. Since

$$1 = F_1(\Psi) \leq (1 + 12/5s)^n\Psi(B(0, 5s/6)) \leq (6/\varepsilon)^n\Psi(B(0, 5s/6))$$

according to 3.2(1),

$$6F_1(\tilde{\Phi}, \Psi)/(\varepsilon s) \le \varepsilon\Psi(B(0, 5s/6))/3.$$

Hence 1.10(3) implies

$$\begin{aligned}\tilde{\Phi}(B(0, s)) &\ge \Psi(B(0, (1 - \varepsilon/6)s)) - 6F_1(\tilde{\Phi}, \Psi)/(\varepsilon s)\\ &\ge (1 - \varepsilon/3)\Psi(B(0, (1 - \varepsilon/6)s)),\\ \tilde{\Phi}(B(0, s)) &\le (1 + \varepsilon/3)\Psi(B(0, (1 + \varepsilon/6)s)), \quad \text{and}\end{aligned}$$

$$\tilde{\Phi}[B(0, 1/2) - B(\operatorname{spt}\Psi, \varepsilon s/6)] \le \varepsilon\Psi(B(0, (1 - \varepsilon/6)s))/3 \le \varepsilon\tilde{\Phi}(B(0, s)).$$

Whenever $y \in B(0, 1/2) \cap B(\operatorname{spt}\Psi, \varepsilon s/6)$, we find $z \in \operatorname{spt}\Psi$ such that $\|y - z\| \le \varepsilon s/6$ and we use 1.10(3) to infer that

$$\begin{aligned}&\tilde{\Phi}(B(y, s/(1 + \varepsilon)))\\ &\quad\le \tilde{\Phi}(B(z, (1 - \varepsilon/3)s)) \le (1 + \varepsilon/3)\Psi(B(0, (1 - \varepsilon/6)s))\\ &\quad\le (3 + \varepsilon)(3 - \varepsilon)^{-1}\tilde{\Phi}(B(0, s)) < (1 + \varepsilon)\tilde{\Phi}(B(0, s)), \quad \text{and}\\ &\tilde{\Phi}(B(y, (1 + \varepsilon)s))\\ &\quad\ge \tilde{\Phi}(B(z, (1 + \varepsilon/3)s)) \ge (1 - \varepsilon/3)\Psi(B(0, (1 + \varepsilon/6)s))\\ &\quad\ge (3 - \varepsilon)(3 + \varepsilon)^{-1}\tilde{\Phi}(B(0, s)) > \tilde{\Phi}(B(0, s)/(1 + \varepsilon)).\end{aligned}$$

4.3. Proposition. *Suppose that Φ is an almost finite measure over $\mathbf{R}^n$, $\varepsilon \in (0, 1)$, $E \subset \mathbf{R}^n$, h is a positive function on $(0, \infty)$, and Φ almost every $x \in E$ fulfils*

(1)
$$\begin{aligned}0 &< \limsup_{r \searrow 0} \Phi\Big(B\big(x, (1 + \varepsilon)^{-1/2} r\big)\Big)/h(r)\\ &< (1 + \varepsilon) \liminf_{r \searrow 0} \Phi\Big(B\big(x, (1 + \varepsilon)^{1/2} r\big)\Big)/h(r) < \infty\end{aligned}$$

or

(2)
$$\begin{aligned}&\limsup_{r \searrow 0} \Phi(B(x, tr))/\Phi(B(x, r))\\ &\quad\le \min\big(1 + \varepsilon/2, t^{\varepsilon/(9\ln(2/\varepsilon))}\big) \liminf_{r \searrow 0} \Phi(B(x, tr))/\Phi(B(x, r))\end{aligned}$$

for some $t > 1$.

Then Φ almost every point of E belongs to the set $\bigcup_{s > 0} \bigcap_{r \in (0, s)} Y_r(\Phi, \varepsilon)$.

Proof. Let c and d be positive numbers and let

$$E = \Big\{x \in \mathbf{R}^n;\ ch(r) \le \Phi\big(B\big(x, (1+\varepsilon)^{1/2} r\big)\big) \quad \text{and}$$
$$\Phi\big(B\big(x, (1+\varepsilon)^{-1/2} r\big)\big) < c(1+\varepsilon)h(r) \quad \text{whenever } 0 < r < d\Big\}.$$

For almost every $x \in E$ we use 1.7 to find $s \in (0, d)$ such that $\Phi(B(x, r) \cap E) > (1 - (\varepsilon/5)^{n+1})\Phi(B(x, r))$ for each $r \in (0, s)$. If $r \in (0, \varepsilon s)$ and $y \in B(x, r/\varepsilon) \cap E$, then

$$\Phi(B(y, r/(1+\varepsilon))) < c(1+\varepsilon)h\big((1+\varepsilon)^{-1/2} r\big) \le (1+\varepsilon)\Phi(B(x, r)).$$

Hence $x \in Y_r(\Phi, \varepsilon)$ according to 4.2(4). Consequently, Φ almost every point at which (1) holds belongs to

$$\bigcup_{s>0} \bigcap_{r \in (0, s)} Y_r(\Phi, \varepsilon).$$

Let

$$\Delta(t) = \min\big(1+\varepsilon, t^{\varepsilon/(8 \ln(2/\varepsilon))}\big).$$

Whenever c, d are positive numbers and $q = 1, 2, \ldots,$ let

$$F = \{x \in \mathbf{R}^n;\ \text{there is } t > 1 \text{ such that } t^{q-1} < 2\varepsilon^{-2} \le t^q \text{ and } c\Phi(B(x, r)) \le \Phi(B(x, tr)) < c\Delta(t)\,\Phi(B(x, r)) \text{ whenever } r \in (0, d)\}.$$

If $x \in F$, we find the corresponding t and we note that

$$\begin{aligned}
\Phi(B(y, r/(1+\varepsilon))) &\le c^{-q}\Phi(B(y, t^q r/(1+\varepsilon))) \\
&\le c^{-q}\Phi\big(B\big(x, \big(\varepsilon^{-1} + (1+\varepsilon)^{-1} t^q\big) r\big)\big) \\
&\le c^{-q}\Phi(B(x, t^q r)) \\
&\le \Delta(t)^q \Phi(B(x, r))
\end{aligned}$$

whenever $r \in (0, t^{-q} d)$ and $y \in B(x, r/\varepsilon) \cap F$. If $q = 1$, $\Delta(t)^q = \Delta(t) \le 1 + \varepsilon$. If $q > 1$ then $t^q \le 4\varepsilon^{-4} \le (\varepsilon/2)^{-4}$ and $\Delta(t)^q \le (\varepsilon/2)^{-\varepsilon/(2 \ln(2/\varepsilon))} = e^{\varepsilon/2} \le 1 + \varepsilon$. Hence 4.2(4) implies that every point of F which is a Φ density point of F belongs to $\bigcup_{s>0}\bigcap_{r \in (0, s)} Y_r(\Phi, \varepsilon)$.

Finally, we use 2.4 to infer that Φ almost all of the set where (2) holds may be covered by countably many sets of the above form.

4.4. (1) We recall the definition of the d-cones $\mathfrak{M}_n$, $\mathfrak{M}_{n,m}$, $\mathfrak{M}_n[\sigma]$, $\mathfrak{M}_{n,m}[\sigma]$ and $\mathfrak{U}(n)[\sigma]$. (See 3.1(2), 3.7(1), and 2.13(1).) To simplify the notation, we shall write $\mathfrak{U}(n, \sigma)$ instead of $\mathfrak{U}(n)[\sigma]$. We also note that, if $\sigma \in [0, 1)$, each of these d-cones has a compact basis. (See 3.7(2) and 2.13(2; ii).)

We prove the following statements.

(2) $\mathfrak{M}_{n,m}[\sigma] \cap \mathfrak{M}_{n,k}[\sigma] = \varnothing$ whenever $0 \le k, m \le n$, $k \ne m$, and $0 \le \sigma \le 1/(6n+6)$.

(3) If Φ is an almost finite measure over $\mathbf{R}^n$, $\sigma \in [0, 1/(6n+6)]$, and $E = \{x;\ d_1(\Psi, \mathfrak{M}_n) \le \sigma$ for every $\Psi \in \operatorname{Tan}(\Phi, x)\}$, then for Φ almost every $x \in E$ there is an integer $m(x) = 0, 1, \ldots, n$ such that $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,m(x)}[\sigma]$.

(4) If Φ measures $\mathbf{R}^n$, $m = 0, 1, \ldots, n$, $V \in G(n, m)$, $\sigma \in (0, 1/5)$, $t > 0$, and $d_t(\Phi, \mathfrak{M}_{n,V}) \le \sigma^{m+3}$, then

$$\Phi[B(x,r) \cap B(V, \sigma^2 t/(m+1))] \ge (1 - 5\sigma)(r/s)^m \Phi(B(y,s))$$

whenever $x, y \in V \cap B(0, (1-\sigma)t)$, $\sigma t \le r \le (1-\sigma)t - \|x\|$, and $\sigma t \le s \le (1-\sigma)t - \|y\|$.

(5) If Φ is an almost finite measure over $\mathbf{R}^n$, $m = 0, 1, \ldots, n$, and $V \in G(n, m)$, then Φ almost all of the set

$$\left\{x \in \mathbf{R}^n;\ \operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,V}\left[(20)^{-m-3}\right]\right\}$$

can be covered by countably many graphs of Lipschitzian maps from V to $V^\perp$.

(6) If Φ is an almost finite measure over $\mathbf{R}^n$ then $\Phi\{x\} > 0$ for Φ almost every x for which $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,0}[1/8000]$.

Proof. (2) See 3.12(8).

(3) Let $\mathfrak{N}_m = \{\Psi;\ \sup_{r>0} d_r(\Psi, \mathfrak{M}_{n,m}) \le \sigma\}$. From 3.12(8) and 2.6(2) we see that for every $x \in E$ there is $m = m(x)$ such that $\operatorname{Tan}(\Phi, x) \subset \mathfrak{N}_m$. Since $\mathfrak{M}_{n,m}[\sigma] = \mathfrak{N}_m[0]$, the statement follows from 2.13(5).

(4) We may assume that $t = 1$ and $F_1(\Phi) = 1$. Let

$$\Psi = (m+1)\alpha(m)^{-1}\mathscr{H}^m \llcorner V.$$

Then $F_1(\Psi) = 1$ and $F_1(\Phi, \Psi) \le \sigma^{m+3}$. Hence, denoting $q = \sigma^2/(m+1)$, we may use 1.10(3) to estimate

$$\begin{aligned}
&\Phi[B(x,r) \cap B(V,q)]/\Phi(B(y,s)) \\
&\quad \ge \left[(m+1)(r-q)^m - \sigma^{m+3}/q\right]/\left[(m+1)(s+q)^m + \sigma^{m+3}/q\right] \\
&\quad = (r/s)^m\left[(1 - q/r)^m - \sigma^{m+1}/r^m\right]/\left[(1+q/s)^m + \sigma^{m+1}/s^m\right] \\
&\quad \ge (r/s)^m(1 - \sigma - mq/\sigma)/(1 + \sigma + 2mq/\sigma) \\
&\quad \ge (r/s)^m(1 - 2\sigma)/(1 + 3\sigma) \ge (1 - 5\sigma)(r/s)^m.
\end{aligned}$$

(5) Φ almost all of the set from (5) may be covered by countably many sets $E \subset \mathbf{R}^n$ for which there is $s > 0$ such that $\operatorname{diam}(E) < s$ and $d_1(T_{x,r}[\Phi], \mathfrak{M}_{n,V}) < (19)^{-m-3}$ whenever $x \in E$ and $r \in (0, 400(m+1)s)$. Let P be the orthogonal projection of $\mathbf{R}^n$ onto V. If $x, y \in E$ and $\|P(x) - P(y)\| < \|x - y\|/2$, we let $a = \|y - x\|$, $\sigma = 1/19$, $t = (m+1)a\sigma^{-2}/3$, $r = t/2$, and we use (4)

to infer that

$$\Phi[B(x,r) - B(x+V, a/3)] \le 5\sigma\Phi(B(x,r)) \quad \text{and}$$

$$\begin{aligned}\Phi[B(y,r+a) - B(y+V,a/3)] &\le 5\sigma\Phi(B(y,r+a)) \\ &\le 5\sigma(1-5\sigma)^{-1}(r+a)^m(r-a)^{-m} \\ &\quad \times\Phi(B(y,r-a)) \\ &\le 5\sigma(1-5\sigma)^{-1}(1+3ma/r)\Phi(B(x,r)).\end{aligned}$$

Since $B(x+V, a/3) \cap B(y+V, a/3) = \varnothing$,

$$\Phi(B(x,r)) \le 5\sigma\left[1 + (1-5\sigma)^{-1}(1+3ma/r)\right]\Phi(B(x,r)) \le \Phi(B(x,r)).$$

This contradiction proves that the inverse of the restriction of P to E exists and is Lipschitzian.

(6) This statement follows easily from (5).

4.5. THEOREM. *For each integer $n = 1, 2, \ldots$ and each $\sigma \in (0,1)$ there is a constant $\omega_2 = \omega_2(n, \sigma) \in (0,1)$ with the following property. Whenever Φ measures $\mathbf{R}^n$ and*

$$\Phi[B(0,r) \cap Y_s(\Phi, \omega_2)] > (1-\omega_2)\Phi(B(0,r))$$

for every r, $s \in (\omega_2, 1/\omega_2)$, then

(1) *$d_1(\Phi, \mathfrak{U}(n)) < \sigma$, and*

(2) *there are $s \in (\omega_2, 1)$ and an integer $m = 0, 1, \ldots, n$ such that*

$$\Phi\{x \in B(0,1);\, d_1(T_{x,s}[\Phi], \mathfrak{M}_{n,m}) \ge \sigma\} < \sigma\Phi(B(0,1)).$$

Proof. Assume that for some $n = 1, 2, \ldots$ and some $\sigma \in (0,1)$ there is no ω_2 with the above property. Then there is a sequence $\Phi_3, \Phi_4, \ldots$ of measures over $\mathbf{R}^n$ such that, for each $k = 3, 4, \ldots$,

$$\Phi_k[B(0,r) \cap Y_s(\Phi_k, 1/k)] > (1-1/k)\Phi_k(B(0,r))$$

for every r, $s \in (1/k, k)$ and

(1′) $d_1(\Phi_k, \mathfrak{U}(n)) \ge \sigma$, or

(2′) $\Phi_k\{x \in B(0,1);\ d_1(T_{x,s}[\Phi_k], \mathfrak{M}_{n,m}) \ge \sigma\} \ge \sigma\Phi_k(B(0,1))$ for each $s \in (1/k, 1)$ and each $m = 0, 1, \ldots, n$.

Clearly, we may also assume that $\Phi_k(B(0,1)) = 1$ for each $k = 3, 4, \ldots$.

Whenever $1/k < s \le r \le (k-1)s < k$, we find $y \in B(0,s) \cap Y_s(\Phi_k, 1/k)$ and we infer from 4.2(2) that

$$(*) \qquad \begin{aligned}\Phi_k(B(0,r)) &\le \Phi_k(B(y, r+s)) \le 5^{n+1}(r+s)^n s^{-n}\Phi_k(B(y,s)) \\ &\le 5^{n+1}(r+s)^n s^{-n}\Phi_k(B(0,2s)).\end{aligned}$$

Using $(*)$ with $s = 1/2$, we conclude from 1.12(1) that the sequence Φ_k has a convergent subsequence. To simplify the notation, we may assume that the whole sequence Φ_k converges to a nonzero locally finite measure Φ over $\mathbf{R}^n$.

Let $x, y \in \operatorname{spt}\Phi$, $s > r > 0$, and $t = \|x\| + \|y\| + r + s$. Since

$$\lim_{k\to\infty} \Phi_k[B(0,t) \cap Y_s(\Phi_k, 1/k)]/\Phi_k(B(0,t)) = 1$$

and

$$\begin{aligned}\limsup_{k\to\infty} \Phi_k[B(0,t) - B(x,\tau)]/\Phi_k(B(0,t))\\ = 1 - \liminf_{k\to\infty} \Phi_k(B(x,\tau))/\Phi_k(B(0,t))\\ \le 1 - \Phi(B^0(x,\tau))/\Phi(B(0,t)) < 1\end{aligned}$$

for every $\tau \in (0, r)$, there is a sequence $x_k \in Y_s(\Phi_k, 1/k)$ such that $x = \lim_{k\to\infty} x_k$. Since the sequence

$$\begin{aligned}\Phi_k\{ z \in B(x_k, 2t);\ \Phi_k(B(z, s/(1+1/k)))\\ \ge (1+1/k)\Phi_k(B(x_k, s))\}/\Phi_k(B(x_k, s))\end{aligned}$$

converges to zero as $k \to \infty$, and since

$$\liminf_{k\to\infty} \Phi_k(B(y,\tau))/\Phi_k(B(x_k,s)) \ge \Phi(B^0(y,\tau))/\Phi(B(0,t)) > 0$$

for every $\tau \in (0, r)$, there is a sequence $y_k \in \mathbf{R}^n$ such that $y = \lim_{k\to\infty} y_k$ and

$$\Phi_k(B(y_k, s/(1+1/k))) < (1+1/k)\Phi_k(B(x_k, s)).$$

Hence

$$\begin{aligned}\Phi(B(y,r)) \le \Phi(B^0(y,(s+r)/2)) \le \liminf_{k\to\infty} \Phi_k(B^0(y,(s+r)/2))\\ \le \liminf_{k\to\infty} \Phi_k(B(y_k, s/(1+1/k)))\\ \le \liminf_{k\to\infty} (1+1/k)\Phi_k(B(x_k,s)) \le \Phi(B(x,s)),\end{aligned}$$

which implies that Φ is uniformly distributed. Moreover, from $(*)$ with $r = 1$, we infer that $0 \in \operatorname{spt}\Phi$. Thus $\Phi \in \mathfrak{U}(n)$.

Let $m = \dim_0 \Phi$. Using 3.11(2), we find $s \in (0,1)$ such that the set

$$E = \left\{ y \in B(0,1) \cap \operatorname{spt}\Phi;\ d_1(T_{y,s}[\Phi], \mathfrak{M}_{n,m}) < \sigma/6 \right\}$$

has Φ measure greater than $\Phi(B(0,1)) - \sigma\Phi(B^0(0,1))$. Let $\tau = 9^{-n-2}\sigma$ and $H = B(0,1) - B^0(E, \tau s)$. Then H is compact and $\Phi(H) < \sigma\Phi(B^0(0,1))$. Hence there is $k > 1/s$ such that

$$F_2(\Phi_k, \Phi) \le \sigma F_s(\Phi)/6 \quad \text{and} \quad \Phi_k(H) < \sigma\Phi_k(B(0,1)).$$

Since $d_1(\Phi_k, \mathfrak{U}(n)) \leq 2F_1(\Phi_k, \Phi)/F_1(\Phi) < \sigma$ according to 1.10(5), (1′) does not hold for this k.

If $x \in B(0,1) - H$, we find $y \in E$ and $\Psi \in \mathfrak{M}_{n,m}$ such that $\|x - y\| < \tau s$, $F_1(T_{y,s}[\Phi]) = F_1(\Psi)$, and

$$F_1(T_{y,s}[\Phi], \Psi) < \sigma F_1(\Psi)/6.$$

Using also 1.9(5; iii, iv) and 3.2(1), we estimate

$$\begin{aligned} &F_1(T_{x,s}[\Phi_k], \Psi) \\ &\quad\leq F_1(T_{x,s}[\Phi_k], T_{x,s}[\Phi]) + F_1(T_{x,s}[\Phi], T_{y,s}[\Phi]) + F_1(T_{y,s}[\Phi], \Psi) \\ &\quad< F_2(\Phi_k, \Phi)/s + \tau\Phi(B(y,2s)) + \sigma F_1(\Psi)/6 \\ &\quad\leq \sigma F_s(\Phi)/6s + 9^n\tau\Phi(B(y, s/2)) + \sigma F_1(\Psi)/6 \\ &\quad\leq \sigma F_1(\Psi)/6 + 9^{n+1}\tau F_1(T_{y,s}[\Phi]) + \sigma F_1(\Psi)/6 \\ &\quad\leq \sigma F_1(\Psi)/2. \end{aligned}$$

Hence 1.10(5) implies that $d_1(T_{x,s}[\Phi_k], \mathfrak{M}_{n,m}) < \sigma$ for every $x \in B(0,1) - H$. Consequently, (2′) also does not hold for our value of k. This contradiction finishes the proof of 4.5.

4.6. Corollary. *If Φ is an $\omega_2(n,\sigma)$-approximately uniformly distributed measure over $\mathbf{R}^n$ then for Φ almost every $x \in \mathbf{R}^n$,*

(1) $\mathrm{Tan}(\Phi, x) \subset \mathfrak{U}(n, \sigma)$, *and*

(2) *there is $\Psi \in \mathrm{Tan}(\Phi, x)$ such that $d_1(\Psi, \mathfrak{M}_n) \leq \sigma$.*

Proof. (1) Whenever $r > 0$ and $x \in \operatorname{spt}\Phi$ is a Φ density point of $\bigcap_{s\in(0,r)} Y_s(\Phi, \omega_2)$, we infer from 4.5(1) that $d_1(\Psi, \mathfrak{U}(n)) \leq \sigma$ for every $\Psi \in \mathrm{Tan}(\Phi, x)$. Hence the statement follows from 2.13(5).

(2) If this statement does not hold, we find $r > 0$ such that the set

$$\left\{x \in \bigcap_{s\in(0,r)} Y_s(\Phi, \omega_2);\ d_1(T_{x,t}[\Phi], \mathfrak{M}_n) > \sigma \quad \text{for every } t \in (0, r)\right\}$$

has positive Φ measure. But 4.5(2) implies that this set cannot contain a Φ density point.

4.7. Corollary. (1) *An almost finite measure Φ over $\mathbf{R}^n$ is approximately uniformly distributed if and only if* $\mathrm{Tan}(\Phi, x) \subset \mathfrak{U}(n)$ *for Φ almost every x.*

(2) *If Φ is an approximately uniformly distributed measure over $\mathbf{R}^n$ then* $\mathrm{Tan}(\Phi, x) \cap \mathfrak{M}_n \neq \varnothing$ *for Φ almost every x.*

Proof. (1) See 4.2(5), 4.6(1), and 2.13(4).

(2) For Φ almost every x we deduce from 4.6(2) that there is $\Psi \in \mathrm{Tan}(\Phi, x)$ such that $d_1(\Psi, \mathfrak{M}_n) = 0$. Hence $\varnothing \neq \mathrm{Tan}(\Psi, 0) \subset \mathrm{Tan}(\Phi, x) \cap \mathfrak{M}_n$.

4.8. THEOREM. *For each integer* $n = 1, 2, \dots$ *and each* $\sigma \in (0, 1/3)$ *there is a constant* $\omega_3 = \omega_3(n, \sigma) \in (0, \sigma)$ *with the following property: If* $m = 0, 1, \dots, n$, Φ *measures* $\mathbf{R}^n$, $0 \in E \subset B(0,1)$, $0 \le a < \omega_3$, *and if for every* $x \in E$ *and every* $r, s \in (\omega_3 a, 1/\omega_3)$

(1) $\Phi[B(x,r) \cap Y_s(\Phi, \omega_3)] > (1 - \omega_3)\Phi(B(x,r))$,

(2) *there is* $t \in [1 + \sigma, 1/\sigma]$ *such that*

$$\Phi(B(x, tr)) \le (1 + \omega_3) t^m \Phi(B(x,r)), \quad \textit{and}$$

(3) *there is* $t \in [1, 1/\sigma]$ *such that*

$$\Phi(B(x, tr)) \ge (1 + \sigma) t^{m-1} \Phi(B(x,r)),$$

then

$$\Phi\{x \in E;\ d_1(T_{x,s}[\Phi], \mathfrak{M}_{n,m}) \ge \sigma \textit{ for some } s \in (a, \omega_3)\} \le \sigma\Phi(B(0,1)).$$

Proof. Assuming that for some $n = 1, 2, \dots$ and some $\sigma \in (0, 1/3)$ there is no constant ω_3 with the above property, we let $\eta_k = \omega_2(n, \sigma/k)/k$ and we find an integer $m = 0, 1, \dots, n$, measures Φ_k over $\mathbf{R}^n$, sets $0 \in E_k \subset B(0,1)$, and numbers $0 \le a_k < \eta_k$ such that for each $k = 1, 2, \dots$ the assumptions (1), (2), and (3) hold with Φ, E, a, and ω_3 replaced by Φ_k, E_k, a_k, and η_k, respectively, and

$$\Phi_k\{x \in E_k;\ d_1(T_{x,s}[\Phi_k], \mathfrak{M}_{n,m}) \ge \sigma \text{ for some } s \in (a_k, \eta_k)\} > \sigma\Phi(B(0,1)).$$

Using 4.5(2), we see that for each $k = 1, 2, \dots$ there are an integer $m_k = 0, 1, \dots, n$, a number $s_k \in (k\eta_k, 1)$, and a point $x_k \in E_k$ such that

(a) $d_1(T_{x_k, s_k}[\Phi_k], \mathfrak{M}_{n, m_k}) < \sigma/k$, and

(b) $d_1(T_{x_k, s}[\Phi_k], \mathfrak{M}_{n,m}) \ge \sigma$ for some $s \in (a_k, \eta_k)$.

Let $\hat{\sigma} = (\sigma/20)^{n+3}\omega(m)$. (See 3.14.) If $k > 1/\hat{\sigma}$, (a) and 4.4(4) imply that

$$\begin{aligned}(1 - \sigma/4) t^{m_k} \Phi_k(B(x_k, s_k/2)) \\ \le \Phi_k(B(x_k, t s_k/2)) \\ \le (1 - \sigma/4)^{-1} t^{m_k} \Phi_k(B(x_k, s_k/2))\end{aligned}$$

for every $t \in [\sigma, 1]$. Using (2) with $r = \sigma s_k/2$, we find $t \in [1 + \sigma, 1/\sigma]$ such that

$$\begin{aligned}(1 - \sigma/4)(t\sigma)^{m_k} \Phi_k(B(x_k, s_k/2)) &\le \Phi_k(B(x_k, t\sigma s_k/2)) \\ &\le (1 + \eta_k) t^m \Phi_k(B(x_k, \sigma s_k/2)) \\ &\le (1 + \eta_k)(1 - \sigma/4)^{-1} t^m \sigma^{m_k} \Phi_k(B(x_k, s_k/2)).\end{aligned}$$

Hence $t^{m - m_k} \ge (1 - \sigma/4)^2/(1 + \eta_k) > (1 + \sigma)^{-1}$, which implies that $m \ge m_k$.

Similarly, using (3) with $r = \sigma s_k/2$, we infer that $m \le m_k$. Hence

(c) $m_k = m$ if $k > 1/\hat{\sigma}$.

For each $k > 1/\hat{\sigma}$ let $r_k \in [a_k, s_k]$ be the smallest number such that $d_1(T_{x_k,r}[\Phi_k], \mathfrak{M}_{n,m}) < \hat{\sigma}$ whenever $r_k < r < s_k$. From (c), (a), and (b) we easily see that $a_k < r_k < s_k$. Hence

(d) $d_1(T_{x_k,r_k}[\Phi_k], \mathfrak{M}_{n,m}) = \hat{\sigma}$.

Clearly, (d), (a), and (c) imply that

(e) $\lim_{k\to\infty} r_k/s_k = 0$.

Because of (1) and of 4.5(1), $d_1(T_{x_k,r}[\Phi_k], \mathfrak{U}(n)) < 1/k$ whenever $r \in (r_k/k, kr_k)$. Hence the sequence

$$T_{x_k,r_k}[\Phi_k]/\Phi_k(B(x_k, r_k))$$

has a subsequence converging to a measure $\Psi \in \mathfrak{U}(n)$. Since (e) implies that $d_r(\Psi, \mathfrak{M}_{n,m}) \le \hat{\sigma}$ for each $r \ge 1$, we conclude from 3.14(2; iii) and 3.12(8) that Ψ is flat at ∞ and $\dim_\infty \Psi = m$. Finally, we use (2) to infer that for each $r > 0$ there is $t \in [1+\sigma, 1/\sigma]$ such that $\Psi(B(0, tr)) \le t^m \Psi(B(0, r))$. Hence $\Psi(B(0, r))/r^m \ge \lim_{s\to\infty} \Psi(B(0, s))/s^m$ for every $r > 0$ and we conclude from 3.16 that $\Psi \in \mathfrak{M}_{n,m}$. But this contradicts (d).

4.9. Corollary. *If Φ is an $\omega_3(n, \sigma)$-approximately uniformly distributed measure over $\mathbf{R}^n$, and $m = 0, 1, \ldots, n$, then* $\mathrm{Tan}(\Phi, x) \subset \mathfrak{M}_{n,m}[\sigma]$ *at Φ almost every x at which*

$$\limsup_{r\searrow 0} \inf\{t^{-m}\Phi(B(x, tr))/\Phi(B(x, r));\ t \in [1+\sigma, 1/\sigma]\} < 1 + \omega_3,$$

and

$$\liminf_{r\searrow 0} \sup\{t^{-m+1}\Phi(B(x, tr))/\Phi(B(x, r));\ t \in [1, 1/\sigma]\} > 1 + \sigma.$$

Proof. If $\tilde{r} > 0$ and if E is the set of those $x \in \bigcap_{s\in(0,r)} Y_s(\Phi, \omega_3)$ such that

$$\inf\{t^{-m}\Phi(B(x, tr))/\Phi(B(x, r));\ t \in [1+\sigma, 1/\sigma]\} < 1 + \omega_3$$

and

$$\sup\{t^{-m+1}\Phi(B(x, tr))/\Phi(B(x, r));\ t \in [1, 1/\sigma]\} > 1 + \sigma$$

whenever $r \in (0, \tilde{r})$, and such that for every $r \in (0, \tilde{r})$ there is $s \in (0, r)$ with $d_1(T_{x,s}[\Phi], \mathfrak{M}_{n,m}) > \sigma$, then 4.8 implies that E cannot have a Φ density point. Hence

$$\mathrm{Tan}(\Phi, x) \subset \{\Psi;\ d_1(\Psi, \mathfrak{M}_{n,m}) \le \sigma\}$$

for Φ almost every x for which the assumptions of 4.9 hold. Because of 2.12, this implies the statement of 4.9.

4.10. Corollary. *For every integer $n = 1, 2, \ldots,$ every $\sigma \in (0, 1)$ and every $t > 0$, $t \ne 1$, there is a constant $\Delta(n, \sigma, t) > 1$ with the following property. Whenever Φ is an almost finite measure over $\mathbf{R}^n$ such that for Φ*

almost every x there is $t > 0$, $t \neq 1$, with

$$\limsup_{r\searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)) \leq \Delta(n, \sigma, t) \liminf_{r\searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)),$$

then $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_n[\sigma]$ *for Φ almost every $x \in \mathbf{R}^n$.*

Proof. Clearly, it suffices to find for every $n = 1, 2, \ldots,$ every $\sigma \in (0,1)$ and every $k = 3, 4, \ldots, k > 1/\sigma$ a constant $\Delta > 1$ having the following property: If Φ is an almost finite measure over $\mathbf{R}^n$ and H is the set of all $x \in \mathbf{R}^n$ for which there is $t \in (1 + 2/k, k)$ such that $\limsup_{r\searrow 0}\Phi(B(x, tr))/\Phi(B(x, r)) \leq \Delta \liminf_{r\searrow 0}\Phi(B(x, tr))/\Phi(B(x, r))$, then $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_n[\sigma]$ for Φ almost every $x \in H$.

Let n, σ, and k be given. Let $\tau \in (0, 1/2k)$ be such that $(1 - \tau)^2 > (1 + 1/k)(1 + 2/k)^{-1}$ and $(1 + \tau)(1 - \tau)^{-1} < 1 + \omega_3(n, 1/k)$. We put

$$\varepsilon = \tfrac{1}{2}\min\left(\omega_3(n, 1/k), \omega_2\left(n, (\tau/5)^{n+3}\right)\right) \quad \text{and}$$

$$\Delta = \min\left(1 + \tau, 1 + \varepsilon/2, (1 + 1/k)^{\varepsilon/(9 \ln(2/\varepsilon))}\right).$$

If Φ and H are as above, we use 4.3(2) to infer that almost every $x \in H$ belongs to $\bigcup_{s>0}\bigcap_{r\in(0,s)} Y_r(\Phi, \varepsilon)$. Hence 4.5(2) and 4.4(4) imply that for Φ almost every $x \in H$ there are $m(x) = 0, 1, \ldots, n$ and a sequence $r_k \searrow 0$ such that

$$(1 - \tau)s^{m(x)} \leq \liminf_{k\to\infty} \Phi(B(x, sr_k))/\Phi(B(x, r_k)) \leq \limsup_{k\to\infty} \Phi(B(x, sr_k))/\Phi(B(x, r_k)) \leq (1 - \tau)^{-1}s^{m(x)}$$

for every $s \in [\tau, 1/2]$. Consequently, for Φ almost every $x \in H$ there is $t \in (1 + 2/k, k)$ such that

$$\limsup_{r\searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)) \leq \Delta(1 - \tau)^{-1}t^{m(x)} < (1 + \omega_3(n, 1/k))t^{m(x)}, \quad \text{and}$$

$$\liminf_{r\searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)) \geq \Delta^{-1}(1 - \tau)t^{m(x)} > (1 + 1/k)t^{m(x)-1}.$$

Since $\varepsilon < \omega_3(n, 1/k)$, the statement follows easily from 4.8 and from 2.12.

4.11. THEOREM. *Let Φ be an almost finite measure over $\mathbf{R}^n$. If one of the following eight statements holds Φ almost everywhere then all of them hold Φ*

almost everywhere,

(1) $\mathrm{Tan}(\Phi, x) \subset \mathfrak{M}_n$.

(2) *There is* $m(x) = 0, 1, \ldots, n$ *such that* $\mathrm{Tan}(\Phi, x) \subset \mathfrak{M}_{n, m(x)}$.

(3) *There is* $m(x) = 0, 1, \ldots, n$ *such that*

$$\lim_{r \searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)) = t^{m(x)} \qquad \textit{for every } t > 0.$$

(4) $\lim_{r \searrow 0} \Phi(B(x, tr))/\Phi(B(x, r))$ *exists for some* (*or for all*) $t > 0$, $t \neq 1$.

(5) $\limsup_{r \searrow 0} \sup_{z \in B(x, r/\varepsilon)} \Phi(B(z, r/(1 + \varepsilon)))/\Phi(B(x, r)) \leq 1$ *for every* $\varepsilon > 0$.

(6) *Whenever* $D \subset \mathbf{R}^n$ *is a bounded, symmetric, convex set such that* $0 \in \mathrm{Int}(D)$, *then*

$$\lim_{r \searrow 0} \sup_{z \in x + rD/\varepsilon} \Phi(z + rD)/\Phi(x + rD) = 1 \qquad \textit{for every } \varepsilon > 0.$$

(7) *Whenever* $D \subset \mathbf{R}^n$ *is a bounded convex set such that* $0 \in \mathrm{Int}(D)$, *then*

$$\lim_{r \searrow 0} \Phi\{z \in x + rD/\varepsilon;\ \Phi(x + rD)/(1 + \varepsilon) \leq \Phi(z + rD) \leq (1 + \varepsilon)\Phi(x + rD)\}/\Phi(x + rD/\varepsilon) = 1 \quad \textit{for every } \varepsilon > 0.$$

(8) $\mathrm{Tan}(\Phi, x) \subset \mathfrak{U}(n)$ *and* $b_1^{(1)}(\Psi) = 0$ *for every* $\Psi \in \mathrm{Tan}(\Phi, x)$.

Proof. (1) $\Rightarrow$ (2). See 4.4(3).

(2) $\Rightarrow$ (3), (6), (7). Whenever $x \in \mathbf{R}^n$ is such that

$$\varnothing \neq \mathrm{Tan}(\Phi, x) \subset \mathfrak{M}_{n, m(x)}$$

and $r_k \searrow 0$, we use 2.7 to find $k_1 < k_2 < \cdots$ and $c_j > 0$ such that the sequence $c_j T_{x, r_{k_j}}[\Phi]$ converges to a measure $\Psi \in \mathfrak{M}_{n, m(x)}$.

(3) We compute

$$\lim_{j \to \infty} \Phi\left(B\left(x, tr_{k_j}\right)\right)/\Phi\left(B\left(x, r_{k_j}\right)\right) = \Psi(B(0, t))/\Psi(B(0, 1)) = t^{m(x)}.$$

(6) If there are $t > 1$ and $z_k \in x + r_k D/\varepsilon$ such that $\Phi(z_k + r_k D) \geq t\Phi(x + r_k D)$, we may also assume that the sequence $T_{x, r_{k_j}}(z_{k_j})$ converges to some $z \in \mathbf{R}^n$. Since D is symmetric, $\Psi(z + \mathrm{Clos}(D)) = \Psi(-z + \mathrm{Clos}(D))$. Hence the Brunn-Minkowski theorem ([11, 3.2.41]) implies

$$\begin{aligned} 2(\Psi(z + \mathrm{Clos}(D)))^{1/m(x)} &= (\Psi(z + \mathrm{Clos}(D)))^{1/m(x)} + (\Psi(-z + \mathrm{Clos}(D)))^{1/m(x)} \\ &\leq (\Psi(2\,\mathrm{Clos}(D)))^{1/m(x)} = 2(\Psi(\mathrm{Clos}(D)))^{1/m(x)}. \end{aligned}$$

Noting that $\Psi(\mathrm{Bdry}(D)) = 0$ ([11, 3.2.35]), we infer that $\Psi(z + \mathrm{Clos}(D)) \leq$

$\Psi(\operatorname{Int}(D))$. Hence

$$t \leq \lim_{j\to\infty} \Phi\big(z_{k_j} + r_{k_j}D\big)/\Phi\big(x + r_{k_j}D\big) \leq \Psi(z + \operatorname{Clos}(D))/\Psi(\operatorname{Int}(D)) \leq 1.$$

(7) Let $\sigma > 0$ be such that

$$\Psi(B(D, 2\sigma)) < (1 + \varepsilon)\Psi\{y \in D;\ B(y, 2\sigma) \subset D\}.$$

Whenever $z_j \in (x + r_{k_j}D/\varepsilon) \cap T^{-1}_{x, r_{k_j}}(B(\operatorname{spt}\Psi, \sigma))$ and $\lim_{j\to\infty} T_{x, r_{k_j}}(z_j) = z$, we estimate

$$\liminf_{j\to\infty} \Phi\big(z_j + r_{k_j}D\big)/\Phi\big(x + r_{k_j}D\big) \geq \Psi(z + \operatorname{Int}(D))/\Psi(\operatorname{Clos}(D)) > (1 + \varepsilon)^{-1}$$

and

$$\limsup_{j\to\infty} \Phi\big(z_j + r_{k_j}D\big)/\Phi\big(x + r_{k_j}D\big) \leq \Psi(z + \operatorname{Clos}(D))/\Psi(\operatorname{Int}(D)) < 1 + \varepsilon.$$

This easily implies (7).

The implications (3) $\Rightarrow$ (4) and (6) $\Rightarrow$ (5) are obvious.

(4) $\Rightarrow$ (1). See 4.10 and 2.13(4).

(5) $\Rightarrow$ (8). Whenever (5) holds for some x, we easily see that $\Psi(B(z, r)) \leq \Psi(B(0, r))$ for every $\Psi \in \operatorname{Tan}(\Phi, x)$, every $z \in \mathbf{R}^n$, and every $r > 0$. Using also 2.12, we infer that $\operatorname{Tan}(\Phi, x) \subset \mathfrak{U}(n)$ for Φ almost every x. Hence (8) follows from 3.18(1; i).

(7) $\Rightarrow$ (8). If (7) holds for some x, $\Psi \in \operatorname{Tan}(\Phi, x)$, $r_k \searrow 0$, $c_k > 0$, $\Psi = \lim_{k\to\infty} c_k T_{x, r_k}[\Phi]$, $z \in \operatorname{spt}\Psi$, and D is a compact convex set with $0 \in \operatorname{Int}(D)$, we find a sequence $z_k \in \mathbf{R}^n$ such that

$$\limsup_{k\to\infty} \Phi(z_k + r_kD)/\Phi(x + r_kD) \leq 1 \quad\text{and}\quad \lim_{k\to\infty} T_{x, r_k}(z_k) = z.$$

Hence $\Psi(z + \operatorname{Int}(D)) \leq \Psi(D)$. This easily implies that $\Psi(z + D) \leq \Psi(D)$ for every $z \in \operatorname{spt}\Psi$ and every compact convex set D such that $0 \in \operatorname{Int}(D)$. Now (8) follows from 2.12 and from 3.18(1, ii).

(8) $\Rightarrow$ (1). Let $x \in \mathbf{R}^n$ be such that $\varnothing \neq \operatorname{Tan}(\Phi, x) \subset \mathfrak{U}(n)$ and $b_1^{(1)}(T_{u,1}[\Psi]) = 0$ for every $\Psi \in \operatorname{Tan}(\Phi, x)$ and every $u \in \operatorname{spt}\Psi$. Let $m = \min\{\dim_\infty\Psi;\ \Psi \in \operatorname{Tan}(\Phi, x)\}$. Whenever $\Psi \in \operatorname{Tan}(\Phi, x)$ and $\dim_\infty\Psi = m$, we use $\operatorname{Tan}(\Psi, 0) \subset \operatorname{Tan}(\Phi, x)$ and 3.12(5) to infer that $\dim_0\Psi \geq m$. Hence 3.18(1, 2) imply that

$$\operatorname{Tan}(\Phi, x) \cap \{\Psi \in U(n);\ \dim_\infty\Psi = m\} \subset \mathfrak{M}_{n,m}.$$

If $\operatorname{Tan}(\Phi, x) \not\subset \mathfrak{M}_{n,m}$, we use 2.6 to find $\varepsilon \in (0, 1/(3m + 6))$ and $\tilde{\Psi} \in \operatorname{Tan}(\Phi, x)$ such that $d_1(\tilde{\Psi}, \mathfrak{M}_{n,m}) = \varepsilon$ and $d_r(\tilde{\Psi}, \mathfrak{M}_{n,m}) \leq \varepsilon$ for every

$r \geq 1$. But then 3.12(8) implies that $\tilde{\Psi} \in \mathfrak{M}_{n,m}$, which is impossible. Hence $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_n$ whenever x fulfils our assumptions. Moreover, according to 2.12, these assumptions hold for Φ almost every $x \in \mathbf{R}^n$.

4.12. (1) Whenever Φ measures $\mathbf{R}^n$ and $x \in \mathbf{R}^n$, we denote by $\operatorname{Dim}_x(\Phi)$ the set of all integers $0 \leq m \leq n$ such that $\operatorname{Tan}(\Phi, x) \cap \mathfrak{M}_{n,m} \neq \varnothing$.

(2) Let Φ be an approximately uniformly distributed measure over $\mathbf{R}^n$. Then the following four statements hold for Φ almost every $x \in \mathbf{R}^n$.

(i) $\operatorname{Dim}_x(\Phi) = \{m = 0, 1, \dots, n$; there is a sequence $r_k \searrow 0$ such that

$$\lim_{k \to \infty} \Phi(B(x, tr_k))/\Phi(B(x, r_k)) = t^m \quad \text{for every } t > 0\Big\}$$

$$= \{\dim_0 \Psi;\ \Psi \in \operatorname{Tan}(\Phi, x) \cap \mathfrak{U}(n)\}$$

$$= \{\dim_\infty \Psi;\ \Psi \in \operatorname{Tan}(\Phi, x) \cap \mathfrak{U}(n)\} \neq \varnothing.$$

(ii) If $n \in \operatorname{Dim}_x(\Phi)$ then $\operatorname{Tan}(\Phi, x) = \mathfrak{M}_{n,n}$.

(iii) If $\operatorname{Dim}_x(\Phi) = \{n-1\}$ then

$$\operatorname{Tan}(\Phi, x) \subset \{c\mathscr{H}^{n-1} \llcorner [V \cup (e + V)];\ V \in G(n, n-1),\ e \in \mathbf{R}^n,\ c > 0\}.$$

(iv) If $\operatorname{Dim}_x(\Phi) = \{n-1\}$ and

$$\limsup_{r \searrow 0} \Phi(B(x, tr))/t^{n-1}\Phi(B(x, r)) < \left(1 + (1 - t^{-2})^{(n-1)/2}\right)$$

for some $t > 1$, then $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,n-1}$.

(3) If Φ is an approximately uniformly distributed measure over $\mathbf{R}^n$ then, according to (2; ii), for Φ almost every x either $\operatorname{Dim}_x(\Phi) = \{n\}$ or $\operatorname{Dim}_x(\Phi) \subset \{0, 1, \dots, n-1\}$. Conversely, whenever $M = \{n\}$ or $\varnothing \neq M \subset \{0, 1, \dots, n-1\}$, one can construct a nonzero approximately uniformly distributed measure over $\mathbf{R}^n$ such that $\operatorname{Dim}_x(\Phi) = M$ for Φ almost every $x \in \mathbf{R}^n$. In fact, considerably more will be proved in 6.9(2).

In connection with (2, ii) one should also consult 5.9(1).

Proof of (2). (i) For $m = 0, 1, \dots, n$ let

$$E_m = \{x \in \mathbf{R}^n;\ \text{there is } \Psi \in \mathfrak{U}(n) \cap \operatorname{Tan}(\Phi, x) \text{ such that}$$
$$\Psi(B(0, r)) = \alpha(m) r^m \quad \text{for each } r > 0\}.$$

Since $\operatorname{Tan}(\Psi, 0) \cup \operatorname{Tan}(\Psi, \infty) \subset \operatorname{Tan}(\Phi, x)$ for every $\Psi \in \operatorname{Tan}(\Phi, x)$, we may use 3.12(4, 5), 3.11, and 3.10(2) to reduce the proof to showing that $m \in \operatorname{Dim}_x(\Phi)$ for Φ almost every $x \in E_m$. But this follows from 3.11(2, iv) and 2.12.

(ii) It suffices to use (i), 4.7, and 3.12(3).

(iii) Let

$$\tilde{\mathfrak{M}} = \{c\mathscr{H}^{n-1} \llcorner [V \cup (e + V)];\ V \in G(n, n-1),\ e \in \mathbf{R}^n,\ c > 0\}.$$

If $x \in \mathbf{R}^n$ is such that $\operatorname{Tan}(\Phi, x) \subset \mathfrak{U}(n)$, $\operatorname{Tan}(\Phi, x) \cap \mathfrak{M}_{n-1} \neq \varnothing$, and $\dim_0 \Psi = n - 1$ for each $\Psi \in \operatorname{Tan}(\Phi, x)$, and if $\operatorname{Tan}(\Phi, x) \not\subset \mathfrak{M}$, we use 2.6(1) to find $0 < \varepsilon < \omega(n-1)/(3n+3)$ and $\Psi \in \operatorname{Tan}(\Phi, x)$ such that $d_1(\Psi, \tilde{\mathfrak{M}}) = \varepsilon$ and $d_r(\Psi, \tilde{\mathfrak{M}}) \le \varepsilon$ for every $r \ge 1$. Then 3.14(2; iii) implies that Ψ is flat at ∞ and hence $\Psi \in \tilde{\mathfrak{M}}$ according to 3.19. Since this is impossible, the statement follows from (i) and from 4.7.

(iv) Suppose that $V \in G(n, n-1)$, $e \in V^{\perp} - \{0\}$ and the measure

$$\Psi = \mathscr{H}^{n-1} \llcorner [V \cup (e + V)]$$

belongs to $\operatorname{Tan}(\Phi, x)$. Since

$$\Psi(B^0(0, t\|e\|))/\Psi(B^0(0, \|e\|)) = t^{n-1}\left(1 + (1 - t^{-2})^{(n-1)/2}\right),$$

there is $0 < a < \|e\|$ so close to $\|e\|$ that

$$\Psi(B(0, ta))/\Psi(B(0, a)) > \limsup_{r \searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)).$$

If $r_k \searrow 0$ and $c_k > 0$ are such that $\Psi = \lim_{k \to \infty} c_k T_{x, r_k}[\Phi]$, we easily see that

$$\begin{aligned}\limsup_{r \searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)) &\ge \Psi(B(0, ta))/\Psi(B(0, a)) \\ &> \limsup_{r \searrow 0} \Phi(B(x, tr))/\Phi(B(x, r)).\end{aligned}$$

This contradiction shows that (iv) follows from (iii).

5. Rectifiability

5.1. Suppose that Φ is an almost finite measure over $\mathbf{R}^n$ and that $0 \le m \le n$ is an integer.

(1) The measure Φ is said to be m rectifiable if it is absolutely continuous with respect to $\mathscr{H}^m$ and if Φ almost all of $\mathbf{R}^n$ can be covered by countably many m dimensional submanifolds of class one of $\mathbf{R}^n$.

(2) We define upper and lower m dimensional densities of Φ at a point $x \in \mathbf{R}^n$ by the formulas

$$\overline{D}_m(\Phi, x) = \limsup_{r \searrow 0} \Phi(B(x, r))/\alpha(m) r^m$$

and

$$\underline{D}_m(\Phi, x) = \liminf_{r \searrow 0} \Phi(B(x, r))/\alpha(m) r^m.$$

If the upper and lower m dimensional densities of Φ at x coincide, we denote their common value by $D_m(\Phi, x)$.

5.2. LEMMA. *Suppose that* $1 \le m \le n$ *are integers,* $r > 0$, $\varepsilon \in (0, 2^{-m-5}]$, $r_1 = (1 - \varepsilon/m) r$, $\mu = 2^{-7} m^{-3} \varepsilon^2$, *and* $\kappa \in [0, 8m\mu]$. *Suppose further that* Φ

measures $\mathbf{R}^n$, $z \in \operatorname{spt}\Phi$, $\Phi(B(z,r)) < \infty$,

$$Z = \{(x,s,V) \in B(z,r_1) \times (\kappa r, r] \times G(n,m);\ \Phi(B(y,t)) \geq (1-\varepsilon)(t/r)^m \Phi(B(z,r)) \text{ whenever } y \in B(x,s) \cap (x+V) \text{ and } t \in [\mu s, s]\},$$

and E is a compact subset of $B(z,r_1)$ *such that* $z \in E$,

$$\Phi[B(z,r_1) - E] \leq \mu^{m+1}\Phi(B(z,r_1)),$$

and such that for every $x \in E$ *and every* $s \in (\kappa r, r - \|x - z\|]$ *there is* $V \in G(n,m)$ *with* $(x,s,V) \in Z$.

Let W be an affine subspace of $\mathbf{R}^n$ *passing through z such that* $(z,r,W-z) \in Z$ *and let P be the orthogonal projection of* $\mathbf{R}^n$ *onto W.*

For every $u \in W \cap B(z,r_1)$ *let* $s(u) \in [\kappa r, r]$ *be the smallest number having the following property*:

Whenever $s(u) < s \leq r$, *then* $E \cap P^{-1}(W \cap B(u, s/4m)) \neq \varnothing$, *and*

$$\Phi[B(z,r) \cap P^{-1}(W \cap B(u,s))] \leq \mu^{-m}(s/r)^m \Phi(B(z,r)).$$

Finally, let

$$A = \{u \in W \cap B(z,r_1);\ s(u) = \kappa r\},$$

$$A_1 = \{u \in W \cap B(z,r_1);\ s(u) > \kappa r \text{ and } \Phi[B(z,r) \cap P^{-1}(W \cap B(u,s(u)))] \geq \varepsilon^{-1}(s(u)/r)^m \Phi(B(z,r))\},$$

and

$$A_2 = \{u \in W \cap B(z,r_1);\ s(u) > \kappa r \text{ and } \Phi[(B(z,r) - E) \cap P^{-1}(W \cap B(u, s(u)/4m))] \geq \tfrac{1}{2}(s(u)/4mr)^m \Phi(B(z,r))\}.$$

Then

(1) $s(u) \leq 8m\mu r$ *for every* $u \in W \cap B(z,r_1)$;

(2) *The function* $u \mapsto s(u)$ *is lower semicontinuous on* $W \cap B(z,r_1)$; *consequently, the set A is compact*;

(3) $W \cap B(z,r_1) \subset A \cup A_1 \cup A_2$;

(4) $\mathscr{H}^m[W \cap B(z,r) - A] \leq 2^{m+3}\varepsilon\alpha(m) r^m$;

(5) *For every* $s \in (\kappa r, \varepsilon r/m]$ *there is* $x \in E$ *such that*

$$\Phi(B(x,s)) \leq (1 + 2^{m+4}\varepsilon) r^{-m}(\kappa r + s)^m \Phi(B(z,r));$$

(6) *If* $\kappa = 0$ *then* $P(E \cap P^{-1}(A)) = A$, $\mathscr{H}^m(E \cap P^{-1}(A)) > 0$, *and there are constants* $0 < c < C < \infty$ *such that*

$$c\mathscr{H}^m \llcorner (E \cap P^{-1}(A)) \leq \Phi \llcorner (E \cap P^{-1}(A)) \leq C\mathscr{H}^m \llcorner (E \cap P^{-1}(A));$$

and

(7) *If $\kappa = 0$ then there is an m dimensional submanifold E_0 of class one of* $\mathbf{R}^n$ *such that* $\mathscr{H}^m(E_0 \cap E \cap P^{-1}(A)) > 0$.

Proof. (1) Let $u \in W \cap B(z, r_1)$ and let $8m\mu r \le s \le r$. Then

(α)
$$\Phi[B(z,r) \cap P^{-1}(W \cap B(u,s))] \le \Phi(B(z,r)) \le \mu^{-m}(s/r)^m \Phi(B(z,r)).$$

Moreover, we may find $v \in W \cap B(z, r_1)$ such that

$$B(v, \mu r) \subset B(z, r_1) \cap B(u, 2\mu r).$$

Since $(z, r, W - z) \in Z$,

$$\Phi(B(v, \mu r)) \ge (1-\varepsilon)\mu^m \Phi(B(z,r)) > \Phi(B(z,r_1) - E).$$

Hence

$$(\beta) \qquad E \cap P^{-1}(W \cap B(u, s/4m)) \supset E \cap B(v, \mu r) \ne \varnothing.$$

Clearly, (α) and (β) imply that $s(u) \le 8m\mu r$.

(2) Let $u \in W \cap B(z, r_1)$ and let $\kappa r < s \le s(u)$ be such that

$$(\alpha) \quad \Phi[B(z,r) \cap P^{-1}(W \cap B(u,s))] > (1+\tau)\mu^{-m}(s/r)^m \Phi(B(z,r))$$

for some $\tau > 0$, or

$$(\beta) \qquad E \cap P^{-1}(W \cap B(u, s/4m)) = \varnothing.$$

If $v \in W \cap B(z, r_1)$ is sufficiently close to u, then $s + \|v - u\| \le r$ according to (1),

$$\Phi[B(z,r) \cap P^{-1}(W \cap B(v, s + \|v-u\|))]$$
$$> \mu^{-m}((s + \|v-u\|)/r)^m \Phi(B(z,r))$$

if (α) holds, and

$$E \cap P^{-1}(W \cap B(v, (s - \|v-u\|)/4m)) = \varnothing$$

if (β) holds. Hence $s(v) \ge s - \|v - u\|$ for v sufficiently close to u.

(3) Suppose that $u \in [W \cap B(z, r_1)] - [A \cup A_1]$. Using the compactness of E, we easily see that $s(u) > \kappa r$,

$$E \cap P^{-1}(W \cap B(u, s(u)/4m)) \ne \varnothing, \quad \text{and}$$
$$E \cap P^{-1}(W \cap B^0(u, s(u)/4m)) = \varnothing.$$

We choose $x \in E \cap P^{-1}(W \cap B(u, s(u)/4m))$ and denote $s = 16ms(u)/\varepsilon$ and $\sigma = (2m-1)\varepsilon/(32m^2)$. Using (1), we infer that

$$(\alpha) \qquad \kappa r < s(u) \le s \le 2^7 m^2 \mu r/\varepsilon = r - r_1, \quad \text{and} \quad \mu \le \sigma \le 1.$$

Let V be an affine subspace of $\mathbf{R}^n$ such that $(x, s, V - x) \in Z$. We claim that

$$(\beta) \qquad \|P(y) - P(x)\| \geq \sigma\|y - x\| \quad \text{for every } y \in V.$$

In fact, if $y \in V$, $\|y - x\| = 1$, and $\|P(y) - P(x)\| < \sigma$, we infer from

$$\begin{aligned}\sigma s + s(u)/4m\sigma &= s(u)[1 - 1/2m + 8m/((2m-1)\varepsilon)] \\ &\leq 2^{-4}m^{-2}\varepsilon^2[1 - 1/2m + 8m/((2m-1)\varepsilon)]r \leq \varepsilon r/m\end{aligned}$$

and from

$$\sigma s + s(u)/4m = (1 - 1/4m)s(u)$$

that

$$(\gamma) \qquad B((1-t)x + ty, \sigma s) \subset B(z, r) \cap P^{-1}(W \cap B(u, s(u)))$$

whenever $|t| \leq s(u)/4m\sigma$. Hence the Fubini theorem implies

$$\begin{aligned}&\Phi[B(z,r) \cap P^{-1}(W \cap B(u, s(u)))] \\ &\quad\geq (2\sigma s)^{-1}\int_{-s(u)/4m\sigma}^{s(u)/4m\sigma} \Phi(B((1-t)x + ty, \sigma s))\,dt \\ &\quad\geq (2\sigma s)^{-1}(s(u)/2m\sigma)(1-\varepsilon)(\sigma s/r)^m\Phi(B(z,r)) \\ &\quad= (1-\varepsilon)(1 - 1/2m)^m 16m^2(2m-1)^{-2}\varepsilon^{-1}(s(u)/r)^m\Phi(B(z,r)) \\ &\quad\geq \varepsilon^{-1}(s(u)/r)^m\Phi(B(z,r)),\end{aligned}$$

which contradicts $u \notin A_1$.

Therefore (β) holds; in particular P maps V onto W and there is $y \in V$ such that $P(y) = u$ and $\|y - x\| \leq s(u)/4m\sigma$. From the inequalities used in the proof of (γ) we easily see that

$$\begin{aligned}B^0(y, s(u)/4m) &\subset B(z,r) \cap P^{-1}(W \cap B^0(u, s(u)/4m)) \\ &\subset (B(z,r) - E) \cap P^{-1}(W \cap B^0(u, s(u)/4m)).\end{aligned}$$

Consequently,

$$\begin{aligned}&\Phi[(B(z,r) - E) \cap P^{-1}(W \cap B^0(u, s(u)/4m))] \\ &\quad\geq (1-\varepsilon)(s(u)/4mr)^m\Phi(B(z,r)).\end{aligned}$$

Hence $u \in A_2$, which proves (3).

(4) For every $\tau > 0$ let $D \subset A_1$ be a countable set such that the family $\{B(u, s(u));\ u \in D\}$ is disjointed and the family $\{B(u, (2+\tau)s(u));\ u \in D\}$

covers A_1. (Cf. [11, 2.8.4].) Then

$$\Phi(B(z,r)) \geq \sum_{u \in D} \Phi[B(z,r) \cap P^{-1}(W \cap B(u, s(u)))]$$
$$\geq \varepsilon^{-1} \sum_{u \in D} (s(u)/r)^m \Phi(B(z,r)).$$

Hence $\sum_{u \in D}(s(u))^m \leq \varepsilon r^m$ and

$$\mathscr{H}^m(A_1) \leq \alpha(m)(2+\tau)^m \sum_{u \in D} (s(u))^m \leq (2+\tau)^m \varepsilon \alpha(m) r^m.$$

Consequently, $\mathscr{H}^m(A_1) \leq 2^m \varepsilon \alpha(m) r^m$.

Similarly, for every $\tau > 0$ we may find a countable set $D \subset A_2$ such that the family $\{B(u, s(u)/4m);\ u \in D\}$ is disjointed and the family $\{B(u,(2+\tau)s(u)/4m);\ u \in D\}$ covers A_2. Then

$$\Phi(B(z,r) - E) \geq \sum_{u \in D} \Phi[(B(z,r) - E) \cap P^{-1}(W \cap B(u, s(u)/4m))]$$
$$\geq \frac{1}{2}\Phi(B(z,r)) \sum_{u \in D} (s(u)/4mr)^m.$$

Hence

$$\sum_{u \in D} (s(u)/4m)^m \leq 2r^m \Phi(B(z,r) - E)/\Phi(B(z,r))$$
$$\leq 2r^m[\varepsilon^2 + \Phi(B(z,r) - B(z,r_1))/\Phi(B(z,r))] \leq 4\varepsilon r^m.$$

Consequently,

$$\mathscr{H}^m(A_2) \leq \alpha(m)(2+\tau)^m \sum_{u \in D} (s(u)/4m)^m \leq (2+\tau)^{m+2} \varepsilon \alpha(m) r^m.$$

Finally, we use (3) to estimate

$$\mathscr{H}^m(W \cap B(z,r) - A)$$
$$\leq \mathscr{H}^m(W \cap B(z,r) - B(z,r_1)) + \mathscr{H}^m(A_1) + \mathscr{H}^m(A_2)$$
$$\leq [\varepsilon + 2^m\varepsilon + 2^{m+2}\varepsilon]\alpha(m) r^m \leq 2^{m+3}\varepsilon\alpha(m) r^m.$$

(5) Assuming that $s \in (\kappa r, \varepsilon r/m]$ and

$$\Phi(B(x,s)) \geq (1 + 2^{m+4}\varepsilon) r^{-m} (\kappa r + s)^m \Phi(B(z,r))$$

for every $x \in E$, we deduce that

$$\Phi[B(z,r) \cap P^{-1}(B(u, \kappa r + s) \cap W)]$$
$$\geq (1 + 2^{m+4}\varepsilon) r^{-m} (\kappa r + s)^m \Phi(B(z,r))$$

for every $u \in A$. Hence

$$\begin{aligned}\Phi(B(z,r)) &\geq \alpha(m)^{-1}(\kappa r+s)^{-m}\\ &\quad\times\int_A \Phi\big[B(z,r)\cap P^{-1}(B(u,\kappa r+s)\cap W)\big]\,d\mathscr{H}^m(u)\\ &\geq \alpha(m)^{-1}(\kappa r+s)^{-m}(1+2^{m+4}\varepsilon)r^{-m}(\kappa r+s)^m\\ &\quad\times\Phi(B(z,r))(1-2^{m+3}\varepsilon)\alpha(m)r^m\\ &> \Phi(B(z,r)).\end{aligned}$$

This contradiction proves (5).

(6) If $\kappa = 0$, $s(u) = 0$ for every $u \in A$ and the compactness of E implies $P(E \cap P^{-1}(A)) = A$. Since $\mathscr{H}^m(A) > 0$, $\mathscr{H}^m(E \cap P^{-1}(A)) > 0$. For every $x \in E \cap P^{-1}(A)$ and every $s \in (0, r - r_1)$ we have

$$\Phi(B(x,s)) \leq \big[\mu^{-m}\alpha(m)^{-1}r^{-m}\Phi(B(z,r))\big]\alpha(m)s^m$$

and

$$\Phi(B(x,s)) \geq \big[(1-\varepsilon)\alpha(m)^{-1}r^{-m}\Phi(B(z,r))\big]\alpha(m)s^m.$$

From these inequalities the statement easily follows. (Cf. [11, 2.10.19(1,3)].)

(7) Let $\tilde{A}$ be the set of all $u \in A$ such that

$$\Phi\big[B(z,r)\cap P^{-1}(B(u,s)\cap W)\big] < 2(1-\varepsilon)^2(s/r)^m\Phi(B(z,r))$$

for all sufficiently small positive numbers s. According to the Vitali covering theorem, $\mathscr{H}^m$ almost all of $A - \tilde{A}$ may be covered by a sequence $B(u_k, s_k)$ $(k = 1, 2, \ldots)$ of disjoint balls such that $u_k \in A - \tilde{A}$ and

$$\Phi\big[B(z,r)\cap P^{-1}(B(u_k,s_k)\cap W)\big] \geq 2(1-\varepsilon)^2(s_k/r)^m\Phi(B(z,r))$$

for each $k = 1, 2, \ldots$. Hence

$$\Phi(B(z,r)) \geq 2(1-\varepsilon)^2\Phi(B(z,r))r^{-m}\sum_{k=1}^{\infty}s_k^m$$

and

$$\mathscr{H}^m(A-\tilde{A}) \leq \alpha(m)\sum_{k=1}^{\infty}s_k^m \leq \frac{1}{2}(1-\varepsilon)^{-2}\alpha(m)r^m \leq \frac{32}{49}\alpha(m)r^m.$$

From (4) we see that

$$\mathscr{H}^m(A) \geq (1-2^{m+3}\varepsilon)\alpha(m)r^m \geq \tfrac{3}{4}\alpha(m)r^m > \tfrac{32}{49}\alpha(m)r^m.$$

Assured by (2) that A is $\mathscr{H}^m$ measurable, we find a compact set $\hat{A} \subset \tilde{A}$ and $\delta \in (0, \varepsilon r/m)$ such that $\mathscr{H}^m(\hat{A}) > 0$, $\operatorname{diam}(\hat{A}) < \delta(1 + m/\varepsilon)^{-1}$, and

$$(\alpha)\quad \Phi\big[B(z,r)\cap P^{-1}(B(u,s)\cap W)\big] < 2(1-\varepsilon)^2(s/r)^m\Phi(B(z,r))$$

for every $u \in \hat{A}$ and every $s \in (0, \delta)$.

Since $P(E \cap P^{-1}(A)) = A \supset \hat{A}$, there is a map $f\colon \hat{A} \mapsto E$ such that $P(f(u)) = u$ for every $u \in \hat{A}$.

If $u, v \in \hat{A}$ and $\|f(u) - f(v)\| > 2m\|u - v\|/\varepsilon$, then

$$B(f(u), m\|u - v\|/\varepsilon) \cap B(f(v), m\|u - v\|/\varepsilon) = \varnothing$$

and, consequently,

$$\begin{aligned}\Phi\big[B(z,r) &\cap P^{-1}(B(u,(1 + m/\varepsilon)\|u - v\|) \cap W)\big]\\ &\geq 2(1-\varepsilon)((1 + m/\varepsilon)\|u - v\|/r)^m\Phi(B(z,r))\\ &> 2(1-\varepsilon)^2((1 + m/\varepsilon)\|u - v\|/r)^m\Phi(B(z,r)).\end{aligned}$$

But this contradicts (α), since $u, v \in \hat{A}$ and $\operatorname{diam}(\hat{A}) < \delta(1 + m/\varepsilon)^{-1}$ imply $(1 + m/\varepsilon)\|u - v\| < \delta$. Hence f is a Lipschitzian map of $\hat{A}$ onto $f(\hat{A})$.

Since $\mathscr{H}^m(f(\hat{A})) \geq \mathscr{H}^m(P(f(\hat{A}))) = \mathscr{H}^m(\hat{A}) > 0$ and since $P(f(u)) = u$ for every $u \in \hat{A}$, there exists a map $g\colon W \mapsto W^{\perp}$ of class one such that $\mathscr{H}^m\{u \in \hat{A};\ f(u) \neq u + g(u)\} < \mathscr{H}^m(\hat{A})$. (See [11, 3.1.16].) Thus (7) holds with $E_0 = \{u + g(u);\ u \in W\}$.

5.3. Theorem. *Suppose that $1 \leq m \leq n$ are integers, Φ is an almost finite measure over $\mathbf{R}^n$,*

(a) $$\liminf_{r \searrow 0} [\Phi(B(x,r))]^{-1}\Phi\{z \in B(x,r);$$

$$\Phi(B(z,s)) \leq (1 - 2^{-m-6})(s/r)^m\Phi(B(x,r)) \text{ for some } s \in (0,r)\} = 0$$

and

(b) $$\liminf_{r \searrow 0} \sup_{V \in G(n,m)} \inf_{z \in (x+V)\cap B(x,r)} [\Phi(B(x,r))]^{-1}\Phi(B(z, 8^{-m-9}m^{-4}r)) > 0$$

at Φ almost every $x \in \mathbf{R}^n$. Then Φ is m rectifiable.

Proof. Let Φ measure $\mathbf{R}^n$, $H_1 \subset H_2 \subset \cdots \subset \mathbf{R}^n$ be a sequence of open sets with finite Φ measure such that $\Phi(\mathbf{R}^n - \bigcup_{k=1}^{\infty} H_k) = 0$, and let Φ fulfil the assumptions (a) and (b) of the theorem.

Let $G_j \subset H_j$ be Borel sets such that $\Phi \llcorner G_j$ is m rectifiable and $\Phi(D) \leq \Phi(G_j) + 2^{-j}$ whenever D is a Borel subset of H_j and $\Phi \llcorner D$ is m rectifiable $(j = 1, 2, \ldots)$. Also, let $G = \bigcup_{j=1}^{\infty} G_j$. Then $\Phi \llcorner G$ is m rectifiable and it suffices to show that $\Phi(\mathbf{R}^n - G) = 0$. To prove this, we assume that $\Phi(\mathbf{R}^n - G) > 0$ and we find $k = 1, 2, \ldots$ such that $\Phi(H_k - G) > 0$. We claim that:

(α) The set $H_k - G$ contains a Borel set M of positive Φ measure such that $\Phi \llcorner M$ is m rectifiable.

To prove (α), we denote $\sigma = 8^{-m-9}m^{-4}$ and

$$Z(d) = \{(x, s, V) \in (H_k - G) \times (0, d) \times G(n, m);\ \Phi(B(y, \sigma s)) \geq d\Phi(B^0(x, s)) \text{ whenever } y \in (x + V) \cap B(x, s)\}.$$

Using (b), we infer that there is $d_1 \in (0, 1)$ such that the set

$$E_1 = \{x \in (H_k - G) \cap \operatorname{spt} \Phi;\ B^0(x, 2d_1) \subset H_k \text{ and for every } s \in (0, d_1) \text{ there is } V \in G(n, m) \text{ such that } (x, s, V) \in Z(d_1)\}$$

has positive Φ measure. We easily see that E_1 is a relatively closed subset of $H_k - G$; hence it is Φ measurable. Let $z_1 \in E_1$ be a Φ density point of E_1 at which (a) holds and let $\tau = (4^{-m-9}m^{-3})^{m+2}d_1^2$. Using (a), we find $d \in (0, d_1/3)$ such that

$$\Phi\{x \in B(z_1, d);\ \Phi(B(x, s)) \leq (1 - 2^{-m-6})(s/d)^m \Phi(B(z_1, d)) \text{ for some } s \in (0, d)\} \leq d_1 \tau^2 \Phi(B(z_1, d))$$

and

$$\Phi[B(z_1, s) - E_1] \leq \tau\Phi(B(z_1, s)) \quad \text{for every } s \in (0, d).$$

Let

$$E_2 = \{x \in B^0(z_1, d);\ \Phi(B(x, s)) \geq (1 - 2^{-m-6})(s/d)^m \Phi(B(z_1, d)) \text{ for all } s \in (0, d)\},$$

and

$$E_3 = \{x \in E_2;\ \Phi[B(x, s) - E_2] \leq \tau\Phi(B(x, s)) \text{ for each } s \in (0, d - \|x - z_1\|)\}.$$

Let $x_j \in E_1 \cap E_2 - E_3$ and $s_j \in (0, d - \|x_j - z_1\|)$ be such that the balls $B(x_j, s_j)$ are disjoint, $\Phi(B(x_j, s_j) - E_2) > \tau\Phi(B(x_j, s_j))$, and

$$E_1 \cap E_2 - E_3 \subset \bigcup_j B^0(x_j, 3s_j).$$

Since $x_j \in E_1$ and $3s_j < d_1$,

$$\Phi(B^0(x_j, 3s_j)) \leq \Phi(B(x_j, 3\sigma s_j))/d_1 \leq \Phi(B(x_j, s_j))/d_1.$$

Hence

$$\begin{aligned}\Phi(E_1 \cap E_2 - E_3) &\leq \sum_j \Phi(B^0(x_j, 3s_j)) \leq d_1^{-1} \sum_j \Phi(B(x_j, s_j)) \\ &\leq d_1^{-1}\tau^{-1} \sum_j (\Phi[B(x_j, s_j) - E_2]) \\ &\leq d_1^{-1}\tau^{-1}\Phi[B(z_1, d) - E_2] \\ &\leq \tau\Phi(B(z_1, d)).\end{aligned}$$

Consequently,

$$\Phi\left[B^0(z_1, d) - (E_1 \cap E_3)\right] \leq 3\tau\Phi(B(z_1, d))$$

and there is a compact set $E_4 \subset E_1 \cap E_3$ such that

$$\Phi\left[B^0(z_1, d) - E_4\right] \leq 4\tau\Phi(B(z_1, d)).$$

We intend to find a point $z \in \mathbf{R}^n$, a positive number r, and a set $E \subset \mathbf{R}^n$ so that the assumptions of 5.2 hold with $\varepsilon = 2^{-m-5}$ and $\kappa = 0$.

From $z_1 \in E_1$ we infer that

$$\Phi\left(B(z_1, 2\sigma^2 d)\right) \geq d_1^2\Phi\left(B^0(z_1, 2d)\right) > 4\tau\Phi(B(z_1, d)).$$

Hence

$$B^0(z_1, \sigma d) \cap E_4 \supset B\left(z_1, 2\sigma^2 d\right) \cap E_4 \neq \varnothing.$$

Let $z \in E_4$ be such that $\|z - z_1\| < \sigma d$ and let $r = (1 - \sigma)d$ and $E = E_4 \cap B(z, (1 - 2^{-m-5}m^{-1})r)$.

Then $z \in E$ and, in the notation of 5.2 with $\varepsilon = 2^{-m-5}$ and $\kappa = 0$,

$$\begin{aligned}\Phi[B(z, r_1) - E] &\leq \Phi\left[B^0(z_1, d) - E_4\right] \leq 4\tau\Phi(B(z_1, d)) \\ &\leq 4\tau(1 - 2^{-m-6})^{-1}(d/r_1)^m\Phi(B(z, r_1)) \leq \mu^{m+1}\Phi(B(z, r_1)).\end{aligned}$$

Whenever $x \in E$ and $s \in (0, r - \|x - z\|]$, we find $V \in G(n, m)$ such that $(x, s, V) \in Z(d_1)$. For every $y \in (x + V) \cap B(x, s)$ we find $\bar{y} \in (x + V) \cap B(x, s)$ such that $B(\bar{y}, \sigma s) \subset B(y, 3\sigma s) \cap B^0(x, s)$. Using $\Phi(B(\bar{y}, \sigma s)) \geq d_1\Phi(B^0(x, s))$, $\Phi[B^0(x, s) - E_2] \leq \tau\Phi(B^0(x, s))$, and $\tau < d_1$, we infer that $E_2 \cap B(\bar{y}, \sigma s) \neq \varnothing$. Let $w \in E_2 \cap B(\bar{y}, \sigma s)$. For each $t \in [\mu s, s]$ we estimate

$$\begin{aligned}\Phi(B(y, t)) &\geq \Phi(B(w, t - 3\sigma s)) \geq (1 - 2^{-m-6})((t - 3\sigma s)/d)^m\Phi(B(z_1, d)) \\ &\geq (1 - 2^{-m-6})(1 - \sigma)^m(1 - 3\sigma/\mu)^m(t/r)^m\Phi(B(z, r)) \\ &\geq (1 - 2^{-m-5})(t/r)^m\Phi(B(z, r)).\end{aligned}$$

Hence the assumptions of 5.2 are satisfied and (α) follows from 5.2(6, 7). Therefore there is a Borel set $M \subset H_k - G$ such that $\Phi \llcorner M$ is m rectifiable and $\Phi(M) > 0$. But then $\Phi \llcorner (G_j \cup M)$ is m rectifiable for every $j = 1, 2, \ldots$ and consequently, $\Phi(G_j \cup M) \leq \Phi(G_j) + 2^{-j}$ for $j = k, k + 1, \ldots$. Since this contradicts $M \cap G = \varnothing$ and $\Phi(M) > 0$, the theorem is proved.

5.4. Corollary. *For $n = 1, 2, \ldots$ let $\nu(n) = (8^{n+10}n^4)^{-n-3}$. Suppose that $m = 0, 1, \ldots, n$, $\nu \in (0, \nu(n)]$, h is a positive nonincreasing function on $(0, \infty)$, and*

$$\limsup_{r \searrow 0} h(\nu r)/h(r) \leq (1 - 2\nu)/\nu.$$

Then, if Φ *measures* $\mathbf{R}^n$, *each of the following three conditions is sufficient to guarantee that* Φ *is m rectifiable.*

(1) $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,m}[\nu]$, *and*

$$0 < \liminf_{r \searrow 0} \Phi(B(x,r))/r^m h(r) < \infty$$

at Φ *almost every* $x \in \mathbf{R}^n$.

(2) $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_n[(\nu/5)^{n+4}]$ *and*

$$0 < \liminf_{r \searrow 0} \Phi(B(x,r))/r^m h(r) < \infty$$

at Φ *almost every* $x \in \mathbf{R}^n$.

(3)
$$0 < \limsup_{r \searrow 0} \Phi(B(x,r))/r^m h(r)$$
$$< (1 + \omega_3(n,\nu)) \liminf_{r \searrow 0} \Phi(B(x,r))/r^m h(r) < \infty$$

at Φ *almost every* $x \in \mathbf{R}^n$.

Proof. (1) If $m = 0$, we easily see that $\Phi\{x\} > 0$ for Φ almost every x. If $m \geq 1$, we verify the assumptions of 5.3.

(a) Whenever s and t are positive numbers, let

$$E(s,t) = \{x \in \operatorname{spt} \Phi;\ th(r) r^m \leq \Phi(B(x,r)) < \infty \quad \text{for each } r \in (0,s)\}.$$

Since the sets $E(s,t)$ are closed, the sets

$$\tilde{E}(s,t) = \bigcap_{p=1}^{\infty} \left[E(s,(1 - 2^{-m-6})t) - E(s/p,t)\right]$$

are Φ measurable, We easily see that 5.3(a) holds at every point of $\tilde{E}(s,t)$ which is a Φ density point of $\tilde{E}(s,t)$. Moreover, since $0 < \liminf_{r \searrow 0} \Phi(B(x,r))/r^m h(r) < \infty$ almost everywhere, Φ almost all of $\mathbf{R}^n$ can be covered by countably many sets $\tilde{E}(s,t)$.

(b) The condition 5.3(b) follows immediately from 4.4(4).

(2) Let $x \in \mathbf{R}^n$ and $p = 0, 1, \ldots, n$ be such that $\varnothing \neq \operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,p}[(\nu/5)^{n+4}]$ and

$$0 < \liminf_{r \searrow 0} \Phi(B(x,r))/r^m h(r) < \infty.$$

Using 4.4(4), we find $s > 0$ such that $\Phi(B(x,s)) < \infty$,

$$(1 - \nu) t^p \Phi(B(x,r)) \leq \Phi(B(x,tr)) \leq (1 - \nu)^{-1} t^p \Phi(B(x,r))$$

and $h(\nu r) \leq \nu^{-1}(1 - 3\nu/2) h(r)$ for each $r \in (0, s]$ and each $t \in [\nu r, r]$. Then, for each $k = 0, 1, \ldots$ and each $r \in [\nu^{k+1}s, \nu^k s]$, we have

$$\begin{aligned}
&(1-\nu)^{-k-1}(r/s)^{p-m}\Phi(B(x,s))/s^m h(s) \\
&\quad \geq (1-\nu)^{-k-1}(r/s)^p \Phi(B(x,s))/r^m h(r) \geq \Phi(B(x,r))/r^m h(r) \\
&\quad \geq (1-\nu)^{k+1}(r/s)^p \Phi(B(x,s))/r^m h(\nu^{k+1}s) \\
&\quad \geq (1-\nu)^{k+1}(1-3\nu/2)^{-k-1}\nu^{k+1}(r/s)^{p-m}\Phi(B(x,s))/s^m h(s).
\end{aligned}$$

Using $0 < \liminf_{r \searrow 0}\Phi(B(x,r))/r^m h(r) < \infty$, we infer that $p = m$. Hence 4.4(3) implies that $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,m}[(\nu/5)^{n+4}]$ for Φ almost every x. Thus (2) follows from (1).

(3) For Φ almost every x we estimate

$$\begin{aligned}
&\limsup_{r \searrow 0} \nu^m \Phi(B(x, r/\nu))/\Phi(B(x,r)) \\
&\quad \leq \limsup_{r \searrow 0}\left[\Phi(B(x, r/\nu))/(r/\nu)^m h(r/\nu)\right]\left[r^m h(r)/\Phi(B(x,r))\right] \\
&\quad < 1 + \omega_3(n, \nu)
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{r \searrow 0} \nu^{m-1}\Phi(B(x, r/\nu))/\Phi(B(x,r)) \\
&\quad \geq \liminf_{r \searrow 0}\left[\nu^{-1}h(r/\nu)/h(r)\right] \\
&\qquad \times \liminf_{r \searrow 0}\left[\Phi(B(x, r/\nu))/(r/\nu)^m h(r/\nu)\right]\left[r^m h(r)/\Phi(B(x,r))\right] \\
&\quad \geq (1-2\nu)^{-1}(1+\omega_3)^{-1} \geq (1-2\nu)^{-1}(1+\nu)^{-1} > 1 + \nu.
\end{aligned}$$

Since 4.3(1) implies that Φ is $\omega_3(n, \nu)$ approximately uniformly distributed, we infer from 4.9 that $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{n,m}[\nu]$ for Φ almost every $x \in \mathbf{R}^n$. Hence, to finish the proof of (3), it suffices to use (1).

5.5. Corollary. *If* $0 \leq m \leq n$ *are integers,* Φ *measures* $\mathbf{R}^n$, *and* $0 < \overline{D}_m(\Phi, x) < (1 + \omega_3(n, \nu(n)))\underline{D}_m(\Phi, x) < \infty$ *at* Φ *almost every* x, *then* Φ *is* m *rectifiable.*

Proof. Use 5.4(3) with $h(r) = \alpha(m)$ and $\nu = \nu(n)$.

5.6. Theorem. *Whenever* Φ *is an almost finite measure over* $\mathbf{R}^n$, *the following conditions are equivalent.*

(1) Φ *is* m *rectifiable.*

(2) $0 < D_m(\Phi, x) < \infty$ *at* Φ *almost every* $x \in \mathbf{R}^n$.

(3) *For* Φ *almost every* $x \in \mathbf{R}^n$

$$\underline{D}_m(\Phi, x) < \infty \quad \textit{and} \quad \operatorname{Tan}(\Phi, x) = \mathfrak{M}_{n,V} \quad \textit{for some } V \in G(n, m).$$

(4) *For* Φ *almost every* $x \in \mathbf{R}^n$

$$0 < \underline{D}_m(\Phi, x) < \infty \text{ and } \operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_n.$$

Proof. (1) ⇒ (2), (3). If E is an m dimensional submanifold of class one of $\mathbf{R}^n$ and if $\Phi = \mathscr{H}^m \llcorner E$, one easily sees that, for every $x \in E$, $D_m(\Phi, x) = 1$ and $\operatorname{Tan}(\Phi, x) = \mathfrak{M}_{n,V}$ for some $V \in G(n, m)$. (Cf. [11, 3.2.19].) Hence (2) follows from 1.7 and (3) follows from 2.3(4).

(2) ⇒ (1). See 5.5.

(3) ⇒ (4). Using 4.4(5) and 2.3(4), we reduce the problem to the special case when spt Φ is a subset of the graph of a Lipschitzian map from $\mathbf{R}^m$ to $\mathbf{R}^{n-m}$. We easily see that $\mathscr{H}^m \llcorner E$ is locally finite and $\underline{D}_m(\mathscr{H}^m \llcorner E, x) > 0$ for every $x \in E$. Hence

$$\{x \in E;\ \underline{D}_m(\Phi, x) = 0\} \subset \left\{x \in E;\ \liminf_{r \searrow 0} \Phi(B(x, r))/\mathscr{H}^m(E \cap B(x, r)) = 0\right\}$$

and 1.7 implies that $\Phi\{x;\ \underline{D}_m(\Phi, x) = 0\} = 0$.

(4) ⇒ (1). See 5.4(2).

5.7. Whenever $1 \le m \le n$ are integers, let $\tilde{\omega}(n, m)$ be the largest number having the property: If Φ measures $\mathbf{R}^n$ and

$$0 < \overline{D}_m(\Phi, x) < (1 + \tilde{\omega}(n, m))\underline{D}_m(\Phi, x) < \infty$$

then Φ is m rectifiable.

According to 5.5, $\tilde{\omega}(n, m) > 0$. Nevertheless, our method does not give any lower estimate of $\tilde{\omega}(n, m)$. Here we show that $\inf_n \tilde{\omega}(n, 2) = 0$. This fact should be compared with the estimate $\tilde{\omega}(n, 1) \ge .01$ given in [21].

Let Z denote the set of integers, $X = \{0, 1\}^Z$,

$$Y = \{x \in X;\ \text{there is } p \in Z \text{ such that } x_k = 0 \quad \text{for each } k < p\},$$

and let Ψ be a Borel measure over Y defined by the requirement

$$\Psi\{x \in Y;\ x_k = y_k \quad \text{for } k \le p\} = 2^{-p} \quad \text{whenever } p \in Z \quad \text{and} \quad y \in Y.$$

If $x \in Y$ and $t > 0$ is irrational, we find $\sigma \in Y$ such that $t = \sum_{k \in Z} \sigma_k 2^{-k}$ and we compute

$$\begin{aligned}
&\Psi\left\{y \in Y;\ \sum_{k \in Z} |y_k - x_k| 2^{-k} < t\right\} \\
&\quad = \sum_{p \in Z} \Psi\{y \in Y;\ |y_k - x_k| = \sigma_k \text{ for } k < p \text{ and } |y_p - x_p| = 0 < 1 = \sigma_p\} \\
&\quad = \sum_{p \in Z,\ \sigma_p = 1} 2^{-p} = t.
\end{aligned}$$

By continuity, we infer that

$$\Psi\left\{y \in Y;\ \sum_{k\in Z} |y_k - x_k| 2^{-k} < t\right\} = t$$

whenever $x \in Y$ and $t > 0$.

Let $n \geq 4$ and let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbf{R}^n$. We define $e_{k+jn} = e_k$ whenever $k = 1, \ldots, n$ and $j \in Z$, and $f(x) = \sum_{k\in Z} 2^{-k/2} x_k e_k$ whenever $x \in Y$.

We easily see that the map $f\colon Y \mapsto \mathbf{R}^n$ is Borel measurable. If $x, y \in Y$, we compute

$$\begin{aligned} \|f(x) - f(y)\|^2 &= \left\| \sum_{k\in Z} (x_k - y_k) 2^{-k/2} e_k \right\|^2 \\ &= \sum_{k\in Z} |x_k - y_k| 2^{-k} \\ &\quad + 2 \sum_{k\in Z} \sum_{j=1}^{\infty} (x_k - y_k)(x_{k+jn} - y_{k+jn}) 2^{-k-jn/2}. \end{aligned}$$

Hence

$$\|f(x) - f(y)\|^2 \leq \sum_{k\in Z} |x_k - y_k| 2^{-k} \left[1 + 2(2^{n/2} - 1)^{-1}\right]$$

and

$$\|f(x) - f(y)\|^2 \geq \sum_{k\in Z} |x_k - y_k| 2^{-k} \left[1 - 2(2^{n/2} - 1)^{-1}\right].$$

Let $\Phi = f[\Psi]$. Whenever $x \in Y$ and $r > 0$, we estimate

$$\Phi(B(f(x), r)) = \Psi\left\{y; \|f(x) - f(y)\|^2 \leq r^2\right\} \leq \left[1 - 2(2^{n/2} - 1)^{-1}\right]^{-1} r^2$$

and

$$\Phi(B(f(x), r)) \geq \left[1 + 2(2^{n/2} - 1)^{-1}\right]^{-1} r^2.$$

Thus

$$(1) \quad \left[1 + 2(2^{n/2} - 1)^{-1}\right]^{-1} r^2 \leq \Phi(B(z, r)) \leq \left[1 - 2(2^{n/2} - 1)^{-1}\right]^{-1} r^2$$

for every $z \in \operatorname{spt} \Phi$ and every $r > 0$.

If $x \in Y$, $r > 0$, $V \in G(n, 2)$, and $d_1(T_{f(x), r}[\Phi], \mathfrak{M}_{n,V}) < 5^{-25}$, we find $k \in Z$ such that $2^{-(k-1)/2} \leq r < 2^{-(k-2)/2}$ and we note that for each $j = 0, 1, 2$ there is $t_j = \pm 1$ such that $f(x) + t_j 2^{-(k+j)/2} e_{k+j} \in \operatorname{spt} \Phi$. Since the vectors e_k, e_{k+1}, e_{k+2} are orthonormal, there is $j = 0, 1, 2$ such that

$$B^0\left(f(x) + t_j s e_{k+j}, s/3\right) \cap (f(x) + V) = \varnothing,$$

where $s = 2^{-(k+j)/2}$. Thus (1) and 4.4(4) with $\sigma = 5^{-5}$ imply

$$\left[1 + 2(2^{n/2} - 1)^{-1}\right]^{-1}(s/4)^2 \le \Phi\big(B\big(f(x) + t_j s e_{k+j}, s/4\big)\big)$$
$$\le \Phi\big(B\big(f(x), 5s/4\big) - B\big(f(x) + V, \sigma^2 r/2\big)\big)$$
$$\le 5\sigma\Phi\big(B\big(f(x), 5s/4\big)\big)$$
$$\le 5\sigma\left[1 - 2(2^{n/2} - 1)^{-1}\right]^{-1}(5s/4)^2.$$

But this implies $\sigma \ge 5^{-3}(2^{n/2} - 3)/(2^{n/2} + 1) \ge 5^{-4}$, which is a contradiction. Thus

(2) $d_1(T_{z,r}[\Phi], \mathfrak{M}_n) \ge 5^{-25}$ whenever $z \in \operatorname{spt} \Phi$ and $r > 0$.

Clearly (1), (2), and 5.6 show that $\tilde{\omega}(n,2) \le 4/(2^{n/2} - 1)$ for $n \ge 4$. Moreover, this example also illustrates the behaviour of the (optimal values) of the constants $\omega_2(n, \sigma)$ from 4.5. Namely, we obtain $\omega_2(n, 5^{-25}) \le 4/(2^{n/2} - 1)$ if $n \ge 4$.

An amusing variant of this example is obtained by replacing $\mathbf{R}^n$ by an infinitely dimensional Hilbert space H and by taking an orthonormal system $\{e_k;\ k \in Z\}$. The above construction then gives a rather singular nonzero Borel measure over H such that $\Phi(B(z, r)) = r^2$ for every $z \in \operatorname{spt} \Phi$ and every $r > 0$.

5.8. The following construction of a measure Φ over $\mathbf{R}^2$ will be used in 5.9 to give examples illustrating the need for the density assumptions in 5.6(3, 4). It will be also used in 6.5.

Let $c = 0$, $1/2$, or ∞ and let $a_1, a_2, \ldots \in [0, 1/400]$ be such that $\limsup_{k\to\infty} k^{1/3} a_k \le c \le \prod_{k=1}^{\infty}(1 + a_k^2)$. Also, let $\tau_k = \sup\{a_k, a_{k+1}, \ldots\}$.

Let e_1, e_2 be the standard basis of $\mathbf{R}^2$ and for every $e \in \mathbf{R}^2$ let $e^\perp = \langle e, e_2\rangle e_1 - \langle e, e_1\rangle e_2$.

Whenever $k = 0, 1, \ldots,$ $i = 0, 1, \ldots, 2^k - 1$, and $t \in (i2^{-k}, (i+1)2^{-k})$, we define $s_k(t) = (-1)^{i(i+1)/2}$. Let $A_k = (0,1) - \{i2^{-k};\ i = 0, 1, \ldots, 2^k\}$ be the domain of the function s_k.

We put $g_0(t) = e_1$ for $t \in (0,1)$ and $\varepsilon_0 = 1/400$. By induction we shall define maps g_k of A_k into $\mathbf{R}^2$, functions $\hat{s}_k$ defined on A_k and constants ε_k as follows:

Suppose that $k \ge 1$ and that g_j and ε_j have been defined for all $0 \le j < k$. If $t \in A_k$, we put

$\hat{s}_k(t) = 0$ if there is $u \in A_k$ such that $|u - t| < 2^{-k+2}$,

$s_{k-1}(u) = s_{k-1}(t)$ and $\|g_j(u)\| \le c + 3\varepsilon_{k-1}$ for each $j = 0, \ldots, k-1$, and

$\hat{s}_k(t) = \varepsilon_{k-1} s_k(t)$ in the opposite case.

Finally, we put

$$g_k(t) = (1 + \hat{s}_k(t))g_{k-1}(t) + a_k s_k(t) g_{k-1}(t)^\perp, \quad \text{and}$$

$$\varepsilon_k = \varepsilon_{k-1} \quad \text{if } \mathscr{L}^1\{t \in A_k;\ \|g_j(t)\| > c + 24\varepsilon_{k-1} \text{ for each } j = 0, 1, \dots, k\} \geq \varepsilon_{k-1},$$

$$\varepsilon_k = \varepsilon_{k-1} \quad \text{if } \mathscr{L}^1\left\{t \in A_k;\ \|g_j(t)\| < \min\left(1/\varepsilon_{k-1}, \prod_{i=k}^{\infty}(1 + a_i^2)\right) \text{ for each } j = 0, 1, \dots, k\right\} \geq \varepsilon_{k-1}, \quad \text{and}$$

$\varepsilon_k = \varepsilon_{k-1}/2$ in all other cases.

We easily see that

(a) the function g_k is constant on $(i2^{-k}, (i+1)2^{-k})$, and

(b) $\|g_k(t)\|^2 = \prod_{j=1}^{k}[(1 + \hat{s}_j(t))^2 + a_j^2]$.

We prove the following statements.

(c) $\|g_k(t)\| \geq \varepsilon_k + \min(c, 1/2)$ for every $t \in A_k$.

(d) If $k \geq q \geq 1$, $i = 0, 1, \dots, 2^{q-1} - 1$, and $(2i+1)2^{-q} - 2^{-k} < t < (2i+1)2^{-q} < \hat{t} < (2i+1)2^{-q} + 2^{-k}$, then

$$\|g_k(\hat{t}) - g_k(t)\| \leq 8(\varepsilon_{q-1} + \tau_q)\min\{\|g_k(v)\|;\ v \in [t, \hat{t}] \cap A_k\}.$$

(e) $\lim_{k\to\infty} \varepsilon_k = 0$.

(f) $\liminf_{k\to\infty} \|g_k(t)\| = c$ for $\mathscr{L}^1$ almost every $t \in (0, 1)$.

(g) If $\prod_{k=1}^{\infty}(1 + a_k^2) = \infty$ then $\limsup_{k\to\infty}\|g_k(t)\| = \infty$ for $\mathscr{L}^1$ almost every $t \in (0, 1)$.

Proof. (c) If $k = 0$, this is obvious. If the statement holds for $k - 1$ then

$$\|g_k(t)\| \geq (1 + a_k^2)^{1/2}\|g_{k-1}(t)\| \geq \|g_{k-1}(t)\| \geq \varepsilon_k + \min(c, 1/2)$$

if $\hat{s}_k(t) = 0$ and

$$\|g_k(t)\| \geq (1 + a_k^2)^{1/2}(1 - \varepsilon_{k-1})\|g_{k-1}(t)\| \geq (1 - \varepsilon_{k-1})(c + 3\varepsilon_{k-1}) \geq \varepsilon_k + \min(c, 1/2)$$

if $\hat{s}_k(t) \neq 0$.

(d) Whenever $v \in [t, \hat{t}] \cap A_k$, we use (a) to infer that $g_{q-1}(t) = g_{q-1}(\hat{t}) = g_{q-1}(v)$ and we recall that

$$g_q(t) = (1 + \hat{s}_q(t))g_{q-1}(v) + a_q s_q(t) g_{q-1}(v)^\perp$$

and

$$g_{q+1}(t) = \left[(1 + \hat{s}_{q+1}(t))(1 + \hat{s}_q(t)) - a_{q+1}a_q s_{q+1}(t)s_q(t)\right]g_{q-1}(v) + \left[(1 + \hat{s}_{q+1}(t))a_q s_q(t) + (1 + \hat{s}_q(t))a_{q+1}s_{q+1}(t)\right]g_{q-1}(v)^\perp.$$

Hence

$$\|g_q(\hat{t}) - g_q(t)\| \le 2(\varepsilon_{q-1} + a_q)\|g_{q-1}(v)\|$$

and

$$\|g_{q+1}(\hat{t}) - g_{q+1}(t)\| \le 7(\varepsilon_{q-1} + \tau_q)\|g_{k-1}(v)\|.$$

Since $\|g_q(v)\| \ge (1 - \varepsilon_{q-1})\|g_{q-1}(v)\|$ and

$$\|g_{q+1}(v)\| \ge (1 - \varepsilon_{q-1})^2\|g_{q-1}(v)\|,$$

and since $(1 - \varepsilon_{q-1})^{-2} \le 8/7$, this implies (d) if $k = q$ or $k = q + 1$. If $k > q + 1$, we note that $s_j(t) = s_j(\hat{t}) = s_j(v)$ for each $j = q + 1, \dots, k$ and that, consequently, $\hat{s}_j(t) = \hat{s}_j(\hat{t}) = \hat{s}_j(v)$ for each $j = q + 2, \dots, k$. Hence, by induction,

$$\begin{aligned} \|g_k(\hat{t}) - g_k(t)\| &= \left[(1 + \hat{s}_k(v))^2 + a_k^2\right]^{1/2}\|g_{k-1}(\hat{t}) - g_{k-1}(t)\| \\ &\le 8(\varepsilon_{q-1} + \tau_q)\left[(1 + \hat{s}_k(v))^2 + a_k^2\right]^{1/2}\|g_{k-1}(v)\| \\ &= 8(\varepsilon_{q-1} + \tau_q)\|g_k(v)\|. \end{aligned}$$

(e) Assume that there is p such that $\varepsilon_p = \varepsilon_{p+1} = \dots$. Then at least one of the sets

$$E = \left\{t \in \bigcap_{k=0}^{\infty} A_k;\ \|g_j(t)\| > c + 24\varepsilon_p \quad \text{for each } j = 0, 1, \dots\right\}$$

and

$$H = \left\{t \in \bigcap_{k=0}^{\infty} A_k;\ \|g_j(t)\| < \min\left(1/\varepsilon_p;\ \prod_{i=k}^{\infty}\left(1 + a_i^2\right)\right) \quad \text{for each } j = 0, 1, \dots \quad \text{and each } k = 1, 2, \dots\right\}$$

has positive Lebesgue measure.

Suppose first that $\mathscr{L}^1(E) > 0$. Then clearly $c < \infty$ and, consequently, $\lim_{k\to\infty} a_k = 0$. Let $\hat{p} > p$ be such that $\tau_{\hat{p}} < \varepsilon_p$. Whenever $k > \hat{p}$, $t \in E$, $(t - 2^{-k+2}, t + 2^{-k+2}) \subset A_{\hat{p}+1}$, $u \in A_k$, $|u - t| < 2^{-k+2}$ and $s_{k-1}(u) = s_{k-1}(t)$, we conclude from (a) and (d) that

$$\|g_k(u) - g_k(t)\| \le 8(\varepsilon_p + \tau_{\hat{p}})\|g_k(t)\| \le 9\varepsilon_p\|g_k(t)\|.$$

Hence $\|g_k(u)\| > c + 3\varepsilon_p$. Consequently, there are a set $\tilde{E} \subset E$ and $q \ge \hat{p}$ such that $\mathscr{L}^1(\tilde{E}) > 0$, $a_k \le k^{-1/3}$, and $\hat{s}_k(t) = \varepsilon_p s_k(t)$ for each $k \ge q$ and each

$t \in \tilde{E}$. Finally, using the law of the iterated logarithm, we conclude that

$$\begin{aligned}\liminf_{k\to\infty} \|g_k(t)\| &= \liminf_{k\to\infty} \prod_{j=1}^{k}\left(1+\hat{s}_j(t)\right)\prod_{j=1}^{k}\left(1+a_j^2/\left(1+\hat{s}_j(t)\right)^2\right)^{1/2}\\ &\le \liminf_{k\to\infty}\exp\left[\sum_{j=1}^{k}\hat{s}_j(t)+\sum_{j=1}^{k}a_j^2\right]\\ &\le \liminf_{k\to\infty}\exp\left[q+3(k+1)^{1/3}+\varepsilon_p\sum_{j=q}^{k}s_j(t)\right]=0\end{aligned}$$

for $\mathscr{L}^1$ almost every $t \in \tilde{E}$. But this obviously contradicts $\mathscr{L}^1(\tilde{E}) > 0$ and $\tilde{E} \subset E$.

Suppose next that $\mathscr{L}^1(H) > 0$. Since $\|g_0(t)\| \ge 1$, we easily see that $\Pi_{j=1}^{\infty}(1 + a_j^2) = \infty$. Let $a \in (0, 1/\varepsilon_p]$, $\hat{p} \ge p$, and $\tilde{H} \subset H$ be such that $\mathscr{L}^1(\tilde{H}) > 0$, $\sup\{\|g_j(t)\|;\ j = \hat{p}, \hat{p}+1, \dots\} \le a$ and $\limsup_{j\to\infty}\|g_j(t)\| > a - \varepsilon_p$ for every $t \in \tilde{H}$.

Let $t \in \tilde{H}$ be an $\mathscr{L}^1$ density point of $\tilde{H}$ and let $q > \hat{p}$ be such that $\|g_q(t)\| > a - \varepsilon_p$ and $\mathscr{L}^1(I - \tilde{H}) < \mathscr{L}^1(I)/2$ whenever I is an interval containing t and $\mathscr{L}^1(I) \le 2^{-q}$.

We claim that $\|g_k(t)\| = \|g_q(t)\|\Pi_{j=q+1}^{k}(1 + a_j^2)^{1/2}$ for each $k = q, q+1, \dots$. If $k = q$, this is obvious. Suppose that our claim holds for some $k \ge q$ and that $\hat{s}_{k+1}(t) \ne 0$. Let I be the component of A_k containing t. Then

$$\mathscr{L}^1\left\{u \in I;\ \hat{s}_{k+1}(u) = \varepsilon_p\right\} = \mathscr{L}^1(I)/2$$

and, consequently,

$$\mathscr{L}^1(I)/2 > \mathscr{L}^1(I - \tilde{H}) \ge \mathscr{L}^1\left\{u \in I;\ g_{k+1}(u) = g_k(t) + \varepsilon_p\right\} = \mathscr{L}^1(I)/2.$$

Hence $\hat{s}_{k+1}(t) = 0$, which implies

$$\|g_{k+1}(t)\| = \|g_k(t)\|\left(1 + a_{k+1}^2\right)^{1/2}.$$

This proves the above claim. But, since $\Pi_{j=1}^{\infty}(1 + a_j^2) = \infty$, the validity of our claim obviously contradicts $t \in H$.

(f) If $c = \infty$, $\Pi_{j=1}^{\infty}(1 + a_j^2) = \infty$ and $\hat{s}_j = 0$ for each j. Hence the statement follows from (b). If $c < \infty$, we see from (e) that $\inf\{\|g_k(t)\|;\ k = 0, 1, \dots\} \le c$ for $\mathscr{L}^1$ almost every $t \in (0, 1)$. Hence the statement follows from (c).

(g) This follows easily from (e).

For each $k = 0, 1, \dots$ let f_k be an indefinite integral of g_k such that $f_k(0) = 0$.

If I is a component of A_k then $f_{k+1} = f_k$ at its end points. Since $\|g_{k+1}(u) - g_k(u)\| \le (\varepsilon_k + \tau_k)\|g_k(t)\|$ whenever $u, t \in I$,

$$\|f_{k+1}(t) - f_k(t)\| \le (\varepsilon_k + \tau_k)\|g_k(t)\|\mathscr{L}^1(I)/2 = 2^{-k-1}(\varepsilon_k + \tau_k)\|g_k(t)\|$$

for every $t \in I$. Since, according to (b), $\|g_k(t)\| \le (3/2)^{k/2}$, the sequence f_k converges to a continuous map f: $[0,1] \mapsto \mathbf{R}^2$. Moreover, for each $k = 0, 1, \ldots$ and each $t \in A_k$:

$$\text{(h)} \qquad \|f(t) - f_k(t)\| \le \sum_{j=k}^{\infty} 2^{-j-1}(\varepsilon_j + \tau_j)\|g_k(t)\|(3/2)^{(j-k)/2}$$
$$\le 2^{-k+1}(\varepsilon_k + \tau_k)\|g_k(t)\|.$$

We prove that:

(i) If $k = 0, 1, \ldots,$ $0 \le t < u \le 1$, $2^{-k-1} < u - t < 2^{-k}$ and q is the smallest integer such that $(t, u) - A_q \neq \varnothing$, then

$$\big(1 - 24(\varepsilon_{q-1} + \tau_{q-1})\big)\|g_k(v)\|(u-t)$$
$$\le \|f(u) - f(t)\|$$
$$\le \big(1 + 24(\varepsilon_{q-1} + \tau_{q-1})\big)\|g_k(v)\|(u-t)$$

for every $v \in A_k \cap [t, u]$.

In fact, we easily see that $q \le k + 1$ and that (a) (if $q = k + 1$) or (d) (if $q \le k$) imply

$$\|f_k(u) - f_k(t) - (u-t)g_k(v)\| \le 8(\varepsilon_{q-1} + \tau_{q-1})\|g_k(v)\|(u-t)$$

for every $v \in (t, u) \cap A_k$. Using also (h), we obtain

$$\|f(u) - f(t) - (u-t)g_k(v)\|$$
$$\le 8(\varepsilon_{q-1} + \tau_{q-1})\|g_k(v)\|(u-t)$$
$$+2^{-k+1}(\varepsilon_{q-1} + \tau_{q-1})\big(\|g_k(u)\| + \|g_k(t)\|\big)$$
$$\le 8(\varepsilon_{q-1} + \tau_{q-1})\|g_k(v)\|(u-t) + 2^{-k+3}(\varepsilon_{q-1} + \tau_{q-1})\|g_k(v)\|$$
$$\le 24(\varepsilon_{q-1} + \tau_{q-1})\|g_k(v)\|(u-t)$$

if $u, t \in A_k$ and $v \in (t, u) \cap A_k$. By continuity, this inequality extends to all t, u and $v \in [t, u] \cap A_k$.

Whenever $t \in \bigcap_{k=0}^{\infty} A_k$ and $r \in (0, 1]$, we note that $\|g_0(t)\| = 1 \ge 2^0 r$ and $\lim_{k\to\infty} 2^{-k}\|g_k(t)\| \le \lim_{k\to\infty} 2^{-k}(3/2)^{k/2} = 0$. Hence we may find the largest integer $k = k(t, r)$ such that $\|g_k(t)\| \ge 2^k r$. We also define $h(t, r) = r/\|g_{k(t,r)}(t)\|$. We shall need the following facts.

(j) $k(t, 1) = 0$.

(k) If $s \in [(3/8)^{1/2}, 1]$ then $k(t, r) \le k(t, sr) \le k(t, r) + 1$.

(m) $2^{-k(t,r)} \le 2r^{1/2}$,

(n) If $q = 0, 1, \ldots,$ $\tau = 30(\varepsilon_q + \tau_q)$, and $r < 4^{-q-2}\text{dist}^2(t, [0,1] - A_q)$, then

$$\begin{aligned}(t - (1 + \tau)h(t, r), t + (1 + \tau)h(t, r))\\ \supset f^{-1}(B(f(t), r))\\ \supset (t - (1 - \tau)h(t, r), t + (1 - \tau)h(t, r)).\end{aligned}$$

Proof. (j) If $j \geq 1$ then $\|g_j(t)\| \leq (3/2)^{j/2} < 2^j$.

In the proof of the remaining statements we shall denote $k = k(t, r)$.

(k) The inequality $k \leq k(t, sr)$ is obvious. If $j \geq k + 2$ then

$$\|g_j(t)\| \leq (3/2)^{(j-k-1)/2}\|g_{k+1}(t)\| < (3/2)^{(j-k-1)/2}2^{k+1}r \leq 2^j sr.$$

Hence $k(t, sr) \leq k + 1$.

(m) Since $(2/3)^{(k+1)/2} \leq \|g_{k+1}(t)\| \leq 2^{k+1}r$,

$$2^{-k-1} \leq 6^{-(k+1)/4} = \left[2^{-k-1}(2/3)^{(k+1)/2}\right]^{1/2} \leq r^{1/2}.$$

(n) Suppose that $j = 0, 1, \ldots,$ $u \in (0,1)$, $2^{-j-1} \leq |u - t| < 2^{-j}$, and $\|f(u) - f(t)\| \leq r$. From (i) we infer that

$$\|g_i(t)\| \leq 2^{i-j}\|g_j(t)\| < 2^{i-j+2}r/(3|u - t|) \leq 2^i(8r/3)$$

whenever $i \geq j$. Hence (k) implies that $j \geq k - 1$. Using this, (m), and (i), we obtain

$$\begin{aligned}|u - t| &\leq \left(1 - 24(\varepsilon_q + \tau_q)\right)^{-1} r/\|g_j(t)\|\\ &\leq \left(1 - 24(\varepsilon_q + \tau_q)\right)^{-1}(1 + \varepsilon_q + \tau_q)r/\|g_k(t)\| < (1 + \tau)h(t, r)\end{aligned}$$

if $j \leq k$, and

$$|u - t| \leq 2^{-k-1} < r/\|g_{k+1}(t)\| \leq \left(1 - \varepsilon_q\right)^{-1} r/\|g_k(t)\| < (1 + \tau)h(t, r)$$

if $j > k$. This proves the first inclusion in (n).

If $|u - t| < (1 - \tau)h(t, r)$ then $|u - t| < 2^{-k}$. Hence $u \in (0,1)$ according to (m) and there is $j \geq k$ such that $2^{-j-1} \leq |u - t| < 2^{-j}$. Using (m) and (i), we obtain

$$\begin{aligned}\|f(u) - f(t)\| &\leq \left(1 + 24(\varepsilon_q + \tau_q)\right)\|g_j(t)\||u - t|\\ &\leq \left(1 + \varepsilon_q + \tau_q\right)\left(1 + 24(\varepsilon_q + \tau_q)\right)\|g_k(t)\||u - t|\\ &\leq \left(1 + \varepsilon_q + \tau_q\right)\left(1 + 24(\varepsilon_q + \tau_q)\right)(1 - \tau)r \leq r\end{aligned}$$

if $j \leq k + 1$, and

$$\begin{aligned}\|f(u) - f(t)\| &\leq \left(1 + 24(\varepsilon_q + \tau_q)\right)(3/2)^{(j-k-1)/2}\|g_{k+1}(t)\|2^{-j}\\ &\leq \left(1 + 24(\varepsilon_q + \tau_q)\right)(3/2)^{(j-k-1)/2}2^{k-j+1}r \leq r\end{aligned}$$

if $j > k + 1$.

Finally, we define $\Phi = f[\mathscr{L}^1 \llcorner (0,1)]$ and we summarize some of the previous results in the following statements.

For $\mathscr{L}^1$ almost every $t \in (0,1)$:

(1)
$$\left(1 - 30 \limsup_{k\to\infty} a_k\right) \le \liminf_{r\searrow 0} \Phi(B(f(t), r))/(2h(t,r))$$
$$\le \limsup_{r\searrow 0} \Phi(B(f(t), r))/(2h(t,r))$$
$$\le \left(1 + 30 \limsup_{k\to\infty} a_k\right),$$

(2) $\overline{D}_1(\Phi, f(t)) = 1/c$,
(3) If $\prod_{k=1}^{\infty}(1 + a_k^2) = \infty$ then $\underline{D}_1(\Phi, f(t)) = 0$, and
(4) If $\lim_{k\to\infty} a_k = 0$ then $\operatorname{Tan}(\Phi, f(t)) \subset \mathfrak{M}_{2,1}$.

Proof. The statement (1) follows immediately from (n). To prove (2) and (3), we use first (j), (k), and (m) to conclude that for $\mathscr{L}^1$ almost every $t \in (0,1)$ the values of $k(t,r)$ run through all nonnegative integers and $\lim_{r\searrow 0} k(t,r) = \infty$. Hence (f) and (g) imply

$$\limsup_{r\searrow 0} h(t,r)/r = 1\Big/\liminf_{k\to\infty} \|g_k(t)\| = 1/c$$

and

$$\liminf_{r\searrow 0} h(t,r)/r = 1\Big/\limsup_{k\to\infty} \|g_k(t)\| = 0$$

if $\prod_{k=1}^{\infty}(1 + a_k^2) = \infty$.

If $c < \infty$ then $\lim_{k\to\infty} a_k = 0$ and (2) and (3) follow from (1). If $c = \infty$ then (1) implies

$$\lim_{r\searrow 0} \Phi(B(f(t), r))/2r$$
$$\le \lim_{r\searrow 0} (h(t,r)/r) \lim_{r\searrow 0} \Phi(B(f(t), r))/(2h(t,r))$$
$$\le 2 \lim_{k\to\infty} \left(1/\|g_k(t)\|\right) = 2 \prod_{k=1}^{\infty} \left(1 + a_k^2\right)^{-1/2} = 0.$$

Finally, to prove (4), we use (k) to infer that

$$(1 + \varepsilon_k + a_k)^{-1} \|g_k(t)\| \le \|g_{k(t,sr)}(t)\| \le (1 + \varepsilon_k + a_k) \|g_k(t)\|$$

if $s \in [3/4, 1]$ and $k = k(t,r)$. Consequently

$$\lim_{r\searrow 0} h(t, sr)/h(t,r) = s \lim_{r\searrow 0} \|g_{k(t,r)}\| / \|g_{k(t,sr)}\| = s$$

for every $s \in [3/4, 1]$. Since this implies that $\lim_{r\searrow 0} h(t, sr)/h(t,r) = s$ for every $s > 0$, (4) follows from (1) and from 4.9.

5.9. *Examples*. (1) There is a finite, nonzero measure Φ over $\mathbf{R}$ such that $D_1(\Phi, x) = \infty$ and $\operatorname{Tan}(\Phi, x) = \mathfrak{M}_{1,1}$ for Φ almost every $x \in \mathbf{R}$.

(2) There is a finite, nonzero measure Φ over $\mathbf{R}^2$ such that $\overline{D}_1(\Phi, x) = 2$, $\underline{D}_1(\Phi, x) = 0$, and $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{2,1}$ for Φ almost every $x \in \mathbf{R}^2$.

(3) There is a finite, nonzero measure Φ over $\mathbf{R}^2$ such that $\overline{D}_1(\Phi, x) = \infty$, $\underline{D}_1(\Phi, x) = 0$, and $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{2,1}$ for Φ almost every $x \in \mathbf{R}^2$.

(4) There is a finite, nonzero measure Φ over $\mathbf{R}^2$ such that $D_1(\Phi, x) = 0$ and $\operatorname{Tan}(\Phi, x) \subset \mathfrak{M}_{2,1}$ for Φ almost every $x \in \mathbf{R}^2$.

Proof. (1) We use 5.8 with $c = 0$ and $a_1 = a_2 = \cdots = 0$. We easily see that the resulting measure is in fact concentrated on the real line. Hence we may consider Φ as a measure over $\mathbf{R}$. This and 5.8(4) imply that $\operatorname{Tan}(\Phi, x) = \mathfrak{M}_{1,1}$ for Φ almost every $x \in \mathbf{R}$. Moreover, from 5.8(2) we see that $\overline{D}_1(\Phi, x) = \infty$ for Φ almost every $x \in \mathbf{R}$ which, in view of 1.7, implies that $D_1(\Phi, x) = \infty$ for Φ almost every $x \in \mathbf{R}$.

(2, 3, 4) We use 5.8 with $c = 1/2$, 0, and ∞, respectively, and with $a_k = k^{-1/2}/400$. The resulting measures have the desired properties according to 5.8(2, 3, 4).

6. Densities

6.1. (1) If h is a positive function on $(0, \infty)$, Φ measures $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, we define the upper and lower h densities of Φ at x by the formulas

$$\overline{D}_h(\Phi, x) = \lim_{t \nearrow 1} \limsup_{r \searrow 0} \Phi(B(x, tr))/h(r)$$

and

$$\underline{D}_h(\Phi, x) = \lim_{t \searrow 1} \liminf_{r \searrow 0} \Phi(B(x, tr))/h(r).$$

If the upper and lower h densities of Φ at x coincide, we denote their common value by $D_h(\Phi, x)$.

(The densities introduced in 5.2 correspond to the functions $h(r) = \alpha(m)r^m$. (Cf. also (3) below.) We shall not use them in this chapter and hence the slight inconsistency in the notation cannot lead to any confusion.)

Whenever h is a positive function on $(0, \infty)$ and Φ measures $\mathbf{R}^n$, we observe that:

(2) $$\overline{D}_h(\Phi, x) \le \limsup_{r \searrow 0} \Phi(B(x, r))/h(r),$$

$$\underline{D}_h(\Phi, x) \ge \liminf_{r \searrow 0} \Phi(B(x, r))/h(r),$$

(3) The inequalities in (2) become equalities provided that $\limsup_{t \searrow 1} \limsup_{r \searrow 0} h(tr)/h(r) \le 1$,

(4) $\underline{D}_h(\Phi, x) \le \overline{D}_h(\Phi, x)$ for Φ almost every $x \in \mathbf{R}^n$, and

(5) If $0 < D_h(\Phi, x) < \infty$ almost everywhere then Φ is approximately uniformly distributed.

Proof. The statements (2) and (3) are obvious.

(4) If $\underline{D}_h(\Phi, x) > a > b > \overline{D}_h(\Phi, x)$ and if $\Psi \in \operatorname{Tan}(\Phi, x)$, we find $\varepsilon > 0$ and $s > 0$ such that

$$\Psi(B(0, s)) > b\Psi\big(B\big(0, (1+\varepsilon)^3 s\big)\big)/a$$

and

$$ah(r/(1+\varepsilon)) \le \Phi(B(x, r)) \le bh((1+\varepsilon)r)$$

for all sufficiently small $r > 0$. Then

$$\Phi(B(x, r)) \le b\Phi\big(B\big(x, (1+\varepsilon)^2 r\big)\big)/a$$

for all sufficiently small $r > 0$. Consequently

$$\Psi(B(0, s)) \le b\Psi\big(B\big(0, (1+\varepsilon)^3 s\big)\big)/a.$$

Hence the statement follows from 2.5.

(5) See 4.3(1).

6.2. (1) A positive function h defined on $(0, \infty)$ is said to be a density function (an exact density function, respectively) in $\mathbf{R}^n$ if there is a nonzero measure Φ over $\mathbf{R}^n$ such that

$$0 < D_h(\Phi, x) < \infty \left(0 < \lim_{r \searrow 0} \Phi(B(x, r))/h(r) < \infty, \text{ respectively}\right)$$

at Φ almost every $x \in \mathbf{R}^n$.

The function h is said to be an (exact) density function if it is an (exact) density function in some $\mathbf{R}^n$.

(2) From 6.1(2, 3) we see that:

(i) Every exact density function (in $\mathbf{R}^n$) is a density function (in $\mathbf{R}^n$), and

(ii) If h is a density function (in $\mathbf{R}^n$) and

$$\limsup_{t \searrow 1} \limsup_{r \searrow 0} h(tr)/h(r) = 1$$

then h is an exact density function (in $\mathbf{R}^n$).

(3) If $\limsup_{r \searrow 0} h(r) > 0$, we easily see that h is an (exact) density function if and only if $0 < \lim_{r \searrow 0} h(r) < \infty$.

(4) A density function h will be called regular if $\lim_{r \searrow 0} h(tr)/h(r)$ exists for each $t > 0$.

(5) From 4.11(4) we see that for irregular density function h the limit $\lim_{r \searrow 0} h(tr)/h(r)$ fails to exist for every $t > 0$, $t \ne 1$.

(6) If h is a positive function on $(0, \infty)$, we define

$$\operatorname{Dim}(h) = \Big\{ m = 0, 1, \ldots; \text{ there is a sequence } r_k \searrow 0 \text{ such that } \lim_{k \to \infty} h(tr_k)/h(r_k) = t^m \quad \text{for each } t > 0 \Big\}.$$

(7) From 6.1(5) and 4.12(2) we see that, whenever Φ is a nonzero measure over $\mathbf{R}^n$ and $0 < D_h(\Phi, x) < \infty$ at Φ almost every x, then $\operatorname{Dim}(h) = \operatorname{Dim}_x(\Phi)$ for Φ almost every $x \in \mathbf{R}^n$.

6.3. THEOREM. *If h is a density function in $\mathbf{R}^n$ then $\varnothing \neq \operatorname{Dim}(h) \subset \{0, 1, \ldots, n\}$.*

Proof. See 6.2(7) and 4.12(2).

6.4. PROPOSITION. *For each $n = 1, 2, \ldots$ and each $\lambda > 1$ there is a constant $\delta = \delta(n, \lambda) > 0$ with the following property:*

For every density function h in $\mathbf{R}^n$ there is $\tilde{r} > 0$ such that, whenever $0 < a < b < \tilde{r}$, $m = 1, \ldots, n$, and $(1 - \delta)2^m h(r) \leq h(2r) \leq (1 + \delta)2^m h(r)$ for every $r \in (\delta a, b/\delta)$, then $h(a) \leq \lambda(a/b)^m h(b)$.

Proof. Let $\varepsilon \in (0, 2^{-n-5})$ and $\sigma \in (0, 2^{-7}n^{-3}\varepsilon^2)$ be such that

$$(1 - \varepsilon)^{-n-10}(1 + 2^{n+4}\varepsilon)(1 + \varepsilon)^n \leq \lambda \quad \text{and} \quad (1 - \sigma)^{10} > 1 - \varepsilon.$$

We also denote $\tilde{\sigma} = (\varepsilon\sigma/60n)^{n+3}$ and $\omega = \omega_3(n, \tilde{\sigma})$. (See 4.8.) We prove that 6.4 holds with $\delta = (\omega/5)^{n+1}$.

Suppose that h is a density function in $\mathbf{R}^n$, Φ is a nonzero measure over $\mathbf{R}^n$, and $0 < D_h(\Phi, x) < \infty$ at Φ almost every $x \in \mathbf{R}^n$. Let $\eta = (1 + \delta)^{1/(2n+1)}$. Multiplying Φ by a suitable constant, we achieve that $\Phi\{x \in \mathbf{R}^n;\ 1 < D_h(\Phi, x) < \eta\} > 0$.

Let $A \subset \operatorname{spt} \Phi$ and $s_0 > 0$ be such that $\Phi(A) > 0$ and $\Phi(B(x, \eta r)) \geq h(r) \geq \eta^{-1}\Phi(B(x, r/\eta))$ for every $x \in A$ and every $r \in (0, s_0)$. Using the density theorem twice, we find $\tilde{A} \subset \hat{A} \subset A$ and $0 < s_2 < s_1 < s_0$ such that $\Phi(\tilde{A}) > 0$, $\Phi(B(x, r) \cap A) > (1 - \delta)\Phi(B(x, r))$ for every $x \in \hat{A}$ and every $r \in (0, s_1)$, and $\Phi(B(x, r) \cap \hat{A}) > (1 - \delta)\Phi(B(x, r))$ for every $x \in \tilde{A}$ and every $r \in (0, s_2)$.

We may assume that $0 \in \tilde{A}$ is a Φ density point of $\tilde{A}$. Hence there is $0 < s_3 < s_2$ such that

$$\Phi\big(B(0, r) \cap \tilde{A}\big) > (1 - \delta)\Phi(B(0, r)) \quad \text{for every } r \in (0, s_3).$$

We prove that the statement of 6.4 holds with $\tilde{r} = \delta^2 s_3$. Assume, to the contrary, that there are $0 < a < b < \tilde{r}$ and $m = 1, \ldots, n$ such that $h(a) >$

$\lambda(a/b)^m h(b)$ and

$$(1-\delta)2^m h(r) \le h(2r) \le (1+\delta)2^m h(r)$$

for every $r \in (\delta a, b/\delta)$.

Let

$$\tilde{E} = \{x \in B(0,b) \cap \tilde{A};\ d_5(T_{x,s}[\Phi], \mathfrak{M}_{n,m}) < \tilde{\sigma} \text{ for every } s \in [a,b]\}.$$

Since $\hat{A} \subset Y_s(\Phi, \omega)$ for every $s \in (0, 6b/\omega^2)$ and since

$$\Phi(B(x, 2\eta^{-2}r)) \le \eta h(2\eta^{-1}r) \le (1+\delta)\eta 2^m h(\eta^{-1}r)$$
$$\le (1+\delta)\eta 2^m \Phi(B(x,r)) \le (1+\omega)(2\eta^{-2})^m \Phi(B(x,r))$$

and

$$\Phi(B(x, 2\eta^2 r)) \ge h(2\eta r) \ge (1-\delta)2^m h(\eta r)$$
$$\ge (1+\tilde{\sigma})(2\eta^2)^{m-1}\Phi(B(x,r))$$

for every $x \in \tilde{A}$ and every $r \in (\omega a, 6b/\omega^2)$, we infer from 4.8 that $\Phi[(\tilde{A} - \tilde{E}) \cap B(0,b)] \le \tilde{\sigma}\Phi(B(0,6b))$. Hence 4.2(2) implies that

$$\Phi(\tilde{E}) \ge \Phi(B(0,b)) - 2\tilde{\sigma}\Phi(B(0,6b))$$
$$\ge \Phi(B(0,b)) - (\sigma/2)^{n+1}\Phi(B(0, \varepsilon b/n)).$$

From 4.4(4) we infer that:

(a) For every $x \in \tilde{E}$ and for every $r \in [a,b]$ there is $V \in G(n,m)$ such that $\Phi(B(y,t)) \ge (1-\sigma)(t/s)^m \Phi(B(z,s))$ whenever $y, z \in B(x,r) \cap (x+V)$ and $\sigma r \le s, t \le 3r$.

Consequently,

(b) $$(1-\sigma)^2 h(r) \le (1-\sigma)^2\Phi(B(x,\eta r)) \le (1-\sigma)\eta^m\Phi(B(x,r))$$
$$\le \Phi(B(x,r)) \le (1-\sigma)^{-1}\eta^m\Phi(B(x,r/\eta))$$
$$\le (1-\sigma)^{-1}\eta^{m+1}h(r) \le (1-\sigma)^{-2}h(r)$$

for every $x \in \tilde{E}$ and every $r \in [a,b]$.

Since $\tilde{E} \ne \varnothing$, we infer from (a) and (b) that

(c) $h(u)/u^m \le (1-\sigma)^{-5}h(v)/v^m$ whenever $a \le u < v \le b$ and $u \ge \sigma v$.

Let

$$r = \inf\{s \in [a,b];\ h(s)/s^m \le h(b)/b^m\}$$

and let $r_1 = r(1 - \varepsilon/m)$. Also, let $\hat{E}$ be the closure of $\tilde{E}$. We claim that

(d) There is $z \in \tilde{E}$ such that $\Phi(B(z, r_1) - \hat{E}) < \sigma^{m+1}\Phi(B(z, r_1))$.

Indeed, let $X \subset \tilde{E} \cap B(0, b - r_1)$ be a maximal set such that $\|x - y\| > 2r_1$ whenever $x, y \in X$ and $x \neq y$. If (d) does not hold, we use also (a) to infer that

$$\begin{aligned}
&(\sigma/2)^{n+1}\Phi(B(0, b - r_1)) \\
&\quad \geq \Phi\big(B(0, b) - \hat{E}\big) \geq \sum_{z\in X} \Phi\big(B(z, r_1) - \hat{E}\big) \\
&\quad \geq \sigma^{m+1} \sum_{z\in X} \Phi(B(z, r_1)) \geq 2^{-m}(1 - \sigma)\sigma^{m+1} \sum_{z\in X} \Phi(B(z, 2r_1)) \\
&\quad \geq 2^{-m}(1 - \sigma)\sigma^{m+1}\Phi\big(B(0, b - r_1) \cap \tilde{E}\big) \\
&\quad \geq 2^{-m}(1 - \sigma)^2\sigma^{m+1}\Phi(B(0, b - r_1)).
\end{aligned}$$

Since $0 \in \operatorname{spt} \Phi$, this proves (d).

Let $z \in \tilde{E}$ be such that (d) holds and let $E = B(z, r_1) \cap \hat{E}$. We prove that the assumptions of 5.2 hold with the above defined r, ε, r_1, and with $\mu = 2^{-7}m^{-3}\varepsilon^2$ and $\kappa = a/r$.

From (c) we easily see that $\kappa \leq \sigma \leq 8m\mu$.

Whenever $x \in \tilde{E}$ and $s \in (\kappa r, r]$, we use (a), (b), and (c) to find $V \in G(n, m)$ such that

$$\begin{aligned}
\Phi(B(y, t)) &\geq (1 - \sigma)(t/s)^m\Phi(B(x, s)) \geq (1 - \sigma)^3(t/s)^m h(s) \\
&\geq (1 - \sigma)^8(t/r)^m h(r) \geq (1 - \sigma)^{10}(t/r)^m\Phi(B(z, r)) \\
&\geq (1 - \varepsilon)(t/r)^m\Phi(B(z, r))
\end{aligned}$$

whenever $y \in B(x, s) \cap (x + V)$ and $t \in [\mu s, s]$. By approximation, we easily see that for every $x \in \hat{E}$ there is $V \in G(n, m)$ such that $\Phi(B(y, t)) \geq (1 - \varepsilon)(t/r)^m\Phi(B(z, r))$ whenever $y \in B(x, s) \cap (x + V)$ and $t \in [\mu s, s]$. (V, of course, depends also upon $s \in (\kappa r, r]$.)

Finally, (d) implies that

$$\Phi\big(B(z, r_1) - E\big) \leq \sigma^{m+1}\Phi(B(z, r_1)) \leq \mu^{m+1}\Phi(B(z, r_1)).$$

Hence the assumptions of 5.2 hold and, since

$$\kappa r = a < a/\varepsilon < \varepsilon a/(8m^2\mu) \leq \varepsilon a/(m\kappa) = \varepsilon r/m,$$

5.2(5) provides us with a point $x \in E$ such that

$$\Phi(B(x, a/\varepsilon)) \leq (1 + 2^{m+4}\varepsilon)(a + a/\varepsilon)^m r^{-m}\Phi(B(z, r)).$$

Let $y \in \tilde{E}$ be such that $\|y - x\| < \sigma a$. Using (a), (b), (c), and the definition of r, we obtain

$$\begin{aligned} h(a) &\le (1-\sigma)^{-2}\Phi(B(y,a)) \le (1-\sigma)^{-m-3}\varepsilon^m\Phi(B(y,(1-\sigma)a/\varepsilon)) \\ &\le (1-\sigma)^{-m-3}\varepsilon^m\Phi(B(x,a/\varepsilon)) \\ &\le (1-\sigma)^{-m-3}(1+2^{m+4}\varepsilon)(1+\varepsilon)^m(a/r)^m\Phi(B(z,r)) \\ &\le (1-\sigma)^{-m-5}(1+2^{m+4}\varepsilon)(1+\varepsilon)^m(a/r)^m h(r) \\ &\le (1-\sigma)^{-m-10}(1+2^{m+4}\varepsilon)(1+\varepsilon)^m(a/b)^m h(b) \le \lambda(a/b)^m h(b). \end{aligned}$$

Since this contradicts our assumptions, 6.4 is proved.

6.5. Theorem. *Let $n = 1, 2, \ldots$ and let h be a positive function defined on $(0, \infty)$. Then the following statements are equivalent.*

(1) *h is a regular density function.*

(2) *h is a regular exact density function.*

(3) (a) *There is an integer $m = 0, 1, \ldots, n$ such that*

$$0 < \lim_{r \searrow 0} h(r)/r^m < \infty, \text{ or}$$

(b) *There is an integer $m = 1, \ldots, n-1$ such that $\lim_{r\searrow 0} h(r)/r^m = 0$, $\lim_{r\searrow 0} h(tr)/h(r) = t^m$ for each $t > 0$, and $\lim_{r\searrow 0}\sup_{t\in(0,1]} t^{-m}h(tr)/h(r) = 1$.*

Moreover, the condition (b) *is equivalent to:*

(b′) *There are an integer $m = 1, \ldots, n-1$ and a positive, nondecreasing function $\tilde{h}$ defined on $(0,\infty)$ such that $\lim_{r\searrow 0}\tilde{h}(r) = 0$, $\lim_{r\searrow 0}\tilde{h}(tr)/\tilde{h}(r) = 1$ for each $t > 0$, and $\lim_{r\searrow 0} h(r)/(r^m\tilde{h}(r)) = 1$.*

Proof. (1) $\Rightarrow$ (3). From 6.3 we see that there is an integer $m = 0, 1, \ldots, n$ such that $\lim_{r\searrow 0} h(tr)/h(r) = t^m$ for every $t > 0$.

If $m = 0$, we infer from 4.9 and from 4.4(6) that $0 < \lim_{r\searrow 0} h(r) < \infty$.

If $m \ge 1$, we infer from 6.4 that $\lim_{r\searrow 0}\sup_{t\in(0,1]} t^{-m}h(tr)/h(r) = 1$. Using this, we easily see that $0 \le \lim_{r\searrow 0} h(r)/r^m < \infty$. If $0 < \lim_{r\searrow 0} h(r)/r^m$, (3; a) holds. If $\lim_{r\searrow 0} h(r)/r^m = 0$ and $m \le n-1$, (3; b) holds. Finally, we note that the case $m = n$ and $\lim_{r\searrow 0} h(r)/r^m = 0$ is impossible because of 1.7.

(3; b) $\Rightarrow$ (3; b′). Let $\tilde{h}(r) = \min(1, \sup h(s)/s^m;\ 0 < s \le r)$. Let $t > 0$, $q > 1$, $\varepsilon \in (0,1)$, and $\hat{r} > 0$ be such that $h(r) \le \varepsilon r^m$, $t^m h(r)/q \le h(tr) \le q t^m h(r)$, and $h(\tau r) \le q\tau^m h(r)$ for every $r \in (0, \hat{r})$ and every $\tau \in (0,1]$. Then $\tilde{h}(r) \le \varepsilon$,

$$\tilde{h}(tr) = \sup\{h(ts)/(ts)^m;\ 0 < s \le r\} \ge \sup\{h(s)/qs^m;\ 0 < s \le r\} = \tilde{h}(r)/q,$$

$$\tilde{h}(tr) \le \sup\{qh(s)/s^m;\ 0 < s \le r\} = q\tilde{h}(r),$$

and

$$h(r)/r^m \le \tilde{h}(r) \le \sup\{h(s)/s^m; 0 < s \le r\} \le qh(r)/r^m$$

for every $0 < r < \hat{r}/(1+t)$.

The implications (3; a) $\Rightarrow$ (2) and (3; b′) $\Rightarrow$ (3; b) are obvious.

(3; b′) $\Rightarrow$ (2). Let $\sigma = (160\,001)^{1/2}/400$ and let $\tilde{r} \in (0,1)$ be such that $\tilde{h}(2r) \le \sigma\tilde{h}(r)$ whenever $r \in (0, \tilde{r}]$. We put $g(r) = \min(1, \tilde{h}(r)/\tilde{h}(\tilde{r}))$, and $s_k = \sup\{r \in (0,1]; \; rg(r) \le 2^{-k}\}$ $(k = 0, 1, \dots)$. Clearly, $s_0 = 1$ and $\lim_{k\to\infty} s_k = 0$. Moreover, since $(r/2)g(r/2) \le rg(r)/2$, $s_k \ge s_{k+1} \ge s_k/2$ for each $k = 0, 1, \dots$.

Let $a_k = [(g(s_{k-1})/g(s_k))^2 - 1]^{1/2}$ $(k = 1, 2, \dots)$. Then

$$0 \le a_k \le \left[(g(2s_k)/g(s_k))^2 - 1\right]^{1/2} \le (\sigma^2 - 1)^{1/2} = 1/400,$$

$$\lim_{k\to\infty} a_k \le \lim_{k\to\infty} \left[(g(2s_k)/g(s_k))^2 - 1\right]^{1/2} = 0,$$

and

$$\prod_{k=1}^{\infty} \left(1 + a_k^2\right) = \lim_{k\to\infty} 1/g(s_k)^2 = \infty.$$

If $k \ge 1$ and $s_{k+1} < r \le s_k$, we note that

$$rg(s_{k-1}) \le \sigma^2 rg(s_{k+1}) \le \sigma^2 2^{-k} < 2^{-(k-1)}$$

and

$$rg(s_j) \ge \sigma^{-2} rg(4s_j) \ge \sigma^{-2} 2^{-(j-1)} > 2^{-j} \qquad \text{if } j \ge k+2.$$

It follows that for each $r \in (0,1]$ there is the largest integer $k(r)$ such that $rg(s_{k(r)}) \le 2^{-k(r)}$ and that $k - 1 \le k(r) \le k + 1$ if $r \in (s_{k+1}, s_k]$. Consequently

$$1 = \lim_{r\searrow 0} g(r/4)/g(r) \le \lim_{r\searrow 0} g(s_{k(r)})/g(r)$$

$$\le \lim_{r\searrow 0} g(4r)/g(r) = 1.$$

Using 5.8 with $c = \infty$ and with the sequence $a_1, a_2, \dots$ defined above, we find a finite, nonzero measure Φ over $\mathbf{R}^2$ such that $\lim_{r\searrow 0} \Phi(B(x,r))/(rg(s_{k(r)})) = 2$ for Φ almost every $x \in \mathbf{R}^2$. (See 5.8(1).) Hence $\lim_{r\searrow 0} \Phi(B(x,r))/(rg(r)) = 2$ for Φ almost every $x \in \mathbf{R}^2$. Identifying $\mathbf{R}^2 \times \mathbf{R}^{m-1} = \mathbf{R}^{m+1} \subset \mathbf{R}^n$, we define a measure Ψ over $\mathbf{R}^{m+1}$ $(\subset \mathbf{R}^n)$ by the formula $\Psi = \Phi \otimes \mathscr{L}^{m-1}$. Then, for Ψ almost every $z = (x, y) \in \mathbf{R}^2 \times \mathbf{R}^{m-1}$, we compute

$$\limsup_{r \searrow 0} \Psi(B_{m+1}(z, r))/h(r)$$
$$= \tilde{h}(\tilde{r})^{-1} \limsup_{r \searrow 0} \int_{B_{m-1}(0,1)} \left[\Phi\left(B_2\left(x, r(1 - \|u\|^2)^{1/2}\right)\right)\Big/(rg(r))\right] d\mathscr{L}^{m-1}(u)$$
$$\leq \tilde{h}(\tilde{r})^{-1}\alpha(m),$$

and, for each $t \in (0, 1)$,

$$\liminf_{r \searrow 0} \Psi(B_{m+1}(z, r))/h(r)$$
$$\geq \tilde{h}(\tilde{r})^{-1} \liminf_{r \searrow 0} \left[g\left((1 - t^2)^{1/2} r\right)/g(r)\right]$$
$$\times \liminf_{r \searrow 0} \int_{B_{m-1}(0,t)} \left[\Phi\left(B_2\left(x, r(1 - \|u\|^2)^{1/2}\right)\right)\Big/\left(rg\left((1 - t^2)^{1/2} r\right)\right)\right] d\mathscr{L}^{m-1}(u)$$
$$\geq \tilde{h}(\tilde{r})^{-1}\alpha(m)t^m.$$

(2) $\Rightarrow$ (1). See 6.2(2; i).

6.6. LEMMA. *There is a decreasing sequence* $\tau_0, \tau_1, \ldots \in (0, 1/8)$ *with the following property. Whenever* $n \geq 1$ *is an integer,* $r_0, r_1, \ldots$ *are positive numbers, and for each* $k = 1, 2, \ldots$

(a) Φ_k *are measures over* $\mathbf{R}^n$,

(b) $E_k, \hat{E}_k, \tilde{E}_k$ *are subsets of* $\mathbf{R}^n$ *such that* $0 \in \tilde{E}_k \subset \hat{E}_k \subset E_k \subset \operatorname{spt} \Phi_k$,

(c) $\Phi_k(B(0, r_{k-1})) < \infty$ and $r_k \leq \tau_{k-1}^2 r_{k-1}$,

(d) $(1 - \tau_k)\Phi_{k+1}(B(y, (1 - \tau_k)r)) \leq \Phi_k(B(x, r))$
$\leq (1 + \tau_k)\Phi_{k+1}(B(y, (1 + \tau_k)r))$

for every $x \in E_k$, $y \in E_{k+1}$, *and* $r \in [\tau_k r_k, r_k]$,

(e) $\Phi_k(B(x, r) \cap E_k) \geq (1 - \tau_k)\Phi_k(B(x, r))$

for every $x \in \hat{E}_k$ *and* $r \in [\tau_k r_k, r_{k-1}]$,

(f) $\Phi_k(B(x, r) \cap \hat{E}_k) \geq (1 - \tau_k)\Phi_k(B(x, r))$

for every $x \in \tilde{E}_k$ *and* $r \in [\tau_k r_k, r_{k-1}]$, *and*

(g) $\Phi_k(B(0, r) \cap \tilde{E}_k) \geq (1 - \tau_k)\Phi_k(B(0, r))$

for every $r \in [\tau_k r_k, r_{k-1}]$,

then there are a finite measure Φ *over* $\mathbf{R}^n$ *and a compact set* $E \subset \mathbf{R}^n$ *of positive* Φ *measure such that for every* $x \in E$ *and every* $k = n, n + 1, \ldots$ *there exists* $\tilde{x} \in E_k$ *fulfilling*

$$2^{-1/k}\Phi_k\left(B(\tilde{x}, 2^{-1/k} r)\right) \leq \Phi(B(x, r)) \leq 2^{1/k}\Phi_k\left(B(\tilde{x}, 2^{1/k} r)\right)$$

for every $r \in [4\tau_k r_k, 4^k \tau_{k-1} r_{k-1}]$.

Proof. Let $t_1, t_2, \ldots \in (1,2)$ be such that $\prod_{j=k}^{\infty} t_j^{12+7j} \le 2^{1/k}$ for each $k = 1, 2, \ldots$. Let $\sigma_k = (t_k - 1)/40$ and let $\varepsilon_k = 2^{-k-1}\sigma_k^{1+3k}$. From 1.10(3) we infer that there are $\eta_k \in (0, 1/2)$ such that, whenever Ψ measures $\mathbf{R}^k$, $m = 0, 1, \ldots, k$, $V \in G(k, m)$, and $d_3(\Psi, \mathfrak{M}_{k,V}) < \eta_k$, then there is $c > 0$ such that

$$t_k^{-1} c \mathscr{H}^m(V \cap M) \le \Psi(B(M, \varepsilon_k)) \le t_k c \mathscr{H}^m(B(M, 2\varepsilon_k) \cap V)$$

whenever $M \subset B(0, 2)$ and $M \cap V \neq \varnothing$.

Let $\tilde{\eta}_k > 0$ be such that, whenever Ψ measures $\mathbf{R}^k$ and $d_3(\Psi, \mathfrak{M}_k) < \tilde{\eta}_k$ then $d_3(T_{0,s}[\Psi], \mathfrak{M}_k) < \eta_k$ for every $s \in [\omega_2(k, \eta_k)/3, 1]$.

Let $a_k = \sigma_k \omega_2(k, \eta_k)\omega_2(k, \tilde{\eta}_k)$ and let $p_k \ge 5$ be integers such that $(p_k - 4)^{-1} 5^{2k+2} a_k^{-k} \le \sigma_k$.

Finally, we find a decreasing sequence $\tau_0, \tau_1, \ldots \in (0, 1/32)$ such that $\tau_k \le 5^{-3k-4}\varepsilon_k^k a_k^{2kp_k}$ for each $k = 1, 2, \ldots$.

Let n, r_k, Φ_k, E_k, $\hat{E}_k$, and $\tilde{E}_k$ fulfil the assumptions of the lemma. By induction, we shall define a sequence Ψ_k of finite measures over $\mathbf{R}^n$ and sequences F_k and $\hat{F}_k$ of subsets of $\mathbf{R}^n$ $(k = n, n+1, \ldots)$ such that, for each $k = n, n+1, \ldots,$

(i) $\hat{F}_k \subset F_k$ are bounded subsets of spt Ψ_k,

(ii) $B(F_k, r_k) \subset B(F_{k-1}, r_{k-1})$ if $k \ge n+1$,

(iii) $\Psi_k(B(x, r) \cap F_k) \ge (1 - \tau_k)\Psi_k(B(x, r))$ whenever $x \in \hat{F}_k$ and $r \in [\tau_k r_k, r_k]$,

(iv) For every $x \in F_k$ there is $\tilde{x} \in E_k$ such that $\Psi_k(B(x, r)) = \Phi_k(B(\tilde{x}, r))$ for every $r \in (0, 4^{k+1}\tau_{k-1} r_{k-1})$,

(v) If $k \ge n+1$, $x \in B(F_k, r_k)$, and $r \ge r_{k-1}$, then

$$t_{k-1}^{-4-2k}\Psi_{k-1}\big(B(x, t_{k-1}^{-4} r)\big) \le \Psi_k(B(x, r)) \le t_{k-1}^{4+2k}\Psi_{k-1}\big(B(x, t_{k-1}^{4} r)\big),$$

(vi) If $k \ge n+1$ then for every $x \in B(F_k, r_k)$ there is $\bar{x} \in F_{k-1}$ such that

$$t_{k-1}^{-4-2k}\Psi_{k-1}\big(B(\bar{x}, t_{k-1}^{-5} r)\big) \le \Psi_k(B(x, r)) \le t_{k-1}^{4+2k}\Psi_{k-1}\big(B(\bar{x}, t_{k-1}^{5} r)\big)$$

whenever $r \ge 2\tau_{k-1} r_{k-1}$, and

(vii) $\Psi_n(\hat{F}_n) > 0$, $\Psi_k(\hat{F}_k) \ge \Psi_k(B(F_k, r_k))/2 > 0$ if $k \ge n+1$, and $\Psi_k(\hat{F}_k) \ge t_{k-1}^{-2-7k}\Psi_{k-1}(F_{k-1})$ if $k \ge n+2$.

We put $\Psi_n = \Phi_n \llcorner B(0, r_{n-1})$, $F_n = E_n \cap B(0, 2r_n)$, and $\hat{F}_n = \hat{E}_n \cap B(0, r_n)$.

Assume that $k \ge n$ and that Ψ_k, F_k, and $\hat{F}_k$ have already been defined. Let B be the family of all balls $B(x, s)$ such that $x \in F_k \cap Y_s(\Psi_k, a_k/5)$, $s \in (a_k^{p_k} r_k, a_k r_k]$, $d_3(T_{x,s}[\Psi_k], \mathfrak{M}_n) < \eta_k$, and there is $y \in \tilde{E}_{k+1}$ such that $d_3(T_{y,s}[\Phi_{k+1}], \mathfrak{M}_n) < \eta_k$. Using [11, 2.8.4], we find a (necessarily finite but, at this stage, possibly empty) disjointed sequence $B(x_i, s_i) \in B$ such that for every $B(x, s) \in B$ there is i for which $s_i > s/2$ and $B(x, s) \cap B(x_i, s_i) \neq \varnothing$.

For each i, let $y_i \in \tilde{E}_{k+1}$, $m_i, \tilde{m}_i = 0, 1, \ldots, n$, $V_i \in G(n, m_i)$, and $W_i \in G(n, \tilde{m}_i)$ be such that

$$d_3\big(T_{x_i, s_i}[\Psi_k], \mathfrak{M}_{n, V_i}\big) < \eta_k \quad \text{and} \quad d_3\big(T_{y_i, s_i}[\Phi_{k+1}], \mathfrak{M}_{n, W_i}\big) < \eta_k.$$

Because of the choice of η_k

(α) There are constants $c_i, \tilde{c}_i > 0$ such that:

$$t_k^{-1} c_i \mathscr{H}^{m_i}((x_i + V_i) \cap M) \leq \Psi_k(B(M, \varepsilon_k s_i))$$
$$\leq t_k c_i \mathscr{H}^{m_i}((x_i + V_i) \cap B(M, 2\varepsilon_k s_i))$$

whenever $M \subset B(x_i, 2s_i)$ and $M \cap (x_i + V_i) \neq \varnothing$, and

$$t_k^{-1} \tilde{c}_i \mathscr{H}^{\tilde{m}_i}\big((y_i + W_i) \cap \tilde{M}\big) \leq \Phi_{k+1}\big(B\big(\tilde{M}, \varepsilon_k s_i\big)\big)$$
$$\leq t_k \tilde{c}_i \mathscr{H}^{\tilde{m}_i}\big((y_i + W_i) \cap B\big(\tilde{M}, 2\varepsilon_k s_i\big)\big)$$

whenever $\tilde{M} \subset B(y_i, 2s_i)$ and $\tilde{M} \cap (y_i + W_i) \neq \varnothing$.

Using the special cases $M = B(x_i, r)$, $\tilde{M} = B(y_i, r)$, (iv), and (d), we infer that

$$t_k^{2k+2} 2^{m_i} \geq t_k^{2m_i + 2} 2^{m_i} \geq \Psi_k(B(x_i, s_i))/\Psi_k(B(x_i, s_i/2))$$
$$\geq t_k^{-2} \Phi_{k+1}(B(y_i, s_i/t_k))/\Phi_{k+1}(B(y_i, t_k s_i/2)) \geq t_k^{-4k-4} 2^{\tilde{m}_i}.$$

Since $t_k^{6k+6} < 2$, $\tilde{m}_i \leq m_i$. Similarly we prove that $m_i \leq \tilde{m}_i$. Hence $m_i = \tilde{m}_i$ and there are isometries Q_i of $\mathbf{R}^n$ such that $Q_i(y_i) = x_i$ and $Q_i(y_i + W_i) = x_i + V_i$.

Let

$$A_i(t) = B(x_i, (1 - \sigma_k - t\varepsilon_k)s_i) \cap B\big(x_i + V_i, \big(\sigma_k^2 - t\varepsilon_k\big)s_i\big)$$

if $t = -1, 0, 1$, $A_i = A_i(0)$, $A = \bigcup_i A_i$, and

$$D_i(t) = B\big(x_i, \big(1 - \sigma_k - t\sigma_k^2\big)s_i\big) \cap B(x_i + V_i, (3 - t)\varepsilon_k s_i)$$

if $t = 1, 2$. We define

$$\Psi_{k+1} = \big[\Psi_k \llcorner (R^n - A)\big] + \sum_i \big(Q_i[\Phi_{k+1}] \llcorner A_i\big),$$

$$F_{k+1} = \bigcup_i \big(D_i(1) \cap Q_i(E_{k+1})\big), \quad \text{and}$$

$$\hat{F}_{k+1} = \bigcup_i \big(D_i(2) \cap Q_i\big(\hat{E}_{k+1}\big)\big).$$

We prove that (i)–(vii) hold with k replaced by $k + 1$. The conditions (i) and (ii) are obvious and (iii) follows from (e) and from $\varepsilon_k s_i \geq \tau_k r_k \geq r_{k+1}$. If $x \in A_i \cap F_{k+1}$, we use $\varepsilon_k s_i \geq 4^{k+2} \tau_k r_k$ to infer that (iv) holds with $\tilde{x} = Q_i^{-1}(x)$.

Before proving the remaining conditions we note that (α), $m_i = \tilde{m}_i$, (iv), and (d) imply

$$\begin{aligned}\tilde{c}_i\alpha(m_i)s_i^{m_i} &\le t_k\Phi_{k+1}(B(y_i, t_k s_i)) \le t_k^2\Psi_k(B(x_i, t_k(1+\tau_k)s_i)) \\ &\le t_k^{3+2m_i}c_i\alpha(m_i)s_i^{m_i}.\end{aligned}$$

This and a similar inequality obtained by interchanging the roles of Φ_{k+1} and Ψ_k show that

$$(\beta) \qquad t_k^{-3-2k}c_i \le \tilde{c}_i \le t_k^{3+2k}c_i.$$

Whenever $x \in \mathbf{R}^n$, $r \ge s_i\sigma_k^2/4$, and $B(x,r) \cap A_i \cap (x_i + V_i) \neq \varnothing$, we note that $B(x, t_k r) \cap A_i(1) \cap (x_i + V_i)$ contains an m_i dimensional ball with radius $\sigma_k^3 s_i$. Hence

$$\begin{aligned}&\mathscr{H}^{m_i}(A_i(-1) \cap B(x, t_k r) \cap (x_i + V_i)) \\ &\quad\le \mathscr{H}^{m_i}(A_i(1) \cap B(x, t_k r) \cap (x_i + V_i)) \\ &\qquad + \mathscr{H}^{m_i}[(A_i(-1) - A_i(1)) \cap (x_i + V_i)] \\ &\quad\le \mathscr{H}^{m_i}(A_i(1) \cap B(x, t_k r) \cap (x_i + V_i)) + \alpha(m_i)2^{k+1}\varepsilon_k s_i^{m_i} \\ &\quad\le (1 + 2^{k+1}\varepsilon_k\sigma_k^{-3k})\mathscr{H}^{m_i}(A_i(1) \cap B(x, t_k r) \cap (x_i + V_i)) \\ &\quad\le t_k\mathscr{H}^{m_i}(A_i(1) \cap B(x, t_k r) \cap (x_i + V_i)).\end{aligned}$$

Consequently, (α) and (β) imply that

$$\begin{aligned}(\gamma) \qquad &\Psi_{k+1}(B(x,r) \cap A_i) \le t_k^{6+2k}\Psi_k(B(x, t_k^3 r) \cap A_i), \quad \text{and} \\ &\Psi_k(B(x,r) \cap A_i) \le t_k^{6+2k}\Psi_{k+1}(B(x, t_k^3 r) \cap A_i).\end{aligned}$$

Let $x \in A_i \cap B(F_{k+1}, r_{k+1})$. Then ($\alpha$) implies that

$$\begin{aligned}\Psi_k(B(x, 5\varepsilon_k s_i)) &\ge t_k^{-1}c_i\mathscr{H}^{m_i}(B(x, 4\varepsilon_k s_i) \cap (x_i + V_i)) \\ &\ge t_k^{-1}c_i\varepsilon_k^k\alpha(m_i)s_i^{m_i} \ge t_k^{-k-1}c_i\varepsilon_k^k\mathscr{H}^{m_i}(B(x_i, t_k s_i) \cap (x_i + V_i)) \\ &\ge \tfrac{1}{2}\varepsilon_k^k\Psi_k(B(x_i, s_i)) > \tau_k\Psi_k(B(x_i, s_i)).\end{aligned}$$

Since $x_i \in \hat{F}_k$ and $B(x, 5\varepsilon_k s_i) \subset B(x_i, s_i)$, we may use (iii) to find a point $\bar{x} \in F_k \cap B(x, 5\varepsilon_k s_i)$. If $r \ge \sigma_k^2 s_i/4$, we observe that $r \ge \sigma_k^2 s_j/4$ and $B(x, t_k r) \cap A_j \cap (x_j + V_j) \neq \varnothing$ whenever $B(x,r) \cap A_j \neq \varnothing$. Denoting by $\sum'$ the summation over those indices j for which $B(x,r) \cap A_j \neq \varnothing$, we use ($\gamma$) to estimate

$$\begin{aligned}\Psi_{k+1}(B(x,r)) &\le \Psi_k(B(x,r) - A) + \textstyle\sum'\Psi_{k+1}(B(x, t_k r) \cap A_j) \\ &\le \Psi_k(B(x,r) - A) + t_k^{6+2k}\textstyle\sum'\Psi_k(B(x, t_k^4 r) \cap A_j) \\ &\le t_k^{6+2k}\Psi_k(B(x, t_k^4 r))\end{aligned}$$

and, similarly,

$$\Psi_k(B(x,r)) \le t_k^{6+2k}\Psi_{k+1}\big(B\big(x, t_k^4 r\big)\big).$$

Thus (v) holds and, since

$$\Psi_k\big(B\big(\bar{x}, t_k^{-5}r\big)\big) \le \Psi_k\big(B\big(x, t_k^{-4}r\big)\big) \le \Psi_k\big(B\big(x, t_k^4 r\big)\big) \le \Psi_k\big(B\big(\bar{x}, t_k^5 r\big)\big),$$

(vi) holds if $r \ge \sigma_k^2 s_i/3$. If $2\tau_k r_k \le r \le \sigma_k^2 s_i/3$, we find $\hat{x} \in F_{k+1} \cap B(x, 2r_{k+1})$ and we use (iv) and (d) to estimate

$$\Psi_{k+1}(B(x,r)) \le \Psi_{k+1}(B(\hat{x}, t_k r)) \le t_k \Psi_k\big(B\big(\bar{x}, t_k^2 r\big)\big)$$

and, similarly,

$$\Psi_{k+1}(B(x,r)) \ge t_k^{-1}\Psi_k\big(B\big(\bar{x}, t_k^{-2}r\big)\big).$$

Thus (vi) is proved.

From $y_i \in \tilde{E}_{k+1}$ and from (f) we infer that

$$(\delta) \qquad \begin{aligned} &\Psi_{k+1}\big(A_i \cap \hat{F}_{k+1}\big) \\ &\quad \ge \Phi_{k+1}\big(B(y_i + W_i, \varepsilon_k s_i) \cap B(y_i, (1 - 2\sigma_k)s_i)\big) \\ &\qquad - \tau_{k+1}\Phi_{k+1}\big(B(y_i, (1 - 2\sigma_k)s_i)\big) \\ &\quad \ge \big(t_k^{-1-k} - t_k\tau_{k+1}\big)\alpha(m_i)\tilde{c}_i s_i^{m_i} \ge t_k^{-2-k}\alpha(m_i)\tilde{c}_i s_i^{m_i} \\ &\quad \ge t_k^{-5-3k}\alpha(m_i)c_i s_i^{m_i} \ge t_k^{-6-4k}\Psi_k(B(x_i, s_i)). \end{aligned}$$

From (δ) and (γ) we conclude that

$$2\Psi_{k+1}\big(\hat{F}_{k+1}\big) \ge 2t_k^{-12-6k}\Psi_{k+1}(B(F_{k+1}, r_{k+1})) \ge \Psi_{k+1}(B(F_{k+1}, r_{k+1})).$$

We shall need to know that

$$(\varepsilon) \qquad F_k \cap B\big(\hat{F}_k, a_k^j r_k\big) \subset Y_r(\Psi_k, a_k/5)$$

if $r \in (a_k^{j+1}r_k, a_k^2 r_k)$ and $j = 3, 4, \dots, p_k - 1$. To prove it, we derive from (iv) and (v) that

$$\Psi_k(B(x,r)) \ge (1 - \tau_k)^2\Psi_k\big(B\big(y, (1 - \tau_k)^2 r\big)\big)$$

whenever $x, y \in F_k$ and $r \in [2\tau_k r_k, r_k]$. Thus (iii) and 4.2(4) show that $\hat{F}_k \subset Y_{br_k}(\Psi_k, b)$, where $b = a_k^{p_k}$. Using these remarks, (iii), and 4.2(2), we see that, whenever $x \in F_k$, $z \in \hat{F}_k$, $\|x - z\| < 2a_k^j r_k$, and $r \in (a_k^{j+1}r_k, a_k^2 r_k)$, then

$$\begin{aligned} \Psi_k(B(x, 5r/a_k) \cap F_k) &\ge \Psi_k(B(x, 5r/a_k)) - \tau_k\Psi_k(B(z, r_k)) \\ &\ge \Psi_k(B(x, 5r/a_k)) - 5^{k+1}\tau_k b^{-k}\Psi_k(B(z, br_k)) \\ &\ge \big(1 - 5^{k+1}\tau_k b^{-k}\big)\Psi_k(B(x, 5r/a_k)) \\ &\ge \big(1 - (a_k/25)^{k+1}\big)\Psi_k(B(x, 5r/a_k)). \end{aligned}$$

Hence (ε) follows from 4.2(4).

Next we observe:

(η) For each $z \in \hat{F}_k$ and each $j = 3, 4, \ldots, p_k - 2$, there are $x \in F_k \cap B(z, a_k^j r_k)$ and $s \in (2a_k^{j+1} r_k, a_k^j r_k)$ with $B(x, s) \in B$.

In fact, using (ε), (iii), and 4.5(2), we find $x \in F_k \cap B(z, a_k^j r_k)$ and $\tilde{s} \in (\omega_2(k, \tilde{\eta}_k) a_k^j r_k/3, a_k^j r_k)$ such that $d_3(T_{x,\tilde{s}}[\Psi_k], \mathfrak{M}_n) < \tilde{\eta}_k$. From (d) and (e) we see that $\hat{E}_{k+1} \subset Y_r(\Phi_{k+1}, a_k)$ for every $r \in (a_k s, s/a_k)$. Hence 4.5(2) and (g) imply that there are $y \in B(0, \tilde{s}) \cap \tilde{E}_{k+1}$ and $s \in (\omega_2(k, \eta_k)\tilde{s}/3, \tilde{s})$ such that $d_3(T_{y,s}[\Phi_{k+1}], \mathfrak{M}_n) < \eta_k$. Since $s \in (2a_k^{j+1} r_k, a_k^j r_k)$, our observation follows from the definition of the constant $\tilde{\eta}_k$ and from (ε).

From (η) it follows that the sequence $B(x_i, s_i)$ is nonempty. Consequently (δ) implies that

$$\Psi_{k+1}\big(B(F_{k+1}, r_{k+1})\big) \geq \Psi_{k+1}\big(\hat{F}_{k+1}\big) > 0.$$

If $k \geq n + 1$, let $G_j(t) = \bigcup\{B(x_i, ts_i);\ s_i \in (a_k^{j+1} r_k, a_k^j r_k]\}$, $G_j = G_j(1)$, and $G = \bigcup_{j=1}^{p_k - 1} G_j$. Using ($\eta$), we see that

$$\hat{F}_k \subset G_j(3 + 1/a_k) \cup G_{j-1}(3) \cup \bigcup_{i=1}^{j-2} G_i(t_k)$$

for every $j = 3, 4, \ldots, p_k - 2$. Since $x_i \in Y_s(\Psi_k, a_k/5)$, we may use 4.2(2) to estimate

$$\begin{aligned}\Psi_k\big(G_j(3 + 1/a_k) \cup G_{j-1}(3)\big) &\leq 5^{k+1} 2^k a_k^{-k} \Psi_k\big(G_j\big) + 5^{k+1} 3^k \Psi_k\big(G_{j-1}\big)\\ &\leq 5^{2k+1} a_k^{-k}\big(\Psi_k\big(G_j\big) + \Psi_k\big(G_{j-1}\big)\big).\end{aligned}$$

Moreover (α) implies that

$$\Psi_k\big(B(x_i, t_k s_i)\big) \leq t_k^{1+2k} c_i \alpha(m_i) s_i^{m_i} \leq t_k^{2+3k} \Psi_k\big(B(x_i, s_i)\big).$$

Hence

$$\Psi_k\big(\hat{F}_k\big) \leq 5^{2k+1} a_k^{-k}\big(\Psi_k\big(G_j\big) + \Psi_k\big(G_{j-1}\big)\big) + t_k^{2+3k} \Psi_k(G)$$

for each $j = 3, 4, \ldots, p_k - 2$. Summing up these inequalities and using $\Psi_k(G) \leq \Psi_k(B(F_k, r_k)) \leq 2\Psi_k(\hat{F}_k)$ (which follows from the induction assumption (vii), since $k \geq n + 1$), we obtain

$$\begin{aligned}\Psi_k\big(\hat{F}_k\big) &\leq 2(p_k - 4)^{-1} 5^{2k+1} a_k^{-k} \Psi_k(G) + t_k^{2+3k} \Psi_k(G)\\ &\leq (p_k - 4)^{-1} 5^{2k+2} a_k^{-k} \Psi_k\big(\hat{F}_k\big) + t_k^{2+3k} \Psi_k(G)\\ &\leq \sigma_k \Psi_k\big(\hat{F}_k\big) + t_k^{2+3k} \Psi_k(G).\end{aligned}$$

Hence $\Psi_k(G) \geq t_k^{-3-3k} \Psi_k(\hat{F}_k)$ and, using also (δ), we conclude that $\Psi_{k+1}(\hat{F}_{k+1}) \geq t_k^{-9-7k} \Psi_k(\hat{F}_k)$. This proves the last inequality in (vii).

Since, according to (v),

$$\Psi_k(\mathbf{R}^n) \le \Psi_n(\mathbf{R}^n) \prod_{j=n}^{\infty} t_j^{6+2j} \quad (< \infty),$$

the sequence Ψ_k has a subsequence converging to a finite measure Φ over $\mathbf{R}^n$. Let $E = \bigcap_{j=n}^{\infty} B(F_j, r_j)$. For every $k \ge n+2$ we infer from (i), (ii), and (vii) that

$$\Psi_k\left(\bigcap_{j=n}^{k} B(F_j, r_j)\right) = \Psi_k(B(F_k, r_k)) \ge \Psi_k(\hat{F}_k)$$
$$\ge \Psi_{n+1}(\hat{F}_{n+1}) \prod_{j=n+1}^{\infty} t_j^{-9-7j} \quad (> 0).$$

Since the sets $B(F_j, r_j)$ are compact, we conclude that $\Phi(E) > 0$.

Let $x \in E$ and let $k = n, n = 1, \dots$. Since $x \in B(F_{k+1}, r_{k+1})$, there is $\bar{x} \in F_k$ such that (vi) holds with k replaced by $k+1$. Using $\bar{x} \in F_k$, we find $\tilde{x} \in E_k$ such that (iv) holds with x replaced by $\bar{x}$. Whenever $r \in [4\tau_k r_k, 4^k \tau_{k-1} r_{k-1}]$, we use (v), (vi), and (iv) to estimate

$$\begin{aligned}\Phi(B(x, r)) &\le \limsup_{j\to\infty} \Psi_j(B(x, t_k r))\\ &\le \Psi_{k+1}\left(B\left(x, t_k r \prod_{j=k+1}^{\infty} t_j^4\right)\right) \prod_{j=k+1}^{\infty} t_j^{6+2j}\\ &\le \Psi_k\left(B\left(\bar{x}, t_k^6 r \prod_{j=k+1}^{\infty} t_j^4\right)\right) \prod_{j=k}^{\infty} t_j^{6+2j} \le 2^{1/k}\Psi_k(B(\bar{x}, 2^{1/k} r))\\ &= 2^{1/k}\Phi_k(B(\tilde{x}, 2^{1/k} r))\end{aligned}$$

and

$$\begin{aligned}\Phi(B(x, r)) &\ge \liminf_{j\to\infty} \Psi_j(B(x, r))\\ &\ge \Psi_{k+1}\left(B\left(x, r \prod_{j=k+1}^{\infty} t_j^{-4}\right)\right) \prod_{j=k+1}^{\infty} t_j^{-6-2j}\\ &\ge \Psi_k\left(B\left(\bar{x}, t_k^{-5} r \prod_{j=k+1}^{\infty} t_j^{-4}\right)\right) \prod_{j=k}^{\infty} t_j^{-6-2j}\\ &\ge 2^{-1/k}\Psi_k(B(\bar{x}, 2^{-1/k} r)) = 2^{-1/k}\Phi_k(B(\tilde{x}, 2^{-1/k} r)).\end{aligned}$$

6.7. THEOREM. *If h is not a density function in $\mathbf{R}^n$, there is $\varepsilon > 0$ (depending upon h and n) such that*

$$\Phi\{x \in \mathbf{R}^n;\ 0 < \overline{D}_h(\Phi, x) < (1+\varepsilon)\underline{D}_h(\Phi, x) < \infty\} = 0$$

for every measure Φ over $\mathbf{R}^n$.

Proof. Let τ_k be the sequence from 6.6 and let $t_k = \min(2^{1/k}, (1+\tau_k)^{1/2})$. Suppose that h is a positive function on $(0, \infty)$ such that for each $\varepsilon > 0$ there is a measure Φ over $\mathbf{R}^n$ with

$$\Phi\left\{x \in \mathbf{R}^n; 0 < \overline{D}_h(\Phi, x) < (1+\varepsilon)\underline{D}_h(\Phi, x) < \infty\right\} > 0.$$

Multiplying these measures by suitable constants and using 6.1(4), we find a sequence $\Phi_0, \Phi_1, \ldots$ of measures over $\mathbf{R}^n$ such that

$$\Phi_k\left\{x \in \mathbf{R}^n; 1 < \underline{D}_h(\Phi, x) \leq \overline{D}_h(\Phi, x) < t_k\right\} > 0.$$

Let $E_k \subset \operatorname{spt} \Phi_k$ and $\delta_k > 0$ be such that $\Phi_k(E_k) > 0$ and $\Phi_k(B(x, t_k r)) \geq h(r)$ and $\Phi_k(B(x, r/t_k)) \leq t_k h(r)$ whenever $x \in E_k$ and $r \in (0, \delta_k)$. Using the density theorem twice, we find sets $E_k \supset \hat{E}_k \supset \tilde{E}_k$ and numbers $\delta_k > \hat{\delta}_k > \tilde{\delta}_k$ such that $\Phi_k(\tilde{E}_k) > 0$, $\Phi_k(B(x, r) \cap E_k) \geq (1-\tau_k)\Phi_k(B(x, r))$ for every $x \in \hat{E}_k$ and every $r \in (0, \hat{\delta}_k)$, and $\Phi_k(B(x, r) \cap \hat{E}_k) \geq (1-\tau_k)\Phi_k(B(x, r))$ for every $x \in \tilde{E}_k$ and every $r \in (0, \tilde{\delta}_k)$.

Shifting the measures Φ_k if necessary, we may assume that $0 \in \tilde{E}_k$ and that 0 is a Φ_k density point of E_k. Hence there are $s_k \in (0, \tilde{\delta}_k/2)$ such that $\Phi_k(B(0, r) \cap \tilde{E}_k) \geq (1-\tau_k)\Phi_k(B(0, r))$ for every $r \in (0, s_k)$.

Let $r_0, r_1, \ldots$ be positive numbers such that $r_k < \min(s_k, s_{k+1})$ and $r_{k+1} \leq \tau_k^2 r_k$. If $x \in E_k$, $y \in E_{k+1}$, and $0 < r \leq r_k$, then

$$\begin{aligned}\Phi_k(B(x, r)) &\leq (1+\tau_k)h(t_k r) \leq (1+\tau_k)\Phi_{k+1}(B(y, t_k t_{k+1} r)) \\ &\leq (1+\tau_k)\Phi_{k+1}(B(y, (1+\tau_k)r))\end{aligned}$$

and

$$\Phi_k(B(x, r)) \geq h(r/t_k) \geq (1-\tau_k)\Phi_{k+1}(B(y, (1-\tau_k)r)).$$

This proves the condition (d) of 6.6. Since the other conditions of 6.6 are obvious, we conclude that there are a finite measure Φ over $\mathbf{R}^n$ and a compact set E of positive Φ measure such that for every $x \in E$ and every $k = n, n+1, \ldots$ there is $\tilde{x} \in E_k$ fulfilling

$$\Phi(B(x, r)) \leq 2^{1/k}\Phi_k(B(\tilde{x}, 2^{1/k} r)) \leq 4^{1/k} h(4^{1/k} r)$$

and

$$\Phi(B(x, r)) \geq 2^{-1/k}\Phi_k(B(\tilde{x}, 2^{-1/k} r)) \geq 2^{-1/k} h(4^{-1/k} r)$$

whenever $r \in [4\tau_k r_k, 4^k \tau_{k-1} r_{k-1}]$.

Thus $D_h(\Phi, x) = 1$ at Φ almost every point of E. Using the density theorem, we conclude that $D_h(\Phi \llcorner E, x) = 1$ at $\Phi \llcorner E$ almost every point of $\mathbf{R}^n$. Hence h is a density function in $\mathbf{R}^n$.

6.8. PROPOSITION. *Let h be a density function in* $\mathbf{R}^n$.

(1) *If* $n \in \operatorname{Dim}(h)$ *then* $0 < \lim_{r \searrow 0} h(r)/r^n < \infty$.

(2) *If* $\operatorname{Dim}(h) = \{n-1\}$ *and there is* $t > 1$ *such that*

$$\limsup_{r \searrow 0} h(tr)/h(r) < \left[1 + (1 - t^{-2})^{(n-1)/2}\right] t^{n-1}$$

then h is regular.

(3) *If* $\operatorname{Dim}(h) = \{n-1\}$ *then* $0 \le \lim_{r \searrow 0} h(r)/r^{n-1} < \infty$.

Proof. (1) From 6.2(7) and 4.12(2; ii) we see that h is regular. Hence it suffices to use 6.5.

(2) See 6.2(7) and 4.12(2; iv).

(3) If $n = 1$, the statement is obvious. If $n > 1$ and h is regular, the statement follows from 6.5. Hence we may assume that $n > 1$ and that h is irregular. Using 6.4, we find $\delta \in (0, (3/4)^{(n-1)/2})$ and $r_0 > 0$ such that, whenever $0 < a < b < r_0$ and $(1-\delta)2^{n-1}h(r) \le h(2r) \le (1+\delta)2^{n-1}h(r)$ for every $r \in (\delta a, b/\delta)$, then $h(a) \le \frac{5}{4}(a/b)^{n-1}h(b)$.

Let $q = n^2/\delta^2$. By induction we define a sequence r_k $(k = 0, 1, \dots)$ of positive numbers as follows. If r_k has been already defined, we use the irregularity of h and (2) to infer that

$$t_k = \sup\left\{ r \in (0, r_k/q);\ 2^{1-n}h(2r)/h(r) \notin (1-\delta, 1+\delta)\right\}$$

is well defined and we choose $r_{k+1} \in (t_k/2, t_k]$ so that $2^{1-n}h(2r_{k+1})/h(r_{k+1}) \notin (1-\delta, 1+\delta)$.

We prove that there is $p \ge 1$ such that for each $k \ge p$

(a) $h(r_k/q) \le \frac{4}{7}q^{-2n+2}h(qr_k)$,

(b) $h(r) \le \frac{5}{4}(r/qr_k)^{n-1}h(qr_k)$ for every $r \in [r_k/q, qr_k]$, and

(c) $2^{1-n}h(2r)/h(r) \in (1-\delta, 1+\delta)$ for every $r \in [r_k/q^2, r_k/q\delta]$.

To prove this statement, let Φ measure $\mathbf{R}^n$ and let $x \in \mathbf{R}^n$ be such that $D_h(\Phi, x) = 1$ and

$$\varnothing \ne \operatorname{Tan}(\Phi, x) \subset \left\{ c\mathscr{H}^{n-1} \llcorner [V \cup (e + V)];\ V \in G(n, n-1),\ e \in R^n,\ c > 0\right\}.$$

(See 6.2(7) and 4.12(2; iii).) Whenever $s_j \searrow 0$ is a subsequence of the sequence r_k, $c_j > 0$, and the sequence $c_j T_{x, s_j}[\Phi]$ converges to a measure $\Psi \in \operatorname{Tan}(\phi, x)$, we find $V \in G(n, n-1)$, $e \in V^{\perp}$, and $c > 0$ such that

$$\Psi = c\mathscr{H}^{n-1} \llcorner [V \cup (e + V)].$$

Since $n \ge 2$, $\lim_{j \to \infty} c_j h(tr_j) = \Psi(B(0, t))$ for every $t > 0$. Moreover, the convergence is uniform for t belonging to a compact subset of $(0, \infty)$. Computing the Ψ measures of balls $B(0, t)$, we see that $2 > \|e\| > \delta/n$ (since

$2^{1-n}\Psi(B(0,2))/\Psi(B(0,1)) \notin (1-\delta, 1+\delta))$ and that, consequently,

$$\Psi(B(0,1/q)) < \tfrac{4}{7}q^{-2n+2}\Psi(B(0,q)),$$

$$\Psi(B(0,t)) \le \tfrac{6}{5}(t/q)^{n-1}\Psi(B(0,q))$$

for every $t \in [1/q, q]$, and

$$2^{1-n}\Psi(B(0,2t))/\Psi(B(0,t)) \in (1-\delta/2, 1+\delta/2)$$

for every $t \in [1/q^2, 1/q\delta]$. This easily implies (a), (b), and (c).

We claim that $qr_{k+1} \le r_k/q$ and

(d) $h(r) \le \frac{5}{4}(qr/r_k)^{n-1}h(r_k/q)$ for every $k \ge p$ and every $r \in [qr_{k+1}, r_k/q]$.

Indeed, from (c) we see that $r_{k+1} \le t_k \le r_k/q^2$. Moreover, since $t_k < 2r_{k+1} \le \delta q r_{k+1} < r_k/q\delta$, (c) and the definition of t_k imply that $2^{1-n}h(2r)/h(r) \in (1-\delta, 1+\delta)$ for every $r \in (\delta q r_{k+1}, r_k/q\delta)$. Hence (d) follows from our choice of δ.

Finally, we use (a) and (d) to infer that $h(qr_{k+1}) \le \frac{5}{7}(r_{k+1}/r_k)^{n-1}h(qr_k)$ if $k \ge p$. Hence $\lim_{k\to\infty} h(qr_k)/(qr_k)^{n-1} = 0$ which, because of (b) and (d), implies that $\lim_{r\searrow 0} h(r)/r^{n-1} = 0$.

6.9. Proposition. (1) *Let* $0 \le m \le n$ *be integers. Then the following statements are equivalent.*

(i) *There is an irregular density function* h *in* $\mathbf{R}^n$ *such that*

$$0 < \liminf_{r\searrow 0} h(r)/r^m \le \limsup_{r\searrow 0} h(r)/r^m < \infty.$$

(ii) *There is an irregular exact density function* h *in* $\mathbf{R}^n$ *such that*

$$0 < \liminf_{r\searrow 0} h(r)/r^m \le \limsup_{r\searrow 0} h(r)/r^m < \infty.$$

(iii) $1 \le m \le n-2$.

(2) *If* $n \ge 1$ *and* $\varnothing \ne M \subset \{0, 1, \dots, n-1\}$, *there is an irregular density function* h *in* $\mathbf{R}^n$ *such that* $\mathrm{Dim}(h) = M$.

(3) *If* $n \ge 1$, $\varnothing \ne M \subset \{0, 1, \dots, n-1\}$, *and* $0 \notin M$ *or there is* $p \in M$ *such that* $1 \le p \le (n-1)/2$, *there is an irregular exact density function in* $\mathbf{R}^n$ *with* $\mathrm{Dim}(h) = M$.

(4) *If* $M \ne \varnothing$ *is a finite set of nonnegative integers and if* $M \ne \{0\}$, *there is an irregular exact density function such that* $\mathrm{Dim}(h) = M$.

Proof. (1; ii) ⇒ (1; i). See 6.2(2; i).

(1; i) ⇒ (1; iii). See 6.2(3), 6.8(1), and 6.8(3).

(1; iii) ⇒ (1; ii); (2); (3). Let $\varnothing \ne M \subset \{0, 1, \dots, n-1\}$. (We let $M = \{m\}$ in the proof of (1; iii) ⇒ (1; ii).) If $0 \notin M$ or if $M \cap [1, (n-1)/2] = \varnothing$, let $p_0, p_1, \dots$ be a sequence of elements of M containing each element of M infinitely many times. If $0 \in M$ and $M \cap [1,(n-1)/2] \neq \varnothing$, let $p_0, p_1, \dots$ be a sequence of elements of $M - \{0\}$ containing each element of $M - \{0\}$ infinitely many times and such that $p_k = p_{k+1} \leq (n-1)/2$ holds for infinitely many values of k. For each $k = 1, 2, \dots$ let Ψ_k be defined as follows. (For convenience we denote $p_{k-1} = p$ and $p_k = q$.)

(a) $$\Psi_k = \mathscr{H}^q \llcorner \{x \in R^n;\ x_1, \dots, x_{p-q} \text{ are integers and } x_j = 0 \text{ for } p+1 \leq j \leq n\}$$

if $p > q$.

(b) $$\Psi_k = \mathscr{H}^q \llcorner \{x \in R^n;\ (x_{p+1} - 1)^2 + x_{p+2}^2 + \cdots + x_{q+1}^2 = 1 \text{ and } x_j = 0 \text{ for } q+2 \leq j \leq n\}$$

if $p < q$.

(c) $$\Psi_k = \mathscr{H}^p \llcorner \{x \in R^n;\ x_1 \in \{0,1\} \text{ and } x_j = 0 \text{ for } p+2 \leq j \leq n\}$$

if $p = q = 0$ or $p = q = n-1$ or $0 \in M$ and $p = q > (n-1)/2$.

(d) $$\Psi_k = \mathscr{H}^p \llcorner \{x \in R^n;\ x_1/2\pi \text{ is an integer}, (x_2 - 1)^2 + x_3^2 = 1, \text{ and } x_j = 0 \text{ for } p+3 \leq j \leq n\}$$

if $1 \leq p = q < (n-1)$ and $0 \notin M$.

(e) $$\Psi_k = \mathscr{H}^p \llcorner \{x \in R^n;\ x_1, \dots, x_p \text{ are integers}, x_j = 0 \text{ for } 2p+2 \leq j \leq n, \text{ and } (x_{p+1} - 1)^2 + x_{p+2}^2 + \cdots + X_{2p+1}^2 = k^{-1/2}\}$$

if $1 \leq p = q \leq (n-1)/2$ and $0 \in M$.

We note that $\Psi_k \in \mathfrak{U}(n)$, $\dim_\infty \Psi_k = p_{k-1}$, and $\dim_0 \Psi_k = p_k$. Let $C_k = \lim_{r \to \infty} \Psi_k(B(0,r))/r^{p_{k-1}}$, and $c_k = \lim_{r \searrow 0} \Psi_k(B(0,r))/r^{p_k}$. Also, let $\tau_0, \tau_1, \dots$ be the sequence with the property described in 6.6 and let $t_k = \min((1+\tau_k)^{1/4}, 1 + 2^{-k})$. We find $s_k \in (0, \tau_k^2/k)$ such that

$$t_k^{-1} C_k r^{p_{k-1}} \leq \Psi_k(B(0,r)) \leq t_k C_k r^{p_{k-1}} \quad \text{if } r \geq \tau_k / k s_k, \quad \text{and}$$
$$t_k^{-1} c_k r^{p_k} \leq \Psi_k(B(0,r)) \leq t_k c_k r^{p_k} \quad \text{if } 0 < r \leq s_k.$$

Let $r_0 = 1$ and let $r_k = s_1 s_2^2, \dots, s_k^2$ for $k \geq 1$. We define $\Phi_1 = \Psi_1$ and

$$\Phi_{k+1} = \Phi_k(B(0,r_k))\left[\Psi_{k+1}(B(0,1/s_{k+1}))\right]^{-1} T_{0,1/r_k s_{k+1}}[\Psi_{k+1}].$$

We shall consider the function h defined by the formulas $h(r) = \Phi_1(B(0, r))$ if $r \geq r_1$, and

$$h(r) = \Phi_{k+1}(B(0, r)) \quad \text{if } k = 1, 2, \ldots \quad \text{and} \quad r \in [r_{k+1}, r_k).$$

Using $\Phi_{k+1}(B(0, r_k)) = \Phi_k(B(0, r_k))$,

$$t_k^{-2}(r/r_k)^{p_k}\Phi_k(B(0, r_k)) \leq \Phi_k(B(0, r)) \leq t_k^2(r/r_k)^{p_k}\Phi_k(B(0, r_k))$$

if $0 < r \leq r_k$, and

$$t_k^{-2}(r/r_k)^{p_k}\Phi_{k+1}(B(0, r_k)) \leq \Phi_{k+1}(B(0, r)) \leq t_k^2(r/r_k)^{p_k}\Phi_{k+1}(B(0, r_k))$$

if $r \geq \tau_k r_k/k$, we observe that h is nondecreasing,

$$t_k^{-2}(r/r_k)^{p_k}h(r_k) \leq h(r) \leq t_k^2(r/r_k)^{p_k}h(r_k) \quad \text{if } r \in [\tau_k r_k/k, r_k], \quad \text{and}$$

$$(*) \qquad t_k^{-4}h(r) \leq \Phi_k(B(0, r)) \leq t_k^4 h(r) \quad \text{if } r \in [\tau_k r_k, r_{k-1}].$$

Using the second observation and the definition of the measures Ψ_k, we easily see that $\operatorname{Dim}(h) = M$ and that h is irregular. Because of $(*)$, 6.6 is applicable with $E_k = \hat{E}_k = \tilde{E}_k = \operatorname{spt} \Phi_k$. Hence there are a measure Φ over $\mathbf{R}^n$ and a compact set $E \subset \mathbf{R}^n$ of positive Φ measure such that

$$2^{-1/k}\Phi_k(B(0, 2^{-1/k}r)) \leq \Phi(B(x, r)) \leq 2^{1/k}\Phi_k(B(0, 2^{1/k}r))$$

whenever $x \in E$, $k = n, n + 1, \ldots$, and $r \in [4\tau_k r_k, 4\tau_{k-1}r_{k-1}]$. Thus $(*)$ implies that

$$(**) \qquad 2^{-1/k}t_k^{-4}h(2^{-1/k}r) \leq \Phi(B(x, r)) \leq 2^{1/k}t_k^4 h(2^{1/k}r)$$

whenever $x \in E$, $k = n, n + 1, \ldots$, and $r \in [4\tau_k r_k, 4\tau_{k-1}r_{k-1}]$.

Let $\Psi = \Phi \llcorner E$. Then $(**)$ and the density theorem imply that $D_h(\Psi, x) = 1$ at Ψ almost every $x \in \mathbf{R}^n$. Thus (2) is proved. Noting that in the remaining cases $p_k \neq 0$ for each k, we use the definition of the measures Ψ_k to find constants $a_k \in (1, \infty)$ such that $\lim_{k \to \infty} a_k = 1$ and $\Psi_k(B(0, 2^{1/k}r)) \leq a_k \Psi_k(B(0, r))$ for each $k = 1, 2, \ldots$ and each $r > 0$. Hence we infer from $(*)$ and $(**)$ that

$$2^{-1/k}a_k^{-1}t_k^{-12}h(r) \leq \Phi(B(x, r)) \leq 2^{1/k}a_k t_k^{12}h(r)$$

whenever $x \in E$, $k = n, n + 1, \ldots$, and $r \in [4\tau_k r_k, 4\tau_{k-1}r_{k-1}]$. Consequently, $\lim_{r \searrow 0} \Psi(B(x, r))/h(r) = 1$ at Ψ almost every $x \in \mathbf{R}^n$, which proves (3).

Finally, assume that $M = \{m\}$ where $1 \leq m \leq n - 2$. We already know that h is an exact density function in $\mathbf{R}^n$. Moreover, all the measures Ψ_k are defined by (d) and do not depend upon k. From (d) we easily see that $C_k = c_k$ and that there is $a \in (1, \infty)$ such that $\Psi_k(B(x, tr)) \leq at^m\Psi_k(B(x, r))$ for every $k = 1, 2, \ldots$, $r > 0$, and $t > 1$. Using also $C_k = c_k$, we infer that

$$t_k^{-2}(r_{k+1}/r_k)^m h(r_k) \leq h(r_{k+1}) \leq t_k^2(r_{k+1}/r_k)^m h(r_k)$$

for each $k = 1, 2, \ldots$. Consequently,

$$a^{-1}\left(\prod_{k=1}^{\infty} t_k^{-2}\right)h(r_1)/r_1^m \le h(r)/r^m \le a\left(\prod_{k=1}^{\infty} t_k^2\right)h(r_1)/r_1^m$$

for each $r \in (0, r_1)$, which proves (1; ii).

(4) If $M \neq \varnothing$ is a finite set of nonnegative integers and $M \neq \{0\}$, we use (3) with $n = 1 + 2\max(M)$ to find an exact density function h in $\mathbf{R}^n$ such that $\operatorname{Dim}(h) = M$.

6.10. I do not know whether 6.9(4) gives a characterization of possible set $\operatorname{Dim}(h)$ for exact density functions. Using 4.12(2; iii), we can easily prove that, if h is an exact density function in R and $\operatorname{Dim}(h) = \{0\}$ then $0 < \lim_{r \searrow 0} h(r) < \infty$. Because of 6.8(1), this assertion is equivalent to:

If h is an exact density function in $\mathbf{R}$ then

$$0 < \lim_{r \searrow 0} h(r) < \infty \quad \text{or} \quad 0 < \lim_{r \searrow 0} h(r)/r < \infty.$$

Since a simple proof of this fact was given by P. Mattila in [20], we do not give the details here.

6.11. The behaviour of the (optimal values) of the constants $\varepsilon = \varepsilon(n, h)$ from 6.7 is not clear even for functions $h(t) = t^\beta$, $\beta \in (1, 2)$. Using 5.4 with $m = 2$ and $h(t) = t^{\beta - 2}$, we see that

$$\hat{\varepsilon}(n, \gamma) = \inf\{\varepsilon(n, t^\beta); \beta \in (\gamma, 2)\} > 0$$

for every $n = 2, 3, \ldots$ and every $\gamma \in (1, 2)$. I do not know whether $\inf_n \hat{\varepsilon}(n, \gamma) > 0$. On the other hand, from 5.8 with $c = \infty$ and with a_j constant we easily see that

$$\lim_{\beta \searrow 1} \varepsilon(2, t^\beta) = 0.$$

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