Math 558

Homework - Part 2

Due March 16

Problem 1 Problem 8, Chapter 7 Evans, Partial Differential Equations.

Problem 2. Let $u \in H^1(B_r(x_0))$ be a weak non-negative supersolution of L in $B_r(x_0) \subset \mathbb{R}^n$, i.e. for any $\zeta \in H_0^1(B_r(x_0))$ with $\zeta \geq 0$ a.e. in $B_r(x_0)$

$$(*) \qquad \int_{B_r(x_0)} \sum_{|\alpha|,|\beta| \leq 1} a_{\alpha\beta} D^\alpha u \, D^\beta \zeta \geq 0,$$

where

$$(E) \qquad \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \eta^\alpha \eta^\beta \geq |\eta|^2, \quad \forall \eta \in \mathbb{R}^n,$$

and

$$(B_\infty) \qquad \sum_{|\alpha|=|\beta|=1} \|a_{\alpha\beta}\|_{L^\infty(B_r(x_0))} + r \sum_{|\alpha|+|\beta|=1} \|a_{\alpha\beta}\|_{L^\infty(B_r(x_0))} + r^2 \|a_{00}\|_{L^\infty(B_r(x_0))} \leq \Lambda.$$

Show that for every $p \in (0, \frac{n}{n-2})$, and every $\theta \in (0,1)$

$$\left(\int_{B_{\theta r}(x_0)} u^p \right) \left(\int_{B_{\theta r}(x_0)} u^{-p} \right) \leq C r^{2n},$$

where C is a constant that depends on n, θ, Λ, and p.

Hint: Let $r = 1$. Show that for $\gamma \in (0,1)$, $w = (u+\epsilon)^\gamma$ is a supersolution of an operator $\tilde{L}$ on $B_1(x_0)$, whose coefficients satisfy (E) and (B_∞). Show that $v = w^q$, for $q \in (0, 1/2)$ satisfies

$$\int_{B_1(x_0)} |D(v\zeta)|^2 \leq C \int_{B_1(x_0)} v^2 (|D\zeta|^2 + |\zeta|^2),$$

for $\zeta \in C_c^\infty(B_1(x_0))$.

Problem 3. Let $u \in H^1(\Omega)$ be a weak non-negative solution of $Lu = \sum_{|\beta|=1} D^\beta f_\beta - f_0$ in $\Omega \subset \mathbb{R}^n$, i.e. for any $\zeta \in H_0^1(\Omega)$

$$\int_\Omega \sum_{|\alpha|,|\beta|\le 1} a_{\alpha\beta} D^\alpha u \, D^\beta \zeta = \int_\Omega \sum_{|\beta|=1} f_\beta \, D^\beta \zeta + \int_\Omega f_0 \zeta, \tag{1}$$

where

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \eta^\alpha \eta^\beta \ge |\eta|^2, \quad \forall \eta \in \mathbb{R}^n. \tag{E}$$

We also assume that if $B_r(x_0) \subset \Omega$ then

$$\sum_{|\alpha|=|\beta|=1} \|a_{\alpha\beta}\|_{L^\infty(B_r(x_0))} + r \sum_{|\alpha|+|\beta|=1} \|a_{\alpha\beta}\|_{L^\infty(B_r(x_0))} + r^2 \|a_{00}\|_{L^\infty(B_r(x_0))} \le \Lambda_0, \tag{B_∞}$$

and

$$\Lambda_1(r) = r \sum_{|\beta|=1} \|f_\beta\|_{L^\infty(B_r(x_0))} + r^2 \|f_0\|_{L^\infty(B_r(x_0))} \le \Lambda_1.$$

Show that for every $p \in (0, \frac{n}{n-2})$, and $B_{4\rho}(y) \subset \Omega$

$$\left(\rho^{-n} \int_{B_{2\rho}(y)} u^p \right)^{1/p} \le C \inf_{B_\rho(y)} u + C\Lambda_1(\rho),$$

where C only depends on n, Λ_0, Λ_1, and p.

Hint: Note that you may assume without loss of generality that $\rho = 1$ and that $\Lambda_1(1) = 1$. Let $\gamma < 1$. Replace ζ above by $(u+1)^{\gamma-1}\zeta^2$. Consider separately the cases $\gamma = 0$, $\gamma < 0$, and $\gamma \in (0, 1)$.