An estimate for supersolutions of second order elliptic operators with bounded coefficients

Lemma. *Let $u \in H^1(B_r(x_0))$ be a weak non-negative supersolution of L in $B_r(x_0) \subset \mathbf{R}^n$, i.e. for any $\zeta \in H^1_0(B_r(x_0))$ with $\zeta \geq 0$ a.e. in $B_r(x_0)$*

$$(*) \qquad \int_{B_r(x_0)} \sum_{|\alpha|,|\beta| \leq 1} a_{\alpha\,\beta} D^\alpha u \, D^\beta \zeta \geq 0,$$

where

$$(E) \qquad \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \eta^\alpha \eta^\beta \geq |\eta|^2, \quad \forall \eta \in \mathbf{R}^n,$$

and

$$(B_\infty) \qquad \sum_{|\alpha|=|\beta|=1} \|a_{\alpha\beta}\|_{L^\infty(B_r(x_0))} + \sum_{|\alpha|+|\beta|=1} r\|a_{\alpha\beta}\|_{L^\infty(B_r(x_0))} + r^2\|a_{00}\|_{L^\infty(B_r(x_0))} \leq \Lambda.$$

Then for $\theta \in (0,1)$, there exists $p_0 > 0$ such that $p \in (0, p_0)$,

$$\left(\int_{B_{\theta r}(x_0)} u^p\right)\left(\int_{B_{\theta r}(x_0)} u^{-p}\right) \leq C r^{2n},$$

where C is a constant that depends on n, θ, Λ, and p.

Proof: Recall that for $A, B > 0$, $\epsilon > 0$, $p > 1$, and $p' = p/p-1$ the following inequality holds

$$(0) \qquad AB \leq \epsilon A^p + C(\epsilon, p) B^{p'}.$$

Without loss of generality assume that $r = 1$ and $x_0 = 0$. By a computation carried out in class we have that, if u is as above for any $\zeta \in H^1_0(B_1)$ with $\zeta \geq 0$ a.e. in B_1,

$$(1) \qquad \int \zeta^2 |Dw|^2 \leq C \int (|D\zeta|^2 + \zeta^2),$$

where $w = \log(u + \epsilon)$ for $\epsilon > 0$. C is a constant independent of ϵ. Choosing $\zeta \in C^\infty_c(B_1)$, $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $B_{\sqrt{\theta}}$, and $|D\zeta| \leq C_\theta$, we have that

$$(2) \qquad \int_{B_{\sqrt{\theta}}} |Dw|^2 \leq C.$$

Let $q \geq 2$, $b > 0$, $a \geq b+1$, $\varphi \in C_c^\infty(B_{\sqrt{\theta}})$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in B_θ, and $|D\varphi| \leq C_\theta$. Let $v = |w - \lambda|$, where λ is a constant to be chosen. Note that $|Dv| = |Dw|$ a.e. If $\zeta = v^{q-1}\varphi^{aq-b}$, (1) yields

$$\int v^{2(q-1)}|Dv|^2\varphi^{2aq-2b} \leq Cq^2 \int v^{2(q-2)}|Dv|^2\varphi^{2aq-2b} + Cq^2 \int v^{2(q-1)}\varphi^{2aq-2b-2}. \tag{3}$$

If $q > 2$, applying (0) with $p = (q-1)/(q-2)$ and $A = v^{2(q-2)}$, we have that

$$Cq^2 v^{2(q-2)} \leq \frac{1}{2}v^{2(q-1)} + C^q q^{2(q-1)}.$$

Then (3) becomes

$$\int v^{2(q-1)}|Dv|^2\varphi^{2aq-2b} \leq C^q q^{2(q-1)} \int |Dv|^2\varphi^{2aq-2b} + Cq^2 \int v^{2(q-1)}\varphi^{2aq-2b-2}. \tag{4}$$

Note that $v^{2(q-1)} \leq 1 + v^{2q}$. Combining the fact that $|Dw| = |Dv|$ a.e., (2) and (4) we have that

$$\int v^{2(q-1)}|Dv|^2\varphi^{2aq-2b} \leq C^q q^{2(q-1)} + Cq^2 \int v^{2q}\varphi^{2aq-2b-2}. \tag{5}$$

Thus

$$\int |D(v^q\varphi^{aq-b})|^2 \leq C^q q^{2q} + Cq^4 \int v^{2q}\varphi^{2aq-2b-2}. \tag{6}$$

Applying Sobolev's inequality in (6) with $\kappa = n/n-2$ if $n \geq 3$, and $\kappa > 1$ if $n = 2$ we obtain

$$\left(\int (v\varphi^a)^{2q\kappa}\varphi^{-2b\kappa}dx\right)^{1/\kappa} \leq (Cq^2)^q + Cq^4 \int (v\varphi^a)^{2q}\varphi^{-2(b+1)}dx. \tag{7}$$

Choose b such that $b+1 = b\kappa$, and let $d\nu = \varphi^{-2b\kappa}dx = \varphi^{-2(b+1)}dx$. Let $q = \kappa^\ell$ for $\ell \geq \ell_0$ where $\ell_0 \geq 1$ is such that $\kappa^{\ell_0 - 1} \leq 2 < \kappa^{\ell_0}$. (7) becomes

$$\left(\int (v\varphi^a)^{2\kappa^{\ell+1}}d\nu\right)^{1/\kappa} \leq (C\kappa^{2\ell})^{\kappa^\ell} + C\kappa^{4\ell} \int (v\varphi^a)^{2\kappa^\ell}d\nu. \tag{8}$$

If

$$\Psi(\ell) = \left(\int (v\varphi^a)^{2\kappa^\ell}d\nu\right)^{1/\kappa^\ell},$$

(8) can be written as follows

$$\Psi(\ell+1) \leq C\kappa^{2(\ell+1)} + C^{1/\kappa^\ell}\kappa^{4\ell/\kappa^\ell}\Psi(\ell). \tag{9}$$

Iterating inequality (9) we conclude that for $\ell \geq \ell_0$

$$\begin{aligned} \Psi(\ell+1) &\leq \left(\sum_{j\geq 1} C^{1/\kappa^j}\kappa^{4j/\kappa^j}\right)\left(\sum_{j=\ell_0}^{\ell}\kappa^{2j}+\Psi(\ell_0)\right) \\ &\leq C\kappa^{2(\ell+1)}+C\Psi(\ell_0). \end{aligned} \tag{10}$$

In order to bound $\Psi(\ell_0)$, recall inequality (3) for $q = 2$

$$\begin{aligned} \int v^2|Dv|^2\varphi^{4a-2b} &\leq C\int |Dv|^2\varphi^{4a-2b}+C\int v^2\varphi^{4a-2b-2} \\ &\leq C+C\int v^2\varphi^{4a-2b-2} \\ &\leq C+C\int v^4\varphi^{4a-2b-2}. \end{aligned} \tag{11}$$

Thus

$$\int |D(v^2\varphi^{2a-b})|^2 \leq C+C\int v^4\varphi^{4a-2b-2}. \tag{12}$$

Applying Sobolev's inequality with κ as above we have that

$$\left(\int (v\varphi^a)^{4\kappa}d\nu\right)^{1/\kappa} \leq C+C\int (v\varphi^a)^4 d\nu. \tag{13}$$

Apply Hölder's inequality with $p = 2\kappa - 1$, then inequality (0) with $p = (2\kappa-1)/\kappa > 1$ we have that for $\delta > 0$ small

$$\begin{aligned} \int (v\varphi^a)^4 d\nu &= \int (v\varphi^a)^{4\kappa/(2\kappa-1)}(v\varphi^a)^{4(1-\kappa/(2\kappa-1))}d\nu \\ &\leq \left(\int (v\varphi^a)^{4\kappa}d\nu\right)^{1/(2\kappa-1)}\left(\int (v\varphi^a)^2 d\nu\right)^{(2\kappa-2)/(2\kappa-1)} \\ &\leq \delta\left(\int (v\varphi^a)^{4\kappa}d\nu\right)^{1/\kappa}+C_\delta\left(\int (v\varphi^a)^2 d\nu\right)^2. \end{aligned} \tag{14}$$

Combining (13) and (14) we conclude that provided $\delta > 0$ is small enough

$$\left(\int (v\varphi^a)^{4\kappa}d\nu\right)^{1/2\kappa} \leq C+C\int (v\varphi^a)^2 d\nu. \tag{15}$$

Note that if $\kappa^{\ell_0-1} \leq 2$, (15) insures that

$$\Psi(\ell_0) \leq C+C\left(\int (v\varphi^a)^2 d\nu\right). \tag{16}$$

We now assume that $\kappa^{\ell_0 - 1} < 2 < \kappa^{\ell_0}$. Let $2\kappa^{\ell_0} = 2\kappa^{\ell_0}(\alpha+\beta)$ where $\alpha = (\kappa^{\ell_0}-2)/\kappa^{\ell_0-1}(\kappa-1)$ and $\beta = 1-\alpha$. Apply Hölder's inequality in $\Psi(\ell_0)$ with $p = (2\kappa-2)/(\kappa^{\ell_0}-2)$, and $p' = p/(p-1)$. Note that $2\kappa^{\ell_0}\alpha p = 4\kappa$ and $2\kappa^{\ell_0}\beta p = 4$. Then the first inequality in (14), (0) applied with the appropriate exponent, and inequality (15) yield

$$
\begin{aligned}
\Psi(\ell_0) &\le \left(\int (v\varphi^a)^{4\kappa} d\nu\right)^{1/p\kappa^{\ell_0}} \left(\int (v\varphi^a)^4 d\nu\right)^{1/p'\kappa^{\ell_0}} \\
&\le \left(\int (v\varphi^a)^{4\kappa} d\nu\right)^{1/p\kappa^{\ell_0}+1/p'(2\kappa-1)\kappa^{\ell_0}} \left(\int (v\varphi^a)^2 d\nu\right)^{(2\kappa-2)/(2\kappa-1)p'\kappa^{\ell_0}} \\
&\le C\left(\int (v\varphi^a)^{4\kappa} d\nu\right)^{1/2\kappa} + C\left(\int (v\varphi^a)^2 d\nu\right) \\
&\le C + C\int (v\varphi^a)^2 d\nu.
\end{aligned} \tag{16}
$$

Since $a \ge b+1$, $\varphi \in C_c^\infty(B_{\sqrt{\theta}})$, $0 \le \varphi \le 1$, and $v = |w-\lambda|$, choosing $\lambda = |B_{\sqrt{\theta}}|^{-1}\int_{B_{\sqrt{\theta}}} w dx$, applying Poincaré's inequality and using inequality (2), (16) becomes

$$
\Psi(\ell_0) \le C + C\int_{B_{\sqrt{\theta}}} |w-\lambda|^2 dx \le C + C\int_{B_{\sqrt{\theta}}} |Dw|^2 dx \le C. \tag{17}
$$

Combining (10), (17), Hölder's inequality, and the fact that $dx \le d\nu$ we conclude that for $\ell \ge 0$

$$
\left(\int (v\varphi^a)^{2\kappa^\ell} dx\right)^{1/2\kappa^\ell} \le C\kappa^\ell. \tag{18}
$$

For $j \ge 1$ there is $\ell \ge 0$ so that $2\kappa^{\ell-1} \le j < 2\kappa^\ell$, thus

$$
\left(\int (v\varphi^a)^j dx\right)^{1/j} \le C\left(\int (v\varphi^a)^{2\kappa^\ell} dx\right)^{1/2\kappa^\ell} \le C\kappa^\ell \le Cj. \tag{19}
$$

Hence

$$
\int_{B_\theta} v^j dx \le C^j j^j. \tag{20}
$$

Dividing (20) by $(2C)^j j^j$, and summing over all $j \ge 1$ we obtain

$$
\int_{B_\theta} \sum_{j\ge 1} \left(\frac{v}{2C}\right)^j \left(\frac{1}{j}\right)^j \le 2. \tag{21}
$$

Recall that $j^{-1}(j!)^{1/j} \to \tau \in (0,1)$ as $j \to \infty$. Assume that for $j > j_0$, $j^{-1}(j!)^{1/j} \geq \tau/2$. Thus for $s > 0$

$$\begin{aligned} \sum_{j\geq 1} \frac{s^j}{j^j} = \sum_{j\geq 1} \frac{s^j}{j!} \cdot \frac{j!}{j^j} &\geq \frac{j_0!}{j_0^{j_0}} \sum_{j=1}^{j_0} \frac{s^j}{j!} + \sum_{j\geq j_0+1} \frac{1}{j!} \left(\frac{s\tau}{2}\right)^j \\ &\geq \frac{j_0!}{j_0^{j_0}} \sum_{j\geq 1} \frac{1}{j!} \left(\frac{s\tau}{2}\right)^j \\ &\geq \frac{j_0!}{j_0^{j_0}} \exp(\frac{s\tau}{2}). \end{aligned} \tag{22}$$

Combining inequalities (21) and (22) we conclude that

$$\int_{B_\theta} \exp\left(\frac{v\tau}{4C}\right) \leq C_0. \tag{23}$$

Recall that $v = |\log(u+\epsilon) - \lambda|$. Let $p_0 = \tau/4C > 0$ and note that for $p \in (0, p_0]$ (23) implies

$$\int_{B_\theta} \exp\left(p\,|\log(u+\epsilon) - \lambda|\right) \leq C_0. \tag{24}$$

Thus

$$\exp(-p\lambda) \int_{B_\theta} \exp\left(p\,\log(u+\epsilon)\right) \leq C_0, \tag{25}$$

and

$$\exp(p\lambda) \int_{B_\theta} \exp\left(-p\,\log(u+\epsilon)\right) \leq C_0. \tag{26}$$

Multiplying (25) and (26) we conclude that

$$\left(\int_{B_\theta} (u+\epsilon)^p\right)\left(\int_{B_\theta} (u+\epsilon)^{-p}\right) \leq C. \tag{27}$$

Since $u + \epsilon \to u$ a.e as $\epsilon \to 0$, Fatou's lemma guarantees that

$$\left(\int_{B_\theta} u^p\right)\left(\int_{B_\theta} u^{-p}\right) \leq C. \qquad \blacksquare$$