Math 524

Homework due 10/14/09

Reading from Stein & Shakarchi: Chapter 1, §2, 3, 4.

Problem 1. Let (X, ρ) be a metric space, and let $\{x_n\}_{n\geq 1} \subset X$ be a Cauchy sequence. Assume that $\{x_n\}$ has a cluster point $x \in X$. Prove that the sequence $\{x_n\}$ converges to x. (Recall x is a cluster point for $\{x_n\}$ if $\forall\ \epsilon > 0$ and $\forall N \in \mathbb{N}$, $\exists n \in \mathbb{N}$, with $n \geq N$ such that $\rho(x, x_n) < \epsilon$.)

Problem 2. Let (X, ρ) be a metric space. Prove that X is complete if and only if for any decreasing sequence $\{A_n\}$ of non-empty closed subsets of X (i.e.$\cdots \subset A_n \subset A_{n-1} \cdots \subset A_2 \subset A_1$, $A_i \neq \emptyset$, A_i closed) such that

$$\lim_{n\to\infty} \operatorname{diam} A_n = 0,$$

then

$$\cap_{n=1}^{\infty} A_n = \{x\} \quad \text{for some} \quad x \in X.$$

Problem 3. Let F be a subset of $[0, 1]$ constructed in the same manner as the Cantor ternary set except that each of the intervals removed at the nth step has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Then F is closed, F^c is dense in $[0, 1]$ and $m_* F = 1 - \alpha$. Such set F is called a *generalized Cantor set*.

Exercise from Stein & Shakarchi: 3, Chapter 1 (page 38).
Note the differences between the last 2 problems.

($*$) Problem from Stein & Shakarchi: 2, Chapter 1 (page 46). This type of decomposition of an open set is called a Whitney decomposition. It is an important tool in Harmonic Analysis.