Math 524

Homework due 12/09/09

Reading from Stein & Shakarchi: Chapter 6, §1, §2, and §3.

Problem 1: Let X be a non-empty set. A *monotone class* $\mathcal{C}$ on X is a subcollection of $\mathcal{P}(X)$ which is closed under countable increasing unions and countable decreasing intersections.

1. Prove that the intersection on any family of monotone classes is a monotone class. Hence for any $\mathcal{E} \subset \mathcal{P}(X)$ there exits a smallest monotone class containing $\mathcal{E}$. This monotone class is the monotone class generated by $\mathcal{E}$.

2. Prove *The monotone class lemma:* If $\mathcal{A}$ is an algebra of subsets of X, the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$.

Problem 2: Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, μ=Lebesgue measure and ν=counting measure. If $D = \{(x,x) : \ x \in [0,1]\}$ is the diagonal in $X \times Y$, show that

$$\iint \chi_D \, d\mu d\nu, \quad \iint \chi_D \, d\nu d\mu, \quad \iint \chi_D \, d(\mu \times \nu)$$

are all unequal. To compute $\iint \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.

Problem 3: Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in \mathcal{L}^+$. Let

$$G_f = \{(x,y) \in X \times [0,\infty] : \ y \le f(x)\}.$$

1. Prove that G_f is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ measurable. (Hint: consider the map $(x,y) \to f(x) - y$).

2. Prove that $\mu \times m(G_f) = \int f \, d\mu$.

Problem 4: Let $(X\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be arbitrary measure spaces (not necessarily σ-finite).

1. If $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{R}$ is $\mathcal{N}$-measurable, and $h(x,y) = f(x)g(y)$, prove that h is $\mathcal{M} \otimes \mathcal{N}$-measurable.
2. If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, prove that $h \in L^1(\mu \times \nu)$ and
$$\int h \, d(\mu \times \nu) = \left[\int f \, d\mu\right] \left[\int g \, d\nu\right].$$

Problem 5: Let (X,d) be a metric space. Let μ be a doubling measure on $\mathcal{B}_X$, i.e. there exists a constant $C \geq 1$ such that $\forall x \in X$ and $r > 0$
$$\mu(B(x,2r)) \leq C\mu(B(x,r)).$$
Assume that $\mu(B(x_0,1)) < \infty$ for some $x_0 \in X$.

1. Show that if K is compact then $\mu(K) < \infty$.
2. Assume $X = \mathbb{R}^n$ and d is the Euclidean distance. Prove that if $E \in \mathcal{B}_{\mathbb{R}^n}$ with $\mu(E) < \infty$ then given $\epsilon > 0$ there exists a compact set K with $K \subset E$ and $\mu(E \backslash K) < \epsilon$.

   Note: μ is not necessarily Lebesgue measure.