Math 524

Homework due 10/06/10

Reading from Stein & Shakarchi: Introduction, Chapter 1, §1,2,3.

Definition: Let (X, ρ) be a metric space. We say that $x \in X$ is a cluster point for $\{x_n\} \subset X$ if $\forall\ \epsilon > 0$ and $\forall N \in \mathbb{N}$, $\exists n \in \mathbb{N}$, with $n \geq N$ such that $\rho(x, x_n) < \epsilon$.

Problem 1. Let (X, ρ) be a metric space, and let $\{x_n\}_{n\geq 1} \subset X$ be a Cauchy sequence. Assume that $\{x_n\}$ has a cluster point $x \in X$. Prove that the sequence $\{x_n\}$ converges to x.

Problem 2. Let $\{x_n\}_n \subset \mathbb{R}$, and $x \in \mathbb{R}$. Show that $x = \lim_{n\to\infty} x_n$ if and only if every subsequence of $\{x_n\}$ has in turn a subsequence which converges to x.

Problem 3. Prove that a metric space (X, ρ) is separable if and only if there exists a countable family of open sets $\{O_i\}$ such that for any open set O,

$$O = \bigcup_{O_i \subset O} O_i.$$

Problem 4. Prove that $E \subset \mathbb{R}^n$ is compact if and only if E is closed and bounded.

Problem 5. Let (X, ρ) be a metric space. Prove that X is complete if and only if for any decreasing sequence $\{A_n\}$ of non-empty closed subsets of X (i.e.$\cdots \subset A_n \subset A_{n-1} \cdots \subset A_2 \subset A_1$, $A_i \neq \emptyset$, A_i closed) such that

$$\lim_{n\to\infty} \operatorname{diam} A_n = 0,$$

then

$$\cap_{n=1}^{\infty} A_n = \{x\} \quad \text{for some} \quad x \in X.$$

Recall

$$\operatorname{diam} A = \sup\{\rho(x, y) :\ x, y \in E\}.$$