Math 524

Homework due 11/17/10

Reading from Stein & Shakarchi: Chapter 2: §1, §2. Chapter 6: §2.

Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Let $\{f_n\}$ be a sequence of real valued measurable functions.

- The sequence $\{f_n\}$ is **Cauchy in measure** if for every $\epsilon > 0$

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \text{ as } m, n \to \infty.$$

- The sequence $\{f_n\}$ **converges in measure** to f if for every $\epsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \text{ as } n \to \infty.$$

Problem 1 (20 points): Suppose that $\{f_n\}$ is Cauchy in measure .

1. Prove that there exists a function f such that $f_n \to f$ in measure.
2. Prove that there exists a subsequence $\{f_{n_j}\}$ that converges to f μ-a.e.
3. Show that if $f_n \to g$ in measure then $g = f$ μ-a.e.

Problem 2: Let μ be the counting measure on $\mathbb{N}$. Then $f_n \to f$ is measure if and only if $f_n \to f$ uniformly.

Problem 3: Suppose μ is σ-finite and $f_n \to f$ μ-a.e.. Show that there exist measurable sets $\{E_n\}$ in X such that $\mu\left((\bigcup_{j=1}^{\infty} E_j)^c\right) = 0$ and $f_n \to f$ uniformly on each E_j.